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Existence of positive solutions for a fourth-order differential system

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Abstract. This paper investigates the existence of positive solutions for a fourth-order differential system using a fixed point theorem of cone expansion and compression type.

1. Introduction. It is well known that the bending of an elastic beam can be described by using fourth-order boundary value problems. An elastic beam with its two ends simply supported can be described by the fourth-order boundary value problem

$$(1.1)\qquad \begin{aligned} &u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1,\\ &u(0) = u(1) = u''(0) = u''(1) = 0.\end{aligned}$$

Existence of solutions for problem (1.1) was established for example by Gupta [G], Liu [L], Ma [M], Ma et al. [MZF], Ma and Wang [MW], Aftabizadeh [A], Yang [Y] and del Pino and Manásevich [DM] (see also the references therein). All of those results are based on the Leray–Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [WA] studied the existence of positive solutions for the second-order boundary value problem

$$(1.2)\qquad \begin{aligned} &-u'' + \lambda u = u\varphi + f(t, u), \quad 0 < t < 1,\\ &-\varphi'' = \mu u, \quad 0 < t < 1,\\ &u(0) = u(1) = 0,\\ &\varphi(0) = \varphi(1) = 0,\end{aligned}$$

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where $\lambda > -\pi^2$, μ is a positive parameter, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

In this paper we shall discuss the existence of positive solutions for the fourth-order boundary value problem

$$(1.3)\qquad \begin{aligned} &u^{(4)} = \varphi u + f(t,u,u''), \quad 0 < t < 1,\\ &-\varphi'' = \mu u, \quad 0 < t < 1,\\ &u(0) = u(1) = u''(0) = u''(1) = 0,\\ &\varphi(0) = \varphi(1) = 0, \end{aligned}$$

where μ is a positive parameter and $f : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous. In fact, as we will see below, in Sections 2 and 3 one could consider $f(t,u,v) = g(t)h(t,u,v)$ with $h : [0,1] \times [0,\infty) \times (-\infty,0] \to [0,\infty)$ continuous and $g \in (C(0,1),\mathbb{R}^+)$ provided

$$\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)g(s)\,ds\,d\tau < \infty;$$

here K is as defined in Section 2.

2. Preliminaries. Let $Y = C[0,1]$ and $Y_+ = \{u \in Y : u(t) \geq 0,\ t \in [0,1]\}$. It is well known that Y is a Banach space equipped with the norm $\|u\|_0 = \sup_{t\in[0,1]} |u(t)|$. We denote the norm $\|u\|_2$ by

$$\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}.$$

It is easy to show that $C^2[0,1]$ is complete with the norm $\|u\|_2$, and $\|u\|_2 \leq \|u\|_0 + \|u''\|_0 \leq 2\|u\|_2$.

Suppose that $K(t,s)$ is the Green function associated with

$$(2.1)\qquad -u'' = f(t), \quad u(0) = u(1) = 0,$$

which is explicitly expressed by

$$K(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1,\\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

We need the following lemmas.

Lemma 2.1. *$K(t,s)$ has the following properties:*

(i) $K(t,s) > 0$ *for all* $t,s \in (0,1)$;
(ii) $K(t,s) \leq K(s,s)$ *for all* $t,s \in [0,1]$;
(iii) $K(t,s) \geq K(t,t)K(s,s)$ *for all* $t,s \in [0,1]$;
(iv) $|K(t_1,s) - K(t_2,s)| \leq |t_1 - t_2|$ *for all* $t_1,t_2,s \in [0,1]$.

Lemma 2.2 ([GL]). *Let E be a real Banach space and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$,* *and let* $Q : P\cap(\overline{\Omega}_2\setminus\Omega_1) \to P$ *be a completely continuous operator such that either*

(i) $\|Qu\| \le \|u\|$, $u \in P\cap\partial\Omega_1$ *and* $\|Qu\| \ge \|u\|$, $u \in P\cap\partial\Omega_2$; *or*
(ii) $\|Qu\| \ge \|u\|$, $u \in P\cap\partial\Omega_1$ *and* $\|Qu\| \le \|u\|$, $u \in P\cap\partial\Omega_2$.

Then Q *has a fixed point in* $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

The boundary value problem

$$-\varphi'' = \mu u, \qquad \varphi(0) = \varphi(1) = 0,$$

can be solved by using the Green function, namely,

$$\varphi(t) = \mu\int_0^1 K(t,s)u(s)\,ds, \qquad 0 < t < 1. \tag{2.2}$$

Inserting (2.2) into the first equation of (1.3), we have

$$\begin{aligned} &u^{(4)} = \mu u(t)\int_0^1 K(t,s)u(s)\,ds + f(t,u,u''),\\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{2.3}$$

Now we consider the existence of a positive solution of (2.3). A function $u \in C^4(0,1)\cap C^2[0,1]$ satisfying (2.3) is a *positive solution* of (2.3) if $u(t) \ge 0$ for all $t \in [0,1]$, and $u \ne 0$.

Then the solution of (2.3) can be expressed as

$$\begin{aligned} u(t) = \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)u(v)\,dv\,ds\,d\tau\\ + \int_0^1\int_0^1 K(t,\tau)K(\tau,s)f(s,u(s),u''(s))\,ds\,d\tau \end{aligned} \tag{2.4}$$

and the second-order derivative u'' can be expressed by

$$\begin{aligned} u''(t) = -\mu\int_0^1 K(t,s)\,u(s)\int_0^1 K(s,v)u(v)\,dv\,ds\\ - \int_0^1 K(t,s)f(s,u(s),u''(s))\,ds. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} P = \{u \in C^2[0,1] : u(0) = u(1) = 0,\ u \ge 0,\ u'' \le 0 \text{ on } [0,1],\\ u(t) \ge \sigma\|u\|_0 \text{ and } -u''(t) \ge \sigma\|u''\|_0 \text{ for } t \in [1/4,3/4]\}, \end{aligned}$$

where $\sigma = \min_{t\in[1/4,3/4]} K(t,t) = 3/16$.

Note that P is a cone in $C^2[0,1]$. For $R>0$, write $B_R=\{u\in C^2[0,1]: \|u\|_2<R\}$.

We now define a mapping $T: P\to C^2[0,1]$ by

$$\tag{2.6} \begin{aligned} Tu(t) &= \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,u(s)\int_0^1 K(s,v)\,u(v)\,dv\,ds\,d\tau \\ &\quad+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f(s,u(s),u''(s))\,ds\,d\tau. \end{aligned}$$

LEMMA 2.3. *Let $w\in P$. Then the following relations hold:*

(a) $(Tw)(t)\ge K(t,t)\|Tw\|_0$ *for* $t\in[0,1]$, *and*
(b) $-(Tw)''(t)\ge K(t,t)\|Tw''\|_0$ *for* $t\in[0,1]$.

Proof. For simplicity we denote

$$\begin{aligned} I &= \mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)w(s)\int_0^1 K(s,v)w(v)\,dv\,ds\,d\tau \\ &\quad+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)h(s)\,ds\,d\tau, \\ J &= \mu\int_0^1 K(s,s)w(s)\int_0^1 K(s,v)w(v)\,dv\,ds+\int_0^1 K(s,s)h(s)\,ds, \\ h(s) &= f(s,w(s),w''(s)). \end{aligned}$$

From Lemma 2.1 it is easy to see that

$$\tag{2.7} K(t,t)I\le Tw(t)\le I,\qquad K(t,t)J\le -(Tw)''(t)\le J,\qquad t\in[0,1].$$

Using (2.7), we have $\|Tw\|_0\le I$ and $\|-(Tw)''\|_0\le J$, hence

$$(Tw)(t)\ge K(t,t)\|Tw\|_0,\quad -(Tw)''(t)\ge K(t,t)\|Tw''\|_0,\quad t\in[0,1]. \ \blacksquare$$

Throughout this paper, we assume additionally that the function f satisfies

$$\tag{H1} f(t,u,v)\le f_1(t)f_2(|u|+|v|),\quad t\in(0,1),\ u\in\mathbb{R}^+,\ v\in\mathbb{R}^-,$$

where $f_1\in C([0,1],\mathbb{R}^+)$, $f_2\in C(\mathbb{R}^+,\mathbb{R}^+)$.

Let us introduce the following constants:

$$\begin{aligned} D_1 &= \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\,ds\,d\tau, \qquad D_2=\int_0^1 K(s,s)\,f_1(s)\,ds, \\ D_3 &= \int_0^1\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)K(s,v)\,dv\,ds\,d\tau, \end{aligned}$$

$$D_4 = \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,ds\,d\tau, \qquad D_5 = \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)\,ds\,d\tau.$$

LEMMA 2.4. *Let* (H1) *hold. Then for all* $u \in C^2[0,1]$ *such that* $u(0) = u(1) = 0$, $u \geq 0$, *and* $u'' \leq 0$, *we have*

$$(Tu)(t) \leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|), \qquad t \in (0,1),$$

$$-(Tu)''(t) \leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|), \qquad t \in (0,1).$$

Proof. It is easy to see that $D_3 \leq D_1$ and $D_4 \leq D_2$.

By Lemma 2.1 and (H1) we have

$$\begin{aligned}Tu(t) &\leq \mu \int_0^1\int_0^1\int_0^1 K(\tau,\tau)\,K(\tau,s)K(s,v)\,dv\,ds\,d\tau \|u\|_0^2 \\ &\quad + \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\,f_1(s)\,ds\,d\tau \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \\ &\leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|),\end{aligned}$$

and similarly

$$\begin{aligned}-(Tu)''(t) &\leq \mu \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\,ds\,d\tau \|u\|_0^2 \\ &\quad + \int_0^1 K(s,s)\,f_1(s)\,ds\,d\tau \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \\ &\leq \mu D_1 \|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|). \ \blacksquare\end{aligned}$$

LEMMA 2.5. $T(P) \subset P$ *and* $T : P \to P$ *is completely continuous.*

Proof. Let $u \in P$. From (2.6), it is clear that

$$\begin{aligned}(Tu)''(t) &= -\mu \int_0^1 K(t,s)u(s) \int_0^1 K(s,v)u(v)\,dv\,ds \\ &\quad - \int_0^1 K(t,s)f(s,u(s),u''(s))\,ds \leq 0.\end{aligned}$$

By Lemma 2.3,

$$Tu(t) \geq K(t,t)\|Tu\|_0 \geq \sigma\|Tu\|_0, \qquad t \in [1/4,3/4],$$

$$-(Tu)''(t) \geq K(t,t)\|(Tu)''\|_0 \geq \sigma\|(Tu)''\|_0, \qquad t \in [1/4,3/4].$$

Hence $T(P) \subset P$.

Let $V \subset P$ be a bounded set. Then $\sup\{\|u\|_2 : u \in V\} =: d > 0$.

First we prove $T(V)$ is bounded. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$, we have $|u(t)| + |u''(t)| \le \|u\|_0 + \|u''\|_0 \le 2d$ for all $t \in [0,1]$. Let $M_d = \sup\{f_2(w) : w \in [0, 2d]\}$. From Lemma 2.4 for any $u \in V$ and $t \in [0,1]$ we have

$$\begin{aligned}|Tu(t)| &= \Big|\mu \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,u(s)\int_0^1 K(s,v)\,u(v)\,dv\,ds\,d\tau \\ &\quad + \int_0^1\int_0^1 K(t,\tau)K(\tau,s)\,f(s,u(s),u''(s))\,ds\,d\tau\Big| \\ &\le \mu D_1\|u\|_0^2 + D_2 \sup_{s\in(0,1)} f_2(|u(s)| + |u''(s)|) \le \mu D_1 d^2 + M_d D_2.\end{aligned}$$

We have a similar inequality for $|(Tu)''(t)|$. Therefore $T(V)$ is bounded.

Next we prove that $T(V)$ is equicontinuous. From Lemma 2.4 for any $u \in V$ and any $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned}&|(Tu)(t_1) - (Tu)(t_2)| \\ &\le \mu \int_0^1\int_0^1 |K(t_1,\tau) - K(t_2,\tau)|K(\tau,s)u(s)\int_0^1 K(s,v)u(v)\,dv\,ds\,d\tau \\ &\quad + \int_0^1\int_0^1 |K(t_1,\tau) - K(t_2,\tau)|K(\tau,s)f(s,u(s),u''(s))\,ds\,d\tau \\ &\le \mu \int_0^1\int_0^1\int_0^1 |K(t_1,\tau) - K(t_2,\tau)|K(\tau,s)K(s,v)\,dv\,ds\,d\tau \|u\|_0^2 \\ &\quad + \int_0^1\int_0^1 |K(t_1,\tau) - K(t_2,\tau)|K(\tau,s)f_1(s)f_2(|u(s)| + |u''(s)|)\,ds\,d\tau \\ &\le \mu|t_1 - t_2| \int_0^1\int_0^1 K(s,s)K(s,v)\,dv\,ds\,\|u\|_0^2 \\ &\quad + M_d|t_1 - t_2| \int_0^1 K(s,s)\,f_1(s)\,ds \\ &\le (\mu D_1 d^2 + M_d D_3)|t_1 - t_2|.\end{aligned}$$

We have a similar inequality for $|(Tu)''(t_1) - (Tu)''(t_2)|$. Therefore $T(V)$ is equicontinuous.

Next we prove that T is continuous. Suppose $u_n, u \in P$ and $\|u_n - u\|_2 \to 0$, which implies that $u_n(t) \to u(t), u_n''(t) \to u''(t)$ uniformly on $[0,1]$ and similarly for $f(t,u,v) = g(t)h(t,u,v)$, $h(t, u_n(t), u_n''(t)) \to h(t, u(t), u''(t))$

uniformly on $[0,1]$. The assertion follows from the estimate

$$|Tu_n(t) - Tu(t)|$$
$$\leq \mu \int_0^1\int_0^1 K(t,\tau)K(\tau,s)|u_n(s)-u(s)| \int_0^1 K(s,v)\,|u_n(v)-u(v)|\,dv\,ds\,d\tau$$
$$+ \int_0^1\int_0^1 K(t,\tau)K(\tau,s)|g(s)|\,|h(s,u_n(s),u_n''(s)) - h(t,u(s),u''(s))|\,ds\,d\tau$$

and a similar estimate for $|(Tu_n)''(t) - (Tu)''(t)|$ by an application of a standard theorem on convergence of integrals. ∎

LEMMA 2.6. *If* $u(0) = u(1) = 0$ *and* $u \in C^2[0,1]$, *then* $\|u\|_0 \leq \|u''\|_0$, *and so* $\|u\|_2 = \|u''\|_0$.

Proof. Since $u(0) = u(1)$, there is a $\alpha \in (0,1)$ such that $u'(\alpha) = 0$, and so $u'(t) = \int_\alpha^t u''(s)\,ds$ for $t \in [0,1]$. Hence $|u'(t)| \leq \int_\alpha^t |u''(s)|\,ds \leq \int_0^1 |u''(s)|\,ds \leq \|u''\|_0$ for $t \in [0,1]$. Thus $\|u'\|_0 \leq \|u''\|_0$. Since $u(0) = 0$, we have $u(t) = \int_0^t u'(s)\,ds$ for $t \in [0,1]$, and so $|u(t)| \leq \int_0^1 |u'(s)|\,ds \leq \|u'\|_0$. Thus $\|u\|_0 \leq \|u'\|_0 \leq \|u''\|_0$. Since $\|u\|_2 = \max\{\|u\|_0, \|u''\|_0\}$ and $\|u\|_0 \leq \|u''\|_0$, we conclude that $\|u\|_2 = \|u''\|_0$. ∎

COROLLARY 2.7. *Let* $r > 0$ *and* $u \in \partial B_r \cap P$. *Then* $\|u\|_2 = \|u''\|_0 = r$.

3. Main results

THEOREM 3.1. *Let* (H1) *hold. Assume that*

$$\text{(H2)} \qquad \limsup_{w\to 0^+} \frac{f_2(w)}{w} = 0, \qquad \liminf_{|v|\to\infty} \min_{t\in[1/4,3/4]} \inf_{u\in[0,\infty)} \frac{f(t,u,v)}{|v|} = \infty.$$

If $\mu \in (0, 1/(4D_1))$, *then problem* (1.3) *has at least one positive solution.*

Proof. Let us choose $0 < c_1 \leq 1/(8D_2)$. Then from (H2), there exists $0 < r < 1/2$ such that

$$f_2(|u|+|v|) \leq c_1(|u|+|v|), \qquad 0 \leq |u|+|v| \leq 2r.$$

Let $u \in \partial B_r \cap P$. Then by Corollary 2.7, $\|u\|_2 = \|u''\|_0 = r$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq r$ and $|u''(t)| \leq \|u''\|_0 = r$, for all $t \in [0,1]$. Thus $0 \leq |u(t)| + |u''(t)| \leq 2r$ for all $t \in [0,1]$.

Hence, by Lemma 2.4, (H1) and (H2), we have

$$(Tu)(t) \leq \mu \int_0^1\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)K(s,v)\,dv\,ds\,d\tau\,\|u\|_0^2$$
$$+ \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)f_2(|u|+|u''|)\,ds\,d\tau$$

$$\begin{aligned}&\le \mu D_3\|u\|_0^2 + c_1 D_4(\|u\|_0 + \|u''\|_0)\\&\le \mu D_1\|u\|_0^2 + c_1 D_2(\|u\|_0 + \|u''\|_0)\\&\le \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\|u\|_2 \le \tfrac{1}{4}\|u\|_2^2 + \tfrac{1}{4}\|u\|_2\\&\le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_r \cap P,\ t \in [0,1].\end{aligned}$$

Consequently,

$$\|Tu\|_0 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_r \cap P. \tag{3.1}$$

Similarly we also have

$$\begin{aligned}(Tu)''(t) = -\mu \int_0^1 K(t,s)u(s)\int_0^1 K(s,v)u(v)\,dv\,ds\,d\tau\\ - \int_0^1 K(t,s) f(t,u(s),u''(s))\,ds\,d\tau.\end{aligned}$$

Hence

$$\begin{aligned}|(Tu)''(t)| &\le \mu \int_0^1\int_0^1 K(s,s)K(s,\tau)\,ds\,d\tau\,\|u\|_0^2 + \int_0^1 K(s,s) f_1(s) f_2(|u| + |u''|)\,ds\\&\le \mu D_1\|u\|_0^2 + c_1 D_2(\|u\|_0 + \|u''\|_0)\\&\le \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\|u\|_2 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_r \cap P,\ t \in [0,1].\end{aligned}$$

Consequently,

$$\|(Tu)''\|_0 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_r \cap P. \tag{3.2}$$

Using (3.1) and (3.2) we have

$$\|Tu\|_2 \le \|Tu\|_0 + \|(Tu)''\|_0 \le \|u\|_2, \quad \forall u \in \partial B_r \cap P. \tag{3.3}$$

Let us choose $c_2 \ge 1/(\sigma D_5)$. Then from condition (H2), there exists $R_1 > 0$ such that

$$f(t,u,v) \ge c_2|v|, \quad \forall u \in R^+,\ \forall |v| \ge R_1,\ t \in [1/4,3/4].$$

Let $R > \max\{R_1/\sigma, r\}$. Let $u \in \partial B_R \cap P$. Then $\|u''\|_0 = R$. Thus we have

$$\min_{t\in[1/4,3/4]} |u''| \ge \sigma\|u''\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 2.1, (H1) and (H2), we have

$$\begin{aligned}(Tu)(1/2) \ge \mu \int_{1/4}^{3/4}\int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)u(s)K(s,v)u(v)\,dv\,ds\,d\tau\\ + \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s) f(t,u(s),u''(s))\,ds\,d\tau\end{aligned}$$

$$\geq c_2 \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)|u''(s)|\,ds\,d\tau$$

$$\geq c_2\sigma \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)\,ds\,d\tau\,\|u''\|_0 \geq \|u''\|_0,$$

so

$$(Tu)(1/2) \geq \|u''\|_0 = \|u\|_2, \qquad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \qquad \forall u \in \partial B_R \cap P.$$

Finally, from Lemma 2.2, by (3.3) and the above inequality we see that the problem (1.3) has a positive solution. ∎

THEOREM 3.2. *Let* (H1) *hold. Assume that*

(H3)
$$\liminf_{|u|+|v|\to 0^+} \min_{t\in[1/4,3/4]} \frac{f(t,u,v)}{|u|+|v|} = \infty,$$
$$\liminf_{|v|\to\infty} \min_{t\in[1/4,3/4]} \inf_{u\in[0,\infty)} \frac{f(t,u,v)}{|v|} = \infty.$$

(H4) *there exists* $0 < \varrho < 1/2$ *such that*

$$\sup_{w\in[0,1]} f_2(w) \leq \frac{\varrho}{4D_2}. \tag{3.4}$$

If $\mu \in (0, 1/(4D_1))$, *then problem* (1.3) *has at least two positive solutions.*

We note for the argument below that $D_4 \leq D_2$.

Proof. By condition (H4) there exists $0 < \varrho < 1/2$ such that (3.4) is fulfilled. Let $u \in \partial B_\varrho \cap P$. Then from Corollary 2.7, $\|u''\|_0 = \varrho$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \leq \|u''\|_0$ we have $u(t) \leq \|u\|_0 \leq \varrho$ and $|u''(t)| \leq \|u''\|_0 = \varrho$, for all $t \in [0,1]$. Thus $0 \leq |u(t)| + |u''(t)| < 1$ for all $t \in [0,1]$. By condition (H4), for all $u \in \partial B_\varrho \cap P$ and $t \in [0,1]$, we have

$$(Tu)(t) \leq \mu \int_0^1\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)K(s,v)\,dv\,ds\,d\tau\,\|u\|_0^2$$
$$+ \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)f_2(|u|+|u''|)\,ds\,d\tau$$
$$\leq \mu D_3\|u\|_0^2 + \frac{\varrho}{4D_2}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,ds\,d\tau$$

$$\begin{aligned}
&\le \mu D_1\|u\|_0^2 + \frac{\varrho}{4D_4}\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,ds\,d\tau \\
&\le \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\varrho = \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\|u''\|_0 = \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\|u\|_2 \\
&\le \tfrac{1}{4}\|u\|_2^2 + \tfrac{1}{4}\|u\|_2 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_\varrho \cap P,\ t\in[0,1].
\end{aligned}$$

Consequently, we get

$$\|Tu\|_0 \le \|u\|_2, \quad \forall u \in \partial B_\varrho \cap P. \tag{3.5}$$

Similarly,

$$\begin{aligned}
(Tu)''(t) = &-\mu\int_0^1 K(t,s)u(s)\int_0^1 K(s,v)u(v)\,dv\,ds\,d\tau \\
&-\int_0^1 K(t,s)f(t,u(s),u''(s))\,ds\,d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
|(Tu)''(t)| &\le \mu\int_0^1\int_0^1 K(s,s)K(s,\tau)\,ds\,d\tau\,\|u\|_0^2 + \int_0^1 K(s,s)f_1(s)f_2(|u|+|u''|)\,ds \\
&\le \mu D_1\|u\|_0^2 + \frac{\varrho}{4D_2}\int_0^1 K(s,s)f_1(s)\,ds \\
&\le \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\varrho = \tfrac{1}{4}\|u\|_0^2 + \tfrac{1}{4}\|u\|_2 \\
&\le \tfrac{1}{4}\|u\|_2^2 + \tfrac{1}{4}\|u\|_2 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_\varrho \cap P,\ t\in[0,1].
\end{aligned}$$

Consequently,

$$\|(Tu)''\|_0 \le \tfrac{1}{2}\|u\|_2, \quad \forall u \in \partial B_\rho \cap P. \tag{3.6}$$

Using (3.5) and (3.6) we have

$$\|Tu\|_2 \le \|Tu\|_0 + \|(Tu)''\|_0 \le \|u\|_2, \quad \forall u \in \partial B_\rho \cap P. \tag{3.7}$$

Let us choose $c_3 \ge 1/(\sigma D_5)$. Then from condition (H3), there exists $0 < r < \varrho$ such that

$$f(t,u,v) \ge c_3(|u|+|v|), \quad \forall u \in [0,r],\ \forall |v| \in [0,r],\ t \in [1/4,3/4].$$

Let $u \in \partial B_r \cap P$. By Corollary 2.7, $\|u''\|_0 = r$ and $u(0) = u(1) = 0$. Also since $\|u\|_0 \le \|u''\|_0$ we have

$$0 \le u(t) \le \|u\|_0 \le r, \quad 0 \le |u''(t)| \le \|u''\|_0 = \|u\|_2 = r, \quad \forall u \in \partial B_r \cap P.$$

Moreover,

$$\min_{t\in[1/4,3/4]} |u''| \ge \sigma\|u''\|_0 = \sigma r, \quad \forall u \in \partial B_r \cap P.$$

The estimate for $(Tu)(1/2)$ is similar to that in the proof of Theorem 3.1: from Lemma 2.1 and (H1) we have

$$(Tu)(1/2) \geq c_3 \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)(|u(s)| + |u''(s)|)\,ds\,d\tau$$
$$\geq c_3\sigma \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)\,ds\,d\tau\,\|u''\|_0 \geq \|u''\|_0.$$

Thus

$$(Tu)(1/2) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_r \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_r \cap P.$$

Finally, we show that for sufficiently large $R > 1/2$, we have

$$\|Tu\|_2 \geq \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

To see this we choose $c_2 \geq 1/(\sigma D_5)$. Then from condition (H4), there exist $R_1 > 0$ such that

$$f(t,u,v) \geq c_2|v|, \quad \forall u \in \mathbb{R}^+,\ \forall |v| \geq R_1,\ t \in [1/4,3/4].$$

Let $R > \max\{R_1/\sigma, 1/2\}$. Let $u \in \partial B_R \cap P$. Then from Corollary 2.7, $\|u''\|_0 = R$. Thus we have

$$\min_{t\in[1/4,3/4]} |u''| \geq \sigma\|u''\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 2.1, (H1) and (H4),

$$(Tu)(1/2) \geq c_2 \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)|u''(s)|\,ds\,d\tau$$
$$\geq c_2\sigma \int_{1/4}^{3/4}\int_{1/4}^{3/4} K(1/2,\tau)K(\tau,s)\,ds\,d\tau\,\|u''\|_0 \geq \|u''\|_0,$$

so

$$(Tu)(1/2) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_R \cap P.$$

Then by Lemma 2.2, we know that T has at least two fixed points in $(\overline{B}_R \setminus B_\rho) \cap P$ and $(\overline{B}_\rho \setminus B_r) \cap P$, i.e. problem (1.3) has at least two positive solutions. ■

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