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# Volume comparison theorem for tubular neighborhoods of submanifolds in Finsler geometry and its applications

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**Abstract.** We consider the distance to compact submanifolds and study volume comparison for tubular neighborhoods of compact submanifolds. As applications, we obtain a lower bound for the length of a closed geodesic of a compact Finsler manifold. When the Finsler metric is reversible, we also provide a lower bound of the injectivity radius. Our results are Finsler versions of Heintze–Karcher's and Cheeger's results for Riemannian manifolds.

**1. Introduction.** The comparison technique is a powerful tool in global differential geometry, and it has been well developed in Riemannian geometry. Volume, as an important geometric invariant, plays a key role in the comparison technique. Recently the comparison technique has been developed for Finsler manifolds, and the relationship between curvature and topology of Finsler manifolds has also been investigated [BCS, S1, S2, W1, W2, W3, WX]. The basic comparison objects are distance and volume. It should be pointed out here that the volume form is uniquely determined by the given Riemannian metric, while there are different choices of volume forms for Finsler metrics. As a result, we usually need to control the S-curvature in order to obtain volume comparison theorems as well as results on curvature and topology. This additional assumption on S-curvature has been removed by the author recently by using the extreme volume forms (the maximal and minimal volume forms) [W1, W2, W3].

Generally distance is taken to a fixed point. In this paper we shall consider the distance to compact submanifolds and study volume comparison for tubular neighborhoods of compact submanifolds. As applications, we obtain a lower bound for the length of a closed geodesic of a compact Finsler manifold. When the Finsler metric is reversible, we also provide a lower bound of the injectivity radius.

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To present our results, let us first introduce some notations of Finsler geometry (see §2 for details). On a compact Finsler manifold $(M,F)$ let $\mathrm{vol}_{\min}(M)$ be the minimal volume of $(M,F)$, $\mu=\mu(x)$ the uniformity function of $M$, and $\|\mathbf{T}\|(x)$ the norm of the T-curvature at $x\in M$. The main results of this paper are the following two theorems.

THEOREM 1.1. *Let $(M,F)$ be an $n$-dimensional compact Finsler manifold with flag curvature satisfying $\mathbf{K}_M\geq -1$, $d=\mathrm{diam}(M)$, the diameter of $M$, and*

$$\Lambda=\max_{x\in M}\mu(x),\qquad \Xi=\max_{x\in M}\|\mathbf{T}\|(x).$$

*Then the length $L(c)$ of any closed geodesic $c$ of $M$ satisfies*

$$L(c)\geq\frac{(n-1)\,\mathrm{vol}_{\min}(M)}{\Lambda^{(n+4)/2}\,\mathrm{vol}(\mathbb{S}^{n-2})(1+\Xi)\sinh^{n-1}d},$$

*where $\mathbb{S}^{n-2}$ denotes the standard unit $(n-2)$-sphere.*

THEOREM 1.2. *Let $(M,F)$ be an $n$-dimensional compact reversible Finsler manifold with flag curvature satisfying $|\mathbf{K}_M|\leq 1$, $d=\mathrm{diam}(M)$, and*

$$\Lambda=\max_{x\in M}\mu(x),\qquad \Xi=\max_{x\in M}\|\mathbf{T}\|(x).$$

*Then the injectivity radius $\mathfrak{i}(M)$ of $M$ satisfies*

$$\mathfrak{i}(M)\geq\min\left\{\pi,\frac{(n-1)\,\mathrm{vol}_{\min}(M)}{2\Lambda^{(n+4)/2}\,\mathrm{vol}(\mathbb{S}^{n-2})(1+\Xi)\sinh^{n-1}d}\right\}.$$

REMARK 1.3. Theorems 1.1 and 1.2 are Finsler versions of Heintze–Karcher's and Cheeger's results for Riemannian manifolds (see, e.g., [C, HK]).

**2. Preliminaries.** In this section, we give a brief description of basic quantities and fundamental formulas of Finsler geometry; for more details see [BCS, S1]. Let $(M,F)$ be a Finsler $n$-manifold with Finsler metric $F:TM\to[0,\infty)$. Let $(x,y)=(x^i,y^i)$ be local coordinates on $TM$, and $\pi:TM\setminus 0\to M$ the natural projection. Unlike the Riemannian case, most Finsler quantities are functions on $TM$ rather than $M$. The *fundamental tensor* $g_{ij}$ and the *Cartan tensor* $C_{ijk}$ are defined by

$$g_{ij}(x,y):=\frac{1}{2}\frac{\partial^2F(x,y)^2}{\partial y^i\partial y^j},\quad C_{ijk}(x,y):=\frac{1}{4}\frac{\partial^3F(x,y)^2}{\partial y^i\partial y^j\partial y^k}.$$

Let $\Gamma^i_{jk}(x,y)$ be the Chern connection coefficients, and write $\mathbf{g}_y=g_{ij}(x,y)dx^i\otimes dx^j$. Let $V=v^i\partial/\partial x^i$ be a nonvanishing vector field on an open subset $\mathcal{U}\subset M$. One can introduce a Riemannian metric $\widetilde{g}=\mathbf{g}_V$ and a linear connection $\nabla^V$ (called the *Chern connection*) on the tangent bundle over $\mathcal{U}$ as follows:

$$\nabla^V_{\partial/\partial x^i}\frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x,v)\frac{\partial}{\partial x^k}.$$

From the torsion-freeness and almost $g$-compatibility of the Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X,Y], \tag{2.1}$$

$$X\cdot \mathbf{g}_V(Y,Z) = \mathbf{g}_V(\nabla^V_X Y, Z) + \mathbf{g}_V(Y,\nabla^V_X Z) + 2\mathbf{C}_V(\nabla^V_X V, Y, Z), \tag{2.2}$$

where $\mathbf{C}_V$ is defined by

$$\mathbf{C}_V(X,Y,Z) = X^iY^jZ^kC_{ijk}(x,v),$$

and it satisfies

$$\mathbf{C}_V(V,X,Y) = 0. \tag{2.3}$$

Let

$$\mathbf{T}_V(U) = \mathbf{g}_V(V,\nabla^V_U U - \nabla^U_U U),\quad \forall V,U\in TM\setminus 0.$$

$\mathbf{T}$ is called the *T-curvature* [S1]. It is clear that $\mathbf{T}\equiv 0$ if and only if $(M,F)$ is a Berwald manifold. The norm $\|\mathbf{T}\|(x)$ of the T-curvature at $x\in M$ is defined by

$$\|\mathbf{T}\|(x) = \max_{v,u\in T_xM\setminus\{0\}}\frac{|\mathbf{T}_v(u)|}{F(v)F(u)^2}.$$

The *Chern curvature* $R^V(X,Y)Z$ for vector fields $X,Y,Z$ on $\mathcal{U}$ is defined by

$$R^V(X,Y)Z := \nabla^V_X\nabla^V_Y Z - \nabla^V_Y\nabla^V_X Z - \nabla^V_{[X,Y]}Z.$$

In the Riemannian case this curvature does not depend on $V$ and coincides with the Riemannian curvature tensor. Let $\mathbf{R}^V(X) = R^V(X,V)V$. Then $\mathbf{g}_V(\mathbf{R}^V(X),Y) = \mathbf{g}_V(X,\mathbf{R}^V(Y))$. For a *flag* $(V;P)$ (or $(V;W)$) consisting of a nonzero tangent vector $V\in T_xM$ and a 2-plane $P\subset T_xM$ with $V\in P$ the *flag curvature* $\mathbf{K}(V;P)$ is defined as follows:

$$\mathbf{K}(V;P) = \mathbf{K}(V;W) := \frac{\mathbf{g}_V(\mathbf{R}^V(W),W)}{\mathbf{g}_V(V,V)\mathbf{g}_V(W,W) - \mathbf{g}_V(V,W)^2}.$$

Here $W$ is a tangent vector such that $V,W$ span the 2-plane $P$, and $V\in T_xM$ is extended to a geodesic field, i.e., $\nabla^V_V V = 0$ near $x$. Let $\gamma:[0,b]\to M$ be a unit speed geodesic with tangent vector field $T$. A vector field $J$ along $\gamma$ is called a *Jacobi field* along $\gamma$ if

$$\nabla^T_T\nabla^T_T J + \mathbf{R}^T(J) = 0. \tag{2.4}$$

Given a Finsler manifold $(M,F)$, the *dual Finsler metric* $F^*$ on $M$ is defined by

$$F^*(\xi_x) := \sup_{Y\in T_xM\setminus\{0\}}\frac{\xi(Y)}{F(Y)},\quad \forall\xi\in T^*M,$$

and the corresponding fundamental tensor is defined by

$$g^{*kl}(\xi) = \frac{1}{2}\frac{\partial^2 F^*(\xi)^2}{\partial\xi_k \partial\xi_l}.$$

The *Legendre transformation* $l : TM \to T^*M$ is defined by

$$l(Y) = \begin{cases} \mathbf{g}_Y(Y, \cdot), & Y \neq 0, \\ 0, & Y = 0. \end{cases}$$

It is well-known that for any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM \setminus \{0\}$ onto $T_x^*M \setminus \{0\}$, and it is norm-preserving, that is, $F(Y) = F^*(l(Y))$ for all $Y \in TM$. Let $f : M \to \mathbb{R}$ be a smooth function on $M$. The *gradient* of $f$ is defined by $\nabla f = l^{-1}(df)$. Thus at points where $df \neq 0$ we have

$$df(X) = \mathbf{g}_{\nabla f}(\nabla f, X), \qquad X \in TM.$$

A *volume form* $d\mu$ on a Finsler manifold $(M, F)$ is nothing but a global nondegenerate $n$-form on $M$. In local coordinates we can express $d\mu$ as $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. The frequently used volume forms in Finsler geometry are the so-called Busemann–Hausdorff volume form and the Holmes–Thompson volume form. In [W1] we introduce the maximal and minimal volume forms for Finsler manifolds which play an important role in the comparison technique in Finsler geometry. They are defined as follows. Let

$$dV_{\max} = \sigma_{\max}(x)dx^1 \wedge \cdots \wedge dx^n$$

and

$$dV_{\min} = \sigma_{\min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{\max}(x) := \max_{y \in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x,y))}, \quad \sigma_{\min}(x) := \min_{y \in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the $n$-forms $dV_{\max}$ and $dV_{\min}$ are well-defined on $M$, and they are called the *maximal volume form* and the *minimal volume form* of $(M, F)$, respectively. The volume with respect to $dV_{\max}$ (resp. $dV_{\min}$) is called the *maximal volume* (resp. *minimal volume*).

The *uniformity function* $\mu : M \to \mathbb{R}$ is defined by

$$\mu(x) = \max_{y,z,u \in T_xM\setminus\{0\}} \frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

$\Lambda = \max_{x\in M} \mu(x)$ is called the *uniformity constant* [E]. It is clear that

$$\mu^{-1}F(u)^2 \leq \mathbf{g}_y(u,u) \leq \mu F(u)^2.$$

Similarly, the *reversible function* $\lambda : M \to \mathbb{R}$ is defined by

$$\lambda(x) = \max_{y \in T_xM\setminus\{0\}} \frac{F(y)}{F(-y)}.$$

$\lambda_F = \max_{x\in M} \lambda(x)$ is called the *reversibility* of $(M,F)$ [R], and $(M,F)$ is called *reversible* if $\lambda_F = 1$. It is clear that $\lambda(x)^2 \le \mu(x)$.

Fix $x \in M$, and let $\mathcal{S}_x = \{v \in T_xM : F(v) = 1\}$ and $\mathcal{B}_x = \{v \in T_xM : F(v) < 1\}$. On $T_xM \setminus \{0\}$ one can consider the singular Riemannian metric $\hat{g}(y) = \mathbf{g}_y$ which induces a Riemannian metric $\dot{g}$ on $\mathcal{S}_x$. Define the *density* $\Theta_x$ at $x \in M$ by [S2]

$$\Theta_x = \frac{\mathrm{vol}_{\dot{g}}(\mathcal{S}_x)}{\mathrm{vol}(\mathbb{S}^{n-1})}.$$

Then by [W1] we have

$$\mu(x)^{-n/2} \le \Theta_x = \frac{\mathrm{vol}_{\dot{g}}(\mathcal{S}_x)}{\mathrm{vol}(\mathbb{S}^{n-1})} = \frac{\mathrm{vol}_{\hat{g}}(\mathcal{B}_x)}{\mathrm{vol}(\mathbb{B}^n)} \le \mu(x)^{n/2}. \tag{2.5}$$

Here $\mathbb{S}^{n-1}$ and $\mathbb{B}^n$ are the unit Euclidean $(n-1)$-sphere and unit $n$-ball, respectively. For $v \in \mathcal{S}_x$, the *cut-value* $\mathfrak{c}(v)$ is defined by

$$\mathfrak{c}(v) := \sup\{t > 0 : d_F(x, \exp_x(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(x)$ of $x$ by $\mathbf{C}(x) := \{\mathfrak{c}(v)v : \mathfrak{c}(v) < \infty,\ v \in \mathcal{S}_x\}$, the *cut locus* $\mathcal{C}(x)$ of $x$ by $\mathcal{C}(x) = \exp_x \mathbf{C}(x)$, and the *injectivity radius* $\mathfrak{i}_x$ at $x$ by $\mathfrak{i}_x = \inf\{\mathfrak{c}(v) : v \in \mathcal{S}_x\}$. When $M$ is compact, the *injectivity radius* of $M$ is defined by $\mathfrak{i}(M) = \min_{x\in M} \mathfrak{i}_x$.

**3. Fermi coordinates for a Minkowski space.** A Finsler manifold $(V,F)$ is called a *Minkowski space* if $V$ is a vector space and the fundamental tensor $g_{ij}(x,y) = g_{ij}(y)$ is independent of the position $x \in V$.

LEMMA 3.1. *Let $(V,F)$ be a Minkowski space with uniformity constant $\mu$, and $u,w,y \in V$ be vectors such that $y = w + u$ and $\mathbf{g}_u(u,w) = 0$. Then*

$$\frac{1}{\mu}F(w)^2 + F(u)^2 \le F(y)^2 \le \mu F(w)^2 + F(u)^2. \tag{3.1}$$

*Proof.* Let

$$f(t) = \tfrac{1}{2}F(u+tw)^2 = \tfrac{1}{2}\mathbf{g}_{u+tw}(u+tw, u+tw), \qquad t \in [0,1].$$

Then

$$f'(t) = \mathbf{g}_{u+tw}(w, u+tw), \tag{3.2}$$

$$f''(t) = \mathbf{g}_{u+tw}(w,w) \le \mu F(w)^2. \tag{3.3}$$

Now consider

$$g(t) = f(t) - \tfrac{1}{2}\mu F(w)^2 t^2 - \tfrac{1}{2}F(u)^2.$$

Then (3.2) and (3.3) imply that

$$g(0) = g'(0) = 0, \qquad g''(t) \le 0.$$

Consequently,

$$2g(1) = F(y)^2 - \mu F(w)^2 - F(u)^2 \le 0,$$

which is equivalent to the right inequality of (3.1). The left inequality can be shown similarly. ■

Now let $W \subset V$ be a $k$-dimensional linear subspace of an $n$-dimensional Minkowski space $(V,F)$. The *orthogonal complement* $W^{\perp}$ of $W$ in $V$ is defined by $W^{\perp} = \{u \in V \setminus \{0\} : \mathbf{g}_u(u,w) = 0, \forall w \in W\} \cup \{0\}$. Let $\mathcal{S}^{\perp}(W) = \{u \in W^{\perp} : F(u) = 1\}$. Then it is easy to see that for any $y \in V$, one has the decomposition

$$y = w + ru, \qquad w \in W,\ r \geq 0,\ u \in \mathcal{S}^{\perp}(W).$$

Consider the diffeomorphism $\Phi : W \times \mathcal{S}^{\perp}(W) \times [0,\infty) \to V$ defined by

$$\Phi(w,u,r) = w + ru, \quad \forall (w,u,r) \in W \times \mathcal{S}^{\perp}(W) \times [0,\infty),$$

which provides the *Fermi coordinates* $(w,u,r)$ on $V$. Fix a basis $e_a$, $1 \leq a \leq k$, for $W$, and let $\theta^{\alpha}$, $k+1 \leq \alpha \leq n-1$, be the local coordinates that are intrinsic to $\mathcal{S}^{\perp}(W)$. We shall use the following convention for the ranges of indices:

$$1 \leq i,j,\ldots \leq n, \quad 1 \leq A,B,C \leq n-1, \quad 1 \leq a,b,\ldots \leq k < \alpha,\beta,\ldots \leq n-1.$$

Then the coordinate vectors of Fermi coordinates are given by

$$\partial_a = d\Phi(e_a) = e_a, \qquad \partial_\alpha = d\Phi\left(\frac{\partial}{\partial\theta^\alpha}\right) = r\frac{\partial}{\partial\theta^\alpha}, \qquad \partial_n = d\Phi\left(\frac{\partial}{\partial r}\right) = u.$$

Consider the singular Riemannian metric $\tilde{g}|_{y=w+ru} = \mathbf{g}_u$ on $V \setminus W$. It can be expressed as

$$\tilde{g} = \mathbf{g}_u(e_a,e_b)dw^a dw^b + r^2\mathbf{g}_u\left(\frac{\partial}{\partial\theta^\alpha},\frac{\partial}{\partial\theta^\beta}\right)d\theta^\alpha d\theta^\beta + dr^2. \tag{3.4}$$

Let $\hat{g}_W = \mathbf{g}_w(e_a,e_b)dw^a dw^b$ and $\dot{g}_{W^\perp} = \mathbf{g}_u(\partial/\partial\theta^\alpha, \partial/\partial\theta^\beta)\, d\theta^\alpha d\theta^\beta$ be the Riemannian metrics on $W\setminus\{0\}$ and $\mathcal{S}^{\perp}(W)$, respectively. The corresponding volume forms are given by

$$dV_{\hat{g}_W} = \sqrt{\det(\mathbf{g}_w(e_a,e_b))}\, dw^1 \wedge \cdots \wedge dw^k, \tag{3.5}$$

$$dV_{\dot{g}_{W^\perp}} = \sqrt{\det(\mathbf{g}_u(\partial/\partial\theta^\alpha, \partial/\partial\theta^\beta))}\, d\theta^1 \wedge \cdots \wedge d\theta^{n-1}. \tag{3.6}$$

For fixed $u$ and $w$ let $e_1,\ldots,e_k$ be a $\mathbf{g}_w$-orthonormal basis for $W$ consisting of eigenvectors of $(\mathbf{g}_u(e_a,e_b))$ with eigenvalues $\rho_1,\ldots,\rho_k$. Then

$$\rho_a = \mathbf{g}_u(e_a,e_a) \geq \mu^{-1}\mathbf{g}_w(e_a,e_a) = \mu^{-1},$$

and consequently

$$\det(\mathbf{g}_u(e_a,e_b)) = \rho_1\cdots\rho_k \geq \mu^{-k}\det(\mathbf{g}_w(e_a,e_b)). \tag{3.7}$$

By (3.4)–(3.7) it is clear that

$$dV_{\tilde{g}} \geq \mu^{-k/2}r^{n-k-1}dV_{\hat{g}_W} \wedge dV_{\dot{g}_{W^\perp}} \wedge dr. \tag{3.8}$$

Let $\mathcal{B}=\{y\in V : F(y)<1\}$, and $z\in V$ be a unit vector such that

$$\sqrt{\det(g_{ij}(z))}=\max_{y\in V\setminus\{0\}}\sqrt{\det(g_{ij}(y))},$$

so that $dV_{\max}=dV_{\mathbf{g}_z}$. By the definition of uniformity constant, one can easily check that $\mathcal{B}\subset B^n(\sqrt{\mu})$, where $B^n(\sqrt{\mu})=\{y\in V:\mathbf{g}_z(y,y)<\mu\}$ denotes the ball of radius $\sqrt{\mu}$ in $V$ with respect to $\mathbf{g}_z$. Hence,

$$\mathrm{vol}_{\tilde g}(\mathcal{B})\le \mathrm{vol}_{\mathbf{g}_z}(\mathcal{B})\le \mathrm{vol}_{\mathbf{g}_z}(B^n(\sqrt{\mu}))=\mu^{n/2}\,\mathrm{vol}(\mathbb{B}^n). \tag{3.9}$$

On the other hand, for any $y=w+ru\in\mathcal{B}$ one has, by (3.1),

$$r^2\le F(y)^2-\frac{1}{\mu}F(w)^2<1,\qquad F(w)^2\ge\frac{F(y)^2-r^2}{\mu}. \tag{3.10}$$

Combining (2.5) and (3.8)–(3.10) we have

$$\begin{aligned}\mu^{n/2}\,\mathrm{vol}(\mathbb{B}^n)&\ge \mathrm{vol}_{\tilde g}(\mathcal{B})=\int_{\mathcal{B}} dV_{\tilde g}\\ &\ge \mu^{-k/2}\,\mathrm{vol}_{\dot g_{W^\perp}}(\mathcal{S}^\perp(W))\int_0^1 r^{n-k-1}\,dr\int_{\mathcal{B}(W,\sqrt{(1-r^2)/\mu})} dV_{\hat g_W}\\ &\ge \mu^{-3k/2}\,\mathrm{vol}_{\dot g_{W^\perp}}(\mathcal{S}^\perp(W))\,\mathrm{vol}(\mathbb{B}^k)\int_0^1 r^{n-k-1}(1-r^2)^{k/2}\,dr,\end{aligned} \tag{3.11}$$

where $\mathcal{B}(W,s)=\{w\in W:F(w)<s\}$. When $F$ is Euclidean, then $\mu=1$, and (3.11) reduces to the equality

$$\mathrm{vol}(\mathbb{B}^n)=\mathrm{vol}(\mathbb{S}^{n-k-1})\,\mathrm{vol}(\mathbb{B}^k)\int_0^1 r^{n-k-1}(1-r^2)^{k/2}\,dr. \tag{3.12}$$

From (3.11) and (3.12) we have the following

PROPOSITION 3.2. *Under the notations as above, the volume of $\mathcal{S}^\perp(W)$ with respect to $\dot g_{W^\perp}$ satisfies*

$$\mathrm{vol}_{\dot g_{W^\perp}}(\mathcal{S}^\perp(W))\le \mu^{(n+3k)/2}\,\mathrm{vol}(\mathbb{S}^{n-k-1}).$$

**4. Jacobi fields with initial submanifolds.** Let $(M,F)$ be an $n$-dimensional Finsler manifold, and $P\subset M$ be a $k$-dimensional compact embedded submanifold. For $p\in P$ let

$$\begin{aligned}T_p^\perp(P)&=\{y\in T_pM\setminus\{0\}:\mathbf{g}_y(y,z)=0,\ \forall z\in T_pP\}\cup\{0\},\\ \mathcal{S}_p^\perp(P)&=\{\xi\in T_p^\perp(P):F(\xi)=1\}.\end{aligned}$$

$T_p^{\perp}(P)$ is called the *normal space* of $P$ in $M$ at $p$. The *normal bundle* $T^{\perp}(P)$ and the *unit normal bundle* $\mathcal{S}^{\perp}(P)$ of $P$ in $M$ are defined by

$$T^{\perp}(P) = \bigcup_{p\in P} T_p^{\perp}(P), \qquad \mathcal{S}^{\perp}(P) = \bigcup_{p\in P} \mathcal{S}_p^{\perp}(P).$$

For $u \in \mathcal{S}_p^{\perp}(P)$ and $x, y \in T_pP$, the *second fundamental form* $B_u(x,y)$ of $x, y$ with respect to $u$ is defined by

$$B_u(x,y) = \mathbf{g}_U(U, \nabla_X^U Y)|_p,$$

where $U \in \mathcal{S}^{\perp}(P)$ and $X, Y \in TP$ are the local extensions of $u, x, y$. By (2.1)–(2.3) it is easy to check that $B_u(x,y)$ is well-defined, and

$$B_u(x,y) = B_u(y,x) = -\mathbf{g}_u(\nabla_X^U U|_p, y). \tag{4.1}$$

$B_u$ determines the *Weingarten transformation* $A_u : T_pP \to T_pP$ via

$$\mathbf{g}_u(A_u x, y) = B_u(x,y), \qquad \forall x, y \in T_pP. \tag{4.2}$$

The *mean curvature* $H(u)$ *with respect to* $u$ is defined by

$$H(u) = \frac{1}{k}\sum_a A_u(e_a, e_a),$$

where $e_1, \ldots, e_k$ is a $\mathbf{g}_u$-orthonormal basis for $T_pP$. It should be noted here that we define the notions of second fundamental form and mean curvature for Finsler submanifold by means of the Chern connection; these are different from the notions defined by volume variation (see e.g. [S3, W4]).

We consider the distance function $r = r_P$ from $P$ which is defined by

$$r_P(q) = d_F(P, q) = \min_{p\in P} d_F(p, q), \quad \forall q \in M.$$

By the first variation of arc length it is clear that a unit speed curve $\gamma : [0, b] \to M$ from $p \in P$ to $q \in M$ realizes the distance $b = r(q)$ only if $\gamma$ is a geodesic with $\xi = \dot\gamma(0) \in \mathcal{S}_p^{\perp}(P)$. For such $\gamma$ extend $\xi = \dot\gamma(0)$ to a local unit normal vector field $\xi(s) \in \mathcal{S}_{c(s)}^{\perp}(P)$ along a curve $c : (-\epsilon, \epsilon) \to P$ with $c(0) = p$, and consider a geodesic variation $\mathrm{Exp} : [0, b] \times (-\epsilon, \epsilon) \to M$ given by $(t, s) \mapsto \exp_{c(s)}(t\xi(s))$. Set

$$\widetilde{T}(t,s) = d\,\mathrm{Exp}\left(\frac{\partial}{\partial t}\right), \quad \widetilde{S}(t,s) = d\,\mathrm{Exp}\left(\frac{\partial}{\partial s}\right), \quad T = \widetilde{T}(t,0), \quad S = \widetilde{S}(t,0).$$

It is well-known that the variation vector field $S$ is a Jacobi field along $\gamma$. It is clear that $S(0) = \dot c(0) \in T_pP$. Noticing that

$$\nabla_{\widetilde{T}}^{\widetilde{T}} \widetilde{S} - \nabla_{\widetilde{S}}^{\widetilde{T}} \widetilde{T} = [\widetilde{T}, \widetilde{S}] = d\,\mathrm{Exp}([\partial/\partial t, \partial/\partial s]) = 0,$$

one has

$$\nabla_T^T S(0) = \nabla_{\widetilde{T}}^{\widetilde{T}} \widetilde{S}|_{(0,0)} = \nabla_{\widetilde{S}}^{\widetilde{T}} \widetilde{T}|_{(0,0)} = \nabla_{S(0)}^T \widetilde{T}(0,s) = \nabla_{S(0)}^T \xi, \tag{4.3}$$

which together with (4.1) and (4.2) yields

$$\nabla_T^T S(0) + A_\xi S(0) \perp_\xi T_pP,$$

where $\perp_\xi$ means perpendicular with respect to $\mathbf{g}_\xi$. Thus it is natural to consider Jacobi fields $J$ along $\gamma$ which satisfy

$$J(0) \in T_pP, \quad \nabla_T^T J(0) + A_\xi J(0) \perp_\xi T_pP, \tag{4.4}$$

called *P-Jacobi fields*. Consider the collection $\Upsilon$ of piecewise smooth *transverse vector fields $X$ along $\gamma$*, that is, those piecewise smooth vector fields $X$ along $\gamma$ with $X(0) \in T_pP$ and $X \perp_T T$. Define the *index form $I = I_\gamma$* on $\Upsilon$ by

$$I_\gamma(X,Y) = -B_\xi(X(0),Y(0)) + \int_0^b \big(\mathbf{g}_T(\nabla_T^T X, \nabla_T^T Y) - \mathbf{g}_T(\mathbf{R}^T(X),Y)\big)\, dt.$$

Since any Jacobi field is the variation vector field of a geodesic variation, by (2.4) and (4.4) it is clear that for any $P$-Jacobi field $J$ and $X \in \Upsilon$ one has

$$I(J,X) = \mathbf{g}_T(\nabla_T^T J, X)(b). \tag{4.5}$$

A point $\gamma(t), 0 < t \le b$, is called a *focal point of $P$ along $\gamma$* if there exists a nonzero $P$-Jacobi field $J$ such that $J(t) = 0$. Let $\Upsilon_0 = \{X \in \Upsilon : X(b) = 0\}$. In the following we are going to prove that the Jacobi criteria still hold for $P$-Jacobi fields. Let us first establish some auxiliary lemmas.

LEMMA 4.1.

(1) *Let $f : [0,b] \to \mathbb{R}$ be a piecewise smooth function such that $f(0) = 0$. Then the function $g(t) = (1/t)f(t)$ is also piecewise smooth.*
(2) *Let $X(t)$ $(0 \le t \le b)$ be a piecewise smooth vector field along $\gamma$ such that $X(0) = 0$. Then $Y(t) = (1/t)X(t)$ is also a piecewise smooth vector field along $\gamma$ with $Y(0) = \nabla_T^T X(0)$.*

*Proof.* (1) If $f(0) = 0$, then

$$f(t) = f(t) - f(0) = \int_0^1 \frac{d}{ds} f(ts)\, ds = t\int_0^1 f'(ts)\, ds$$

for small $t$, thus $g(t) = (1/t)f(t) = \int_0^1 f'(ts)\, ds$ is smooth for small $t$, which implies the result.

To prove (2), let the vectors $e_1, \ldots, e_n$ be a basis for $T_pM$, and extend them to smooth vector fields $E_1(t), \ldots, E_n(t)$ along $\gamma$ by parallel translation. Express $X(t)$ as $X(t) = X^i(t)E_i(t)$ for $0 \le t \le b$. If $X(0)$=0, then $X^i(0) = 0$ for $1 \le i \le n$. By (1), $Y^i(t) = (1/t)X^i(t)$ is piecewise smooth, which implies the piecewise smoothness of $Y(t) = (1/t)X(t)$. Furthermore,

$$\nabla_T^T X(0) = \nabla_T^T (tY(t))|_{t=0} = Y(0). \ \blacksquare$$

Next, let $e_1, \dots, e_n = \xi$ be a $\mathbf{g}_\xi$-orthonormal basis for $T_pM$ such that $e_1, \dots, e_k$ is a basis for $T_pP$. Let $J_1, \dots, J_{n-1}$ be $P$-Jacobi fields along $\gamma$ determined by the following initial conditions:

$$J_a(0) = e_a, \quad \nabla_T^T J_a(0) = -A_\xi e_a, \quad J_\alpha = 0, \quad \nabla_T^T J_\alpha(0) = e_\alpha. \tag{4.6}$$

A Jacobi field $J$ is called a *normal $P$-Jacobi field along $\gamma$* if $J \perp_T T$. The Jacobi fields determined by (4.6) are clearly perpendicular to $T$ with respect to $\mathbf{g}_T$, so they are normal $P$-Jacobi fields. Furthermore, from (2.4), (4.6) and the symmetry of $A_\xi$ we see that

$$\mathbf{g}_\xi(\nabla_T^T J_A(0), J_B(0)) - \mathbf{g}_\xi(J_A(0), \nabla_T^T J_B(0)) = 0, \quad \forall 1 \le A, B \le n-1,$$

and $\frac{d}{dt}(\mathbf{g}_T(\nabla_T^T J_A, J_B) - \mathbf{g}_T(J_A, \nabla_T^T J_B)) = 0$ since $J_A$ are Jacobi fields. Therefore,

$$\mathbf{g}_T(\nabla_T^T J_A, J_B) - \mathbf{g}_T(J_A, \nabla_T^T J_B) = 0. \tag{4.7}$$

On the other hand, if $\gamma(t)$ is not a focal point of $P$ for any $t \in (0, b]$, then $\{J_A(t)\}$ are linearly independent and thus form a basis for $T(t)^\perp := \{y \in T_{\gamma(t)}M : T(t) \perp_{T(t)} y\}$.

LEMMA 4.2. *Suppose that $P$ has no focal point along $\gamma$ on $(0, b]$, and $J_1, \dots, J_{n-1}$ are $P$-Jacobi fields determined by (4.6). For $X \in \Upsilon$ write $X(t) = X^A(t) J_A(t)$ for $t \in (0, b]$. Then each $X^A$ can be extended to $[0, b]$.*

*Proof.* Since $X(0) \in T_pP$, and the $J_a(0) = e_a$ form a basis for $T_pP$, $X^a(0)$ are well-defined, and thus $X^a(t)$ can be extended to $[0, b]$. On the other hand, by Lemma 4.1, $J_\alpha(t) = t\widetilde{J}_\alpha(t)$, where $\widetilde{J}_\alpha(t)$ is a smooth vector field on $\gamma$ with $\widetilde{J}_\alpha(0) = \nabla_T^T J_\alpha(0) = e_\alpha$, which implies that $J_1(t), \dots, J_k(t), \widetilde{J}_{k+1}(t), \dots, \widetilde{J}_{n-1}(t)$ is a basis for $T(t)^\perp$ for all $t \in [0, b]$. Also, since $X^\alpha(t) J_\alpha(t)|_{t=0} = 0$, again by Lemma 4.1, $X^\alpha(t) J_\alpha(t) = tY(t)$ for some piecewise smooth vector field $Y(t)$ on $\gamma$. Write $Y(t) = Y^a(t) J_a(t) + Y^\alpha(t) \widetilde{J}_\alpha(t)$ for $t \in [0, b]$. Then

$$X^\alpha(t) J_\alpha(t) = tY(t) = tY^a(t) J_a(t) + tY^\alpha(t) \widetilde{J}_\alpha(t) = tY^a(t) J_a(t) + Y^\alpha(t) J_\alpha(t).$$

By comparison we have $X^\alpha(t) = Y^\alpha(t)$ whenever $t \ne 0$, thus $X^\alpha$ can be extended to $[0, b]$. ∎

Now we are ready to prove the following Finsler version of Jacobi criteria (see e.g. [O] for the Riemannian case).

PROPOSITION 4.3. *With notations as above:*

(1) *If there is no focal point of $P$ along $\gamma$, then the index form $I$ is positive definite on $\Upsilon_0$.*
(2) *If there is a focal point $\gamma(r)$, $0 < r < b$, of $P$ along $\gamma$, then there exists $X \in \Upsilon_0$ such that $I(X, X) < 0$, and consequently $d_P(\gamma(b)) < b$.*

*Proof.* (1) For any $X \in \Upsilon_0$ write $X = X^A(t)J_A(t)$ on $(0,b]$; it can be extended to $[0,b]$ by Lemma 4.2. We claim that

$$\mathbf{g}_T(\nabla_T^T X, \nabla_T^T X) - \mathbf{g}_T(\mathbf{R}^T(X), X) = \mathbf{g}_T(A,A) + T\cdot \mathbf{g}_T(X,B), \tag{4.8}$$

where $A = (X^i)'J_i$ and $B = X^i\nabla_T^T J_i$. In fact, since $\nabla_T^T X = A + B$, by (2.1)–(2.4) and (4.7) one has

$$\begin{aligned}
T\cdot \mathbf{g}_T(X,B) &= \mathbf{g}_T(\nabla_T^T X, B) + \mathbf{g}_T(X, \nabla_T^T B)\\
&= \mathbf{g}_T(A,B) + \mathbf{g}_T(B,B) + X^i(X^j)'\mathbf{g}_T(J_i, \nabla_T^T J_j) + X^iX^j\mathbf{g}_T(J_i, \nabla_T^T\nabla_T^T J_j)\\
&= 2\mathbf{g}_T(A,B) + \mathbf{g}_T(B,B) - \mathbf{g}_T(\mathbf{R}^T(X), X),
\end{aligned}$$

which clearly implies (4.8). Now observe that $X(b) = 0$, and

$$X^i(0)\mathbf{g}_\xi(X(0), \nabla_T^T J_i(0)) = X^i(0)\mathbf{g}_\xi(X(0), \nabla_{J_i}^T \xi) = -B_\xi(X(0), X(0)),$$

so it follows from (4.8) that

$$\begin{aligned}
I(X,X) &= -B_\xi(X(0),X(0)) - X^i(0)\cdot \mathbf{g}_\xi(X(0), \nabla_T^T J_i(0)) + \int_0^b \mathbf{g}_T(A,A)\,dt\\
&= \int_0^b \mathbf{g}_T(A,A)\,dt \ge 0,
\end{aligned}$$

and the last inequality becomes an equality if and only if $A = 0$, which is equivalent to $X = 0$, thus (1) is proved.

(2) Since $\gamma(r)$, $0 < r < b$, is a focal point of $P$ along $\gamma$, there is a nonzero $P$-Jacobi field $J$ along $\gamma$ with $J(r) = 0$. Note that $J(0) \perp_T T$ and $J(r) \perp_T T$, which implies $J \perp_T T$. Thus for

$$J_1(t) = \begin{cases} J(t), & t \in [0,r],\\ 0, & t \in [r,b],\end{cases}$$

one clearly has $J_1 \in \Upsilon_0$, and $I(J_1,J_1) = 0$. Certainly $\nabla_T^T J(r) \ne 0$. Let $Y(t)$ be the parallel vector field along $\gamma$ such that $Y(r) = -\nabla_T^T J(r)$, and $\phi : [0,b] \to \mathbb{R}$ be a smooth function with $\phi(0) = \phi(b) = 0$ and $\phi(r) = 1$. Write

$$X_\epsilon = J_1 + \epsilon\phi Y.$$

Then $X_\epsilon \in \Upsilon_0$, and

$$\begin{aligned}
I(X_\epsilon, X_\epsilon) &= I(J_1,J_1) + 2\epsilon I(J_1, \phi Y) + \epsilon^2 I(\phi Y, \phi Y)\\
&= 2\epsilon \int_0^r \big(\mathbf{g}_T(\nabla_T^T J_1, \nabla_T^T(\phi Y)) - \mathbf{g}_T(\mathbf{R}^T(J_1), \phi Y)\big)\,dt + \epsilon^2 I(\phi Y, \phi Y)\\
&= 2\epsilon \mathbf{g}_T(\nabla_T^T J_1, \phi Y)(r) + \epsilon^2 I(\phi Y, \phi Y) = -2\epsilon \mathbf{g}_T(\nabla_T^T J, \nabla_T^T J)(r) + \epsilon^2 I(\phi Y, \phi Y).
\end{aligned}$$

Now it is clear that $I(X_\epsilon, X_\epsilon)$ is negative for sufficiently small positive $\epsilon$. ∎

Corollary 4.4 (Index Lemma). *Suppose that there is no focal point of $P$ along $\gamma$, let $J$ be a normal $P$-Jacobi field along $\gamma$, and $X \in \Upsilon$ with $X(b) = J(b)$. Then $I(X,X) \geq I(J,J)$, with equality if and only if $X = J$.*

*Proof.* It is clear that $X - J \in \Upsilon_0$, thus by Proposition 4.3 we have

$$0 \leq I(X-J, X-J) = I(X,X) - 2I(X,J) + I(J,J). \tag{4.9}$$

On the other hand, (4.5) implies that

$$I(J,X) = \mathbf{g}_T(\nabla_T^T J, X)(b) = \mathbf{g}_T(\nabla_T^T J, J)(b) = I(J,J),$$

which together with (4.9) yields $0 \leq I(X,X) - I(J,J)$, and equality holds if and only if $X = J$. ∎

**5. Fermi coordinates and focal cut locus.** We keep the notations of §4. Let $\mathrm{Exp} : T^\perp(P) \to M$ be defined by

$$\mathrm{Exp}(r\xi) = \exp_p(r\xi), \quad \forall \xi \in \mathcal{S}^\perp(P), r \geq 0.$$

Here $p = \pi(\xi)$. Notice that $d(\exp_p)_0 = \mathrm{id}$ for any $p \in P$ and $P$ is compact, hence there exists $\epsilon > 0$ such that Exp is a diffeomorphism from $\{r\xi : \xi \in \mathcal{S}^\perp(P),\, 0 \leq r < \epsilon\}$ onto its image. Let $\mathbf{D}(P) \subset T^\perp(P)$ be the largest open star-like subset containing the zero section such that $\mathrm{Exp}|_{\mathbf{D}(P)}$ is a diffeomorphism, and set $\mathcal{D}(P) = \mathrm{Exp}(\mathbf{D}(P))$. For any $r \geq 0$, $p \in P$, and $\xi \in \mathcal{S}_p^\perp(P)$ such that $r\xi \in \mathbf{D}(P)$, $(p, \xi, r)$ is called the *Fermi coordinate* of $\exp_p(r\xi) \in \mathcal{D}(P)$.

In the following we consider the coordinate vector fields of Fermi coordinates. Let us restrict ourselves to $\xi \in \mathcal{S}_p^\perp(P)$, and let $\theta^\alpha$, $k+1 \leq \alpha \leq n-1$, be the local coordinates that are intrinsic to $\mathcal{S}_p^\perp(P)$. First note that $r_P(\exp_p(r\xi)) = r$ when $\exp_p(r\xi) \in \mathcal{D}(P)$, so

$$\partial_n := d\,\mathrm{Exp}\left(\frac{\partial}{\partial r}\right) = \nabla r = T \tag{5.1}$$

is just the tangent vector of $\gamma : r \mapsto \exp_p(r\xi)$. Next, to calculate $d\,\mathrm{Exp}(\partial/\partial\theta^\alpha)$, let $\eta_\alpha : (-\epsilon, \epsilon) \to \mathcal{S}_p^\perp$ be curves such that $\eta(0) = \xi$ and $\dot\eta(0) = \partial/\partial\theta^\alpha$. Then $(r, u) \mapsto \exp_p(r\eta_\alpha(u))$ is a geodesic variation of the geodesic $\gamma$, and the corresponding variation vector field is just the coordinate vector

$$\partial_\alpha|_{\exp_p(r\xi)} := d\,\mathrm{Exp}\left(\frac{\partial}{\partial\theta^\alpha}\right)\Big|_{\exp_p(r\xi)} = d(\exp_p)_{r\xi}\left(r\frac{\partial}{\partial\theta^\alpha}\right), \tag{5.2}$$

which is the $P$-Jacobi field along $\gamma$ determined by

$$\partial_\alpha(0) = 0, \quad \nabla_T^T \partial_\alpha(0) = \frac{\partial}{\partial\theta^\alpha}. \tag{5.3}$$

Finally, let $p^1, \dots, p^k$ be the local coordinates of $P$ near $p \in P$. Then by §4 we see that the coordinate vector

$$\partial_a = d\,\mathrm{Exp}\left(\frac{\partial}{\partial p^a}\right) \tag{5.4}$$

is the $P$-Jacobi field satisfying

$$\partial_a(0) = \frac{\partial}{\partial p^a}, \qquad \nabla_T^T \partial_a(0) + A_\xi \partial_a(0) \perp_\xi T_pP. \tag{5.5}$$

By the definition of focal point it is clear that if there is no focal point of $P$ along $\gamma(t) = \exp_p(t\xi), 0 < t \le r$, then $\{\partial_a(t), \partial_\alpha(t)\}$ is a basis for $\dot\gamma(t)^\perp$, which implies that Exp is nonsingular at $t\xi$ for $0 < t \le r$. In other words, $\exp_p(r\xi)$ ($\xi \in \mathcal{S}_p^\perp(P)$) is a focal point of $P$ if and only if Exp is singular at $r\xi$. Also, let

$$\overline{\sigma}(p,\xi,r) = \sqrt{\det(\overline{g}_{AB})}\big|_{\exp_p(r\xi)}, \qquad \overline{g}_{AB} = \mathbf{g}_{\dot\gamma(t)}(\partial_A, \partial_B).$$

Then $\exp_p(r\xi)$ ($\xi \in \mathcal{S}_p^\perp(P)$) is a focal point of $P$ if and only if $\overline{\sigma}(p,\xi,r) = 0$. Thus the set of focal points of $P$ is closed.

For $\xi \in \mathcal{S}_p^\perp(\alpha)$, the *focal cut-value* $\mathfrak{c}_P(\xi)$ *of* $\xi$ *with respect to* $P$ is defined by

$$\mathfrak{c}_P(\xi) = \sup\{s > 0 : r_P(\exp_p(s\xi)) = s\},$$

and $\exp_p(\mathfrak{c}_P(\xi)\xi)$ is called the *focal cut point* of $P$.

Proposition 5.1. *$x \in M$ is a focal cut point of $P$ if and only if either*

(1) *$x$ is a focal point of $P$, or*
(2) *there are two distinct vectors $\xi, \eta \in \mathcal{S}^\perp(P)$ such that $x = \mathrm{Exp}(r_P(x)\xi) = \mathrm{Exp}(r_P(x)\eta)$.*

*Proof.* The sufficiency is obvious by Proposition 4.3, so we only need to prove the necessity. Suppose that $x$ is a focal cut point of $P$. By definition, one can find $p \in P$ and $\xi \in \mathcal{S}_p^\perp(P)$ such that $r_0 := r_P(x) = \sup\{s > 0 : r_P(\exp_p(s\xi)) = s\}$. Consider a strictly decreasing sequence $\{s_j\}$ with $s_j > r_0$ for all $j$, and $s_j \to r_0$ as $j \to \infty$. By definition of $r_0$, there are $p_j \in P$, $\xi_j \in \mathcal{S}_{p_j}^\perp(P)$ and $r_j < s_j$ such that $\exp_p(s_j\xi) = \exp_{p_j}(r_j\xi_j)$ for all $j$. Note that $\mathcal{S}^\perp(P)$ is compact, thus $\{(p_j,\xi_j)\}$ has a convergent subsequence. Without loss of generality, we may assume that $(p_j,\xi_j) \to (q,\eta) \in \mathcal{S}^\perp(P)$ as $j \to \infty$. If $(q,\eta) = (p,\xi)$, then Exp is not a diffeomorphism on a neighborhood of $(p, r_0\xi)$ in $T^\perp(P)$, which implies that Exp is singular at $r_0\xi$ and thus $x = \exp_p(r_0\xi)$ is a focal point of $P$, as in (1). On the other hand, the case when $(q,\eta) \ne (p,\xi)$ is just described by (2). ∎

Let $\mathbf{C}(P) = \{\mathfrak{c}_P(\xi)\xi : \xi \in \mathcal{S}^\perp(P)\}$ and $\mathcal{C}(P) = \mathrm{Exp}(\mathbf{C}(P))$ be the *tangent focal cut locus* and *focal cut locus* of $P$, respectively.

PROPOSITION 5.2. *The focal cut-value function* $\mathfrak{c}_\alpha : \mathcal{S}^\perp(\alpha) \to (0,\infty]$ *defined by* $\xi \mapsto \mathfrak{c}_\alpha(\xi)$ *is continuous. Consequently,* $\mathbf{D}(P) = \{t\xi : \xi \in \mathcal{S}^\perp(P),$ $t \in [0, \mathfrak{c}_P(\xi))\}$, $\mathcal{D}(P) = M \setminus \mathcal{C}(P)$, *and* $\mathcal{C}(P)$ *has zero Hausdorff measure in* $M$.

*Proof.* Suppose we are given $\xi \in \mathcal{S}^\perp(P)$ with a sequence $\{\xi_j\} \subset \mathcal{S}^\perp(P)$ with $\xi_j \to \xi$ as $j \to \infty$. Set

$$p = \pi(\xi), \quad p_j = \pi(\xi_j), \quad d_j = \mathfrak{c}_P(\xi_j).$$

We must prove

$$\mathfrak{c}_P(\xi) \geq \limsup_{j\to\infty} d_j \quad \text{and} \quad \liminf_{j\to\infty} d_j \geq \mathfrak{c}_P(\xi).$$

To prove $\mathfrak{c}_P(\xi) \geq \limsup_{j\to\infty} d_j$, we may assume $\mathfrak{c}_P(\xi) < \infty$ (otherwise, there is nothing to prove). Given any $\epsilon > 0$, the number of $d_j = \mathfrak{c}_P(\xi_j)$ that exceed $\mathfrak{c}_P(\xi)+\epsilon$ must be finite. Otherwise, for those $\xi_j$ in question, relabeled as a sequence $\{\xi_i\}$, we have

$$d_P\big(P, \mathrm{Exp}((\mathfrak{c}_P(\xi) + \epsilon)\xi_i)\big) = \mathfrak{c}_P(\xi) + \epsilon.$$

Upon letting $i \to \infty$, this says that the distance from $P$ to $\exp_p((\mathfrak{c}_P(\xi)+\epsilon)\xi)$ is $\mathfrak{c}_P(\xi)+\epsilon$. In other words, the geodesic $\exp_p(t\xi)$ continues to minimize the distance from $P$ beyond the focal cut point of $P$, which is a contradiction. Thus, for all but finitely many $\xi_j$, we must have $d_k \leq \mathfrak{c}_P(\xi) + \epsilon$. Hence $\limsup_{j\to\infty} d_j \leq \mathfrak{c}_P(\xi)$.

Next we will prove $\liminf_{j\to\infty} d_j \geq \mathfrak{c}_P(\xi)$. It suffices to assume that $\{d_j = \mathfrak{c}_P(\xi_j)\}$ converges to $\delta < \infty$ as $j \to \infty$. From Proposition 5.1 we see that by passing to a subsequence if necessary, we may assume that either (i) $\mathrm{Exp}(d_j\xi_j)$ is a focal point of $P$ for all $j$, or (ii) for each $j$ there exists $\eta_j \in \mathcal{S}^\perp(P)$, $\eta_j \neq \xi_j$, for which $\mathrm{Exp}(d_j\xi_j) = \mathrm{Exp}(d_j\eta_j)$. In case (i), $\mathrm{Exp}(\delta\xi)$ is certainly a focal point of $P$ since the set of focal points is closed, so $\mathfrak{c}_P(\xi) \leq \delta$. In case (ii), by passing to a subsequence if necessary, we may assume the existence of $\eta \in \mathcal{S}^\perp(P)$ for which $\eta_j \to \eta$ as $j \to \infty$. But then $\mathrm{Exp}(\delta\xi) = \mathrm{Exp}(\delta\eta)$. If $\eta \neq \xi$, then certainly $\mathfrak{c}_P(\xi) \leq \delta$. If $\eta = \xi$, then Exp is not a diffeomorphism on a neighborhood of $\delta\xi$ in $T^\perp(P)$, which implies $\mathrm{Exp}(\delta\xi)$ is a focal point, as in case (i). Thus we are done. ∎

**6. Volume comparison theorem.** Let $(M, F)$ be an $n$-dimensional Finsler manifold, $P \subset M$ be a $k$-dimensional compact embedded submanifold, and $(p, \xi, r)$ be Fermi coordinates with coordinate vectors $\partial_1, \ldots, \partial_n$ constructed in (5.1)–(5.5). Consider the singular Riemannian metric $\overline{g} = \mathbf{g}_{\nabla r}$ on $\mathcal{D}(P) \setminus P$. Then the corresponding Riemannian volume form of $\overline{g}$ is

$$dV_{\overline{g}} = \overline{\sigma}(p, \xi, r) dV_{T_pP}(\xi) \wedge dV_{\mathcal{S}_p^\perp(P)}(\xi) \wedge dr, \tag{6.1}$$

where

$$\overline{\sigma}(p,\xi,r)=\sqrt{\det(\overline{g}_{AB})}\big|_{\exp_p(r\xi)},\qquad \overline{g}_{AB}=\mathbf{g}_{\nabla r}(\partial_A,\partial_B),$$
$$dV_{T_pP}(\xi)=\sqrt{\det(\mathbf{g}_\xi(\partial/\partial p^a,\partial/\partial p^b))}\,dp^1\wedge\cdots\wedge dp^k,$$
$$dV_{\mathcal{S}_p^\perp(P)}(\xi)=\sqrt{\det(\mathbf{g}_\xi(\partial/\partial\theta^\alpha,\partial/\partial\theta^\beta))}\,d\theta^1\wedge\cdots\wedge d\theta^{n-1}.$$

Let $w\in T_pP$ be a unit vector such that

$$\sqrt{\det(\mathbf{g}_w(\partial/\partial p^a,\partial/\partial p^b))}=\min_{u\in T_pP\setminus\{0\}}\sqrt{\det(\mathbf{g}_u(\partial/\partial p^a,\partial/\partial p^b))},$$

so $dV_{\min}(T_pP)=dV_{\mathbf{g}_w}(T_pP)$. Fix $\xi\in\mathcal{S}_p^\perp(P)$ and let $e_1,\dots,e_k$ be a $\mathbf{g}_w$-orthonormal basis for $T_pP$ consisting of eigenvectors of $(\mathbf{g}_\xi(e_a,e_b))$ with eigenvalues $\rho_1,\dots,\rho_k$. Then

$$\rho_a=\mathbf{g}_\xi(e_a,e_a)\le\mu(p)\mathbf{g}_w(e_a,e_a)=\mu(p),$$

and consequently

$$\det(\mathbf{g}_\xi(e_a,e_b))=\rho_1\cdots\rho_k\le\mu(p)^k\det(\mathbf{g}_w(e_a,e_b)),$$

which together with (6.1) yields

$$dV_{\overline{g}}\le\mu(p)^{k/2}\overline{\sigma}(p,\xi,r)dV_{\min}(T_pP)\wedge dV_{\mathcal{S}_p^\perp(P)}\wedge dr. \tag{6.2}$$

In order to deduce the volume comparison theorem, we have to estimate the quantity $\overline{\sigma}(p,\xi,r)$. For this purpose, assume that the flag curvature of $(M,F)$ satisfies $\mathbf{K}_M\ge\kappa$. Fix $\xi\in\mathcal{S}^\perp(P)$ with $p=\pi(\xi)\in P$ such that there is no focal cut point of $P$ along $\gamma(t)=\exp_p(t\xi)$ for $t\in(0,r]$. Furthermore, we may assume that $\mathbf{g}_{T(r)}(\partial_A(r),\partial_B(r))=\delta_{AB}$ for fixed $r$. Then by (2.1)–(2.3) and (4.5) we have

$$\begin{aligned}\frac{\partial\overline{g}_{AB}}{\partial r}(r)&=T\cdot\mathbf{g}_{T(r)}(\partial_A,\partial_B)(r)\\&=\mathbf{g}_{T(r)}(\nabla_T^T\partial_A(r),\partial_B(r))+\mathbf{g}_{T(r)}(\partial_A(r),\nabla_T^T\partial_B(r))=2I_r(\partial_A,\partial_B),\end{aligned}$$

where $I_r$ is the index form on $\gamma|[0,r]$. Consequently,

$$\frac{\partial}{\partial r}\log\overline{\sigma}(p,\xi,r)=\frac{1}{2}\overline{g}^{AB}\frac{\partial\overline{g}_{AB}}{\partial r}(r)=\sum_A I_r(\partial_A,\partial_A).$$

Let $E_A(t),0\le t\le r$, be the parallel vector fields along $\gamma$ such that $E_A(r)=\partial_A(r)$, $A=1,\dots,n-1$. Now $\{E_A(0)\}$ is an orthonormal basis for $\xi^\perp$. By using the orthogonal transformation if necessary, we may assume that $E_a(0)\in T_pP$, $E_\alpha(0)\perp_\xi T_pP$, and $A_\xi E_a(0)=\lambda_aE_a(0)$. Set

$$X_a=\frac{(\mathfrak{c}_\kappa-\lambda_a\mathfrak{s}_\kappa)(t)}{(\mathfrak{c}_\kappa-\lambda_a\mathfrak{s}_\kappa)(r)}E_a(t),\qquad X_\alpha=\frac{\mathfrak{s}_\kappa(t)}{\mathfrak{s}_\kappa(r)}E_\alpha(t),$$

where

$$\mathfrak{s}_\kappa(t)=\begin{cases}\frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}t, & \kappa>0,\\ t, & \kappa=0,\\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}t, & \kappa<0,\end{cases}\qquad \mathfrak{c}_\kappa=\mathfrak{s}'_\kappa.$$

By construction it is clear that $X_A\in\Upsilon$ with $X_A(r)=\partial_A(r)$, and

$$\nabla_T^T\nabla_T^T X_A+\kappa X_A=0.$$

By the Index Lemma (Corollary 4.4) one has

$$\begin{aligned}\frac{\partial}{\partial r}\log\overline{\sigma}(p,\xi,r)&=\sum_A I_r(\partial_A,\partial_A)\le\sum_A I_r(X_A,X_A)\\ &=-\sum_A \mathbf{g}_\xi(A_\xi(X_A(0)),X_A(0))\\ &\quad+\sum_A\int_0^r\big(\mathbf{g}_T(\nabla_T^T X_A,\nabla_T^T X_A)-\mathbf{g}_T(\mathbf{R}^T(X_A),X_A)\big)\,dt\\ &\le-\sum_A \mathbf{g}_\xi\big(A_\xi(X_A(0)),X_A(0)\big)\\ &\quad+\sum_A\int_0^r\big(\mathbf{g}_T(\nabla_T^T X_A,\nabla_T^T X_A)-\kappa\mathbf{g}_T(X_A,X_A)\big)\,dt\\ &=-\sum_A \mathbf{g}_\xi\big(A_\xi(X_A(0)),X_A(0)\big)+\sum_A\mathbf{g}_T(\nabla_T^T X_A,X_A)|_0^r\\ &=\sum_A\mathbf{g}_T(\nabla_T^T X_A,X_A)(r)\\ &=\frac{\partial}{\partial r}\log\Big(\mathfrak{s}_\kappa^{n-k-1}(r)\prod_a(\mathfrak{c}_\kappa(r)-\lambda_a\mathfrak{s}_\kappa(r))\Big).\end{aligned}$$

By (5.2) it is clear that $\overline{\sigma}(p,\xi,r)\to 0$ as $r\to 0$, thus

$$\begin{aligned}\overline{\sigma}(p,\xi,r)&\le\mathfrak{s}_\kappa^{n-k-1}(r)\prod_a(\mathfrak{c}_\kappa(r)-\lambda_a\mathfrak{s}_\kappa(r))\\ &\le\mathfrak{s}_\kappa^{n-k-1}(r)(\mathfrak{c}_\kappa(r)-H(\xi)\mathfrak{s}_\kappa(r))^k,\end{aligned}\tag{6.3}$$

where $H(\xi)=(1/k)\sum_a\lambda_a$ is the mean curvature of $P$ with respect to $\xi$. Now (6.2) and (6.3) give

$$dV_{\overline{g}}\le\mu(p)^{k/2}\mathfrak{s}_\kappa^{n-k-1}(r)(\mathfrak{c}_\kappa(r)-H(\xi)\mathfrak{s}_\kappa(r))^k dV_{\min}(T_pP)\wedge dV_{\mathcal{S}_p^\perp(P)}\wedge dr.\tag{6.4}$$

Let $B(P;R)=\{x\in M: r_P(x)<R\}$ be the tubular neighborhood of $P$ of radius $r$. We have the following comparison theorem for the minimal volume of $B(P;R)$.

THEOREM 6.1. *Let $(M,F)$ be an $n$-dimensional forward complete Finsler manifold with flag curvature satisfying $\mathbf{K}_M \geq \kappa$, and let $P \subset M$ be a $k$-dimensional compact embedded submanifold with mean curvature satisfying $|H(\xi)| \leq \Theta$ for some $\Theta \geq 0$. Suppose also that the uniformity constant of $M$ satisfies $\mu \leq \Lambda$. Then the minimal volume of $B(P;R)$ satisfies*

$$\mathrm{vol}_{\min}(B(P;R)) \leq \Lambda^{(n+4k)/2}\,\mathrm{vol}(\mathbb{S}^{n-k-1})\,\mathrm{vol}_{\min}(P) \times \int_0^R \mathfrak{s}_\kappa^{n-k-1}(r)(\mathfrak{c}_\kappa(r)+\Theta\mathfrak{s}_\kappa(r))^k\,dr.$$

*Proof.* Noting that the focal cut locus $\mathcal{C}(P)$ has zero Hausdorff measure in $M$, by Proposition 3.2 and (6.4) we get

$$\begin{aligned}(6.1)\qquad \mathrm{vol}_{\min}(B(P;R)) &= \int_{B(P;r)} dV_{\min} \leq \int_{B(P;r)} dV_{\overline{g}} \leq \Lambda^{k/2}\int_P dV_{\min}(T_pP)\\ &\quad\times \int_{\mathcal{S}_p^\perp(P)} dV_{\mathcal{S}_p^\perp(P)}(\xi) \int_0^{\min\{R,\mathfrak{c}_P(\xi)\}} \mathfrak{s}_\kappa^{n-k-1}(r)(\mathfrak{c}_\kappa(r)-H(\xi)\mathfrak{s}_\kappa(r))^k\,dr\\ &\leq \Lambda^{(n+4k)/2}\,\mathrm{vol}(\mathbb{S}^{n-k-1})\,\mathrm{vol}_{\min}(P)\int_0^R \mathfrak{s}_\kappa^{n-k-1}(r)(\mathfrak{c}_\kappa(r)+\Theta\mathfrak{s}_\kappa(r))^k\,dr. \qquad\blacksquare\end{aligned}$$

*Proof of Theorem 1.1.* Let $c:[0,L(c)]\to M$ be a closed geodesic with unit speed. Noting that in this case $\nabla_{\dot c}^{\dot c}\dot c = 0$, for any $\xi \in \mathcal{S}^\perp(c)$ one has

$$H(\xi) = \mathbf{g}_\xi(\xi, \nabla_{\dot c}^{\xi}\dot c) = \mathbf{g}_\xi(\xi, \nabla_{\dot c}^{\xi}\dot c - \nabla_{\dot c}^{\dot c}\dot c) = \mathbf{T}_\xi(\dot c),$$

and thus

$$|H(\xi)| \leq \max_{t\in[0,L(c)]} \|\mathbf{T}\|(c(t)) \leq \Xi.$$

Hence, by Theorem 6.1 we get

$$\begin{aligned}\mathrm{vol}_{\min}(M) &= \mathrm{vol}_{\min}(B(c;d))\\ &\leq \Lambda^{(n+4)/2}\,\mathrm{vol}(\mathbb{S}^{n-2})L(c)\int_0^d \sinh^{n-2} r\cdot(\cosh r + \Xi\sinh r)\,dr\\ &\leq \Lambda^{(n+4)/2}\,\mathrm{vol}(\mathbb{S}^{n-2})L(c)(1+\Xi)\int_0^d \sinh^{n-2} r\cosh r\,dr\\ &= \Lambda^{(n+4)/2}\,\mathrm{vol}(\mathbb{S}^{n-2})L(c)(1+\Xi)\cdot\frac{\sinh^{n-1} d}{n-1},\end{aligned}$$

which clearly implies the desired result. ■

**7. Proof of Theorem 1.2.** In this last section we shall complete the proof of Theorem 1.2. For this purpose we need some auxiliary lemmas. First by using Wirtinger's inequality we can prove the following lemma in almost the same way as Theorem 2.14 in [C].

Lemma 7.1. *Let $(M, F)$ be a Finsler manifold with flag curvature satisfying $\mathbf{K}_M \le 1$, and $\gamma : [0, b] \to M$ be a unit speed geodesic such that $\gamma(b)$ is conjugate to $\gamma(0)$ along $\gamma$. Then $b \ge \pi$.*

The next lemma of [D] is crucial to proving Theorem 1.2. We provide the proof for completeness.

Lemma 7.2 ([D]). *Let $(M, F)$ be an $n$-dimensional compact reversible Finsler manifold, $p \in M$, and $q \in \mathcal{C}(p)$ such that $\mathfrak{i} = \mathfrak{i}(M) = d_F(p, q)$. Then either*

1. *$p$ and $q$ are mutually conjugate along some minimizing geodesic connecting $p$ and $q$, or*
2. *there is a closed geodesic $\tau$ passing through $p$ and $q$ with $L(\tau) = 2\mathfrak{i}$.*

*Proof.* By the definition of injectivity radius we can certainly find $q \in \mathcal{C}(p)$ such that $\mathfrak{i} = d_F(p, q)$, and in fact $q$ is the nearest point to $p$ in $\mathcal{C}(p)$. If $p$ and $q$ are not mutually conjugate along any minimizing geodesic connecting $p$ and $q$, it is well-known that there exist at least two distinct minimizing geodesics $\gamma_1, \gamma_2 : [0, \mathfrak{i}] \to M$ connecting $p$ and $q$ (see [BCS]). It follows that $\exp_p : T_pM \to M$ is nonsingular on some open disjoint domains $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathfrak{i}\dot\gamma_1(0)$ and $\mathfrak{i}\dot\gamma_2(0)$, respectively. Write $U_i = \exp_p(\mathcal{U}_i)$ for $i = 1, 2$. It is clear that $q \in U_1$ and $q \in U_2$, and thus $U = U_1 \cap U_2 \ne \emptyset$. Define $f_i : U \to \mathbb{R}$ by setting

$$f_i(x) = F((\exp_p |U_i)^{-1}(x)), \quad i = 1, 2.$$

These functions are differentiable and of maximal rank with $f_1(q) = f_2(q) = \mathfrak{i}$. We claim that $\dot\gamma_1(\mathfrak{i}) + \dot\gamma_2(\mathfrak{i}) = 0$. Otherwise, since $F$ is reversible, we have

$$l(\dot\gamma_1(\mathfrak{i})) \ne l(-\dot\gamma_2(\mathfrak{i})) = -l(\dot\gamma_2(\mathfrak{i})),$$

where $l : T_qM \to T_q^*M$ is the Legendre transformation. Hence, one can find $X \in T_qM$ such that

$$l(\dot\gamma_1(\mathfrak{i}))(X) + l(\dot\gamma_2(\mathfrak{i}))(X) < 0,$$

or equivalently,

$$\mathbf{g}_{\dot\gamma_1(\mathfrak{i})}(\dot\gamma_1(\mathfrak{i}), X) + \mathbf{g}_{\dot\gamma_2(\mathfrak{i})}(\dot\gamma_2(\mathfrak{i}), X) < 0.$$

Let $c : (-\epsilon, \epsilon) \to U$ be the curve determined by $c(0) = q$ and $\dot c(0) = X$. For any $u \in (-\epsilon, \epsilon)$ let

$$\gamma_{i,u}(t) = \exp_p(t(\exp_p |U_i)^{-1}(c(u))), \quad i = 1, 2.$$

Then $\gamma_{1,u}$ and $\gamma_{2,u}$ are two distinct geodesics from $p$ to $c(u)$ with $L(\gamma_{1,u}) = f_1(c(u))$ and $L(\gamma_{2,u}) = f_2(c(u))$. We see from the first variation formula of arc length that

$$\left.\frac{d}{du}L(\gamma_{i,u})\right|_{u=0}=\mathbf{g}_{\dot\gamma_i(\mathfrak{i})}(\dot\gamma_i(\mathfrak{i}),X),\qquad i=1,2,$$

and consequently

$$\left.\frac{d}{du}(L(\gamma_{1,u})+L(\gamma_{2,u}))\right|_{u=0}=\mathbf{g}_{\dot\gamma_1(\mathfrak{i})}(\dot\gamma_1(\mathfrak{i}),X)+\mathbf{g}_{\dot\gamma_2(\mathfrak{i})}(\dot\gamma_2(\mathfrak{i}),X)<0.$$

Therefore, for sufficiently small $u>0$, one has

$$L(\gamma_{1,u})+L(\gamma_{2,u})<L(\gamma_{1,0})+L(\gamma_{2,0})=2\mathfrak{i},$$

and there is a cut point $q'$ of $p$ on $\gamma_{1,u}$ or $\gamma_{2,u}$ with $d_F(p,q')<\mathfrak{i}$, which contradicts the choice of $q$. Thus, $\dot\gamma_1(\mathfrak{i})+\dot\gamma_2(\mathfrak{i})=0$, and we can prove similarly that $\dot\gamma_1(0)+\dot\gamma_2(0)=0$. Now let $\tau=\gamma_1\cup\tilde\gamma_2$ with $\tilde\gamma_2(t)=\gamma_2(\mathfrak{i}-t)$; it is clear that $\tau$ is the desired closed geodesic. ■

*Proof of Theorem 1.2.* Let $p,q\in M$ be as in Lemma 7.2. If $p$ and $q$ are mutually conjugate along some minimizing geodesic connecting $p$ and $q$, then Lemma 7.1 implies that $\mathfrak{i}(M)=d_F(p,q)\geq\pi$. Otherwise, there is a closed geodesic $\tau$ with $L(\tau)=2\mathfrak{i}$; then by Theorem 1.1, one gets

$$\mathfrak{i}(M)=\frac{1}{2}L(\tau)\geq\frac{(n-1)\operatorname{vol}_{\min}(M)}{2\Lambda^{(n+4)/2}\operatorname{vol}(\mathbb{S}^{n-2})(1+\Xi)\sinh^{n-1}d},$$

which clearly implies the result. ■

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