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Existence of positive solutions for a class of arbitrary order boundary value problems involving nonlinear functionals

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Abstract. We give conditions which guarantee the existence of positive solutions for a variety of arbitrary order boundary value problems for which all boundary conditions involve functionals, using the well-known Krasnosel'skiĭ fixed point theorem. The conditions presented here deal with a variety of problems, which correspond to various functionals, in a uniform way. The applicability of the results obtained is demonstrated by a numerical application.

1. Introduction. There are quite a few papers dealing with the existence of positive solutions for boundary value problems where at least one condition involves functionals. Usually these functionals are linear, for example Riemann–Stieltjes integrals, and must satisfy restrictions posed by the natural problem under study. Additionally, the corresponding boundary value problems are usually, at the most, of fourth order. For problems involving linear functionals and techniques which cover various boundary conditions in a uniform way, the reader may refer to [[1]–[5], [7]–[9], [11]–[14]].

In this paper, we study the existence of positive solutions for arbitrary order boundary value problems, for which all boundary conditions involve functionals, using the well-known Krasnosel'skiĭ fixed point theorem (see [6, 10]). We do not focus on the way this theorem is applied to each problem separately—the results we present deal with various boundary value problems in a uniform way. It is remarkable that the boundary conditions involve functionals which are not necessarily linear, but are required to satisfy a specific form of invertibility. Our purpose is to pose conditions on the functionals which will guarantee that the Krasnosel'skiĭ fixed point theorem can be applied. However, other fixed point theorems can be used as well, yielding analogous results.

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The results presented here can be regarded as a generalization of [11]. For this reason, we tried to keep the overall approach and notation similar between the two papers.

The sectioning is typical for the papers discussing similar problems. Section 2 contains the setting of the problem we study, as well as the necessary preliminary results, while the main result is presented in Section 3. The applicability of the new results is demonstrated in Section 4.

2. Preliminaries. Let $a, b \in (0, \infty)$, $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ for $i \in \{0, 1, \ldots, n-1\}$, and let $J_a = [0, a]$. Also, let $E = C^{n-1}(J_a, \mathbb{R})$, and let $\|\cdot\| : E \to [0, \infty)$ be defined as

$$\|x\| = \sum_{k=0}^{n-2} |x^{(k)}(0)| + \sup_{t \in J_a} |x^{(n-1)}(t)|.$$

It is well-known that the functional $\|\cdot\|$ is a norm in E and that $(E, \|\cdot\|)$ is a Banach space.

Definition 2.1. Let B be a Banach space. A *cone* in B is a nonempty closed set $K \subseteq B$ such that

(i) $\kappa u + \lambda v \in K$ for $u, v \in K$ and $\kappa, \lambda \in [0, \infty)$.
(ii) $u, -u \in K$ implies $u = 0$.

It is easy to see that the set

$$K = \{x \in E : x(t) \geq 0,\ t \in J_a\}$$

is a cone in E. Let

$$K_b = \{x \in K : \|x\| < b\},$$

and let $\overline{K}_b$ be the closure of K_b in K.

For $S \subseteq \mathbb{R}$, let $\mathcal{H}_S$ be the set of all continuous functions $h : C(J_a, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ with the following properties:

- for all $h_1, h_2 \in \mathcal{H}_S$, $x \in C(J_a, \mathbb{R})$ and $s_1, s_2 \in S$, we have
$$h_1(x, s_1) = h_2(x, s_2) \ \Rightarrow\ s_1 = s_2;$$
- for all $h \in \mathcal{H}_S$ and $x_1, x_2 \in C(J_a, \mathbb{R})$, we have
$$x_1(t) \leq x_2(t) \text{ for every } t \in J_a \ \Rightarrow\ h(x_1, \xi) \geq h(x_2, \xi).$$

For any continuous function $u : J_a \times \overline{K}_b \to \mathbb{R}$ and any $v \in \overline{K}_b$, we will use $I(i, u, \sigma, v)$ to denote the function

$$\begin{cases} t \mapsto \int_0^t u(\sigma, v)\, d\sigma,\ t \in J_a, & \text{if } i = 0, \\ t \mapsto \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_{n-i}} u(\sigma, v)\, d\sigma\, ds_{n-i} \cdots ds_{n-1},\ t \in J_a, & \text{if } i \in \{1, \ldots, n\}. \end{cases}$$

Also, for any $c \in \mathbb{R}$ and any $i \in 0, 1, \ldots, n$, we will use $P(i, c)$ to denote the function

$$t \mapsto c\frac{t^i}{i!}, \qquad t \in J_a.$$

For $x \in \overline{K}_b$, consider the nth order differential equation

$$x^{(n)}(t) + f(t, x) = 0, \qquad t \in J_a, \tag{2.1}$$

along with the conditions

$$\alpha_i(x^{(i)}) = \xi_i, \qquad i \in \{0, 1, \ldots, n-1\}, \tag{2.2}$$

where

- $f : J_a \times \overline{K}_b \to \mathbb{R}$ is a continuous function such that
$$|f(t, x)| \le g(t, x) \qquad \text{for every } (t, x) \in J_a \times \overline{K}_b,$$
and $g : J_a \times \overline{K}_b \to [0, \infty)$ is a continuous function with the following properties:
    - ◦ for every $t \in J_a$ we have
$$g(t, x) \le g\Big(t, \sum_{k=0}^{n-1} P(k, b)\Big) \qquad \text{for every } x \in \overline{K}_b;$$
    - ◦ there exists $w : [0, \infty) \to [0, \infty)$ such that
$$\int_0^a g\Big(s, \sum_{k=0}^{n-1} P(k, t)\Big)\, ds \le w(t) \qquad \text{for every } t \in [0, \infty).$$
- For every $i \in \{0, 1, \ldots, n-1\}$, $\alpha_i : C(J_a, \mathbb{R}) \to \mathbb{R}$ is a functional for which there exists a nonempty set $\mathcal{A}_{\alpha_i} \subseteq \mathcal{H}_{\mathcal{R}(\alpha_i)}$ such that for all $x \in C(J_a, \mathbb{R})$ and $c \in \mathbb{R}$,
$$\text{if } \xi_i = \alpha_i(x + c) \text{ then there exists } h_i \in \mathcal{A}_{\alpha_i} \text{ with } c = h_i(x, \xi_i).$$

LEMMA 2.2. *Let $x \in \overline{K}_b$. Then*

$$|x^{(i)}| \le \sum_{k=0}^{n-i-1} P(k, b) \qquad \textit{for every } i \in \{0, 1, \ldots, n-1\}. \tag{2.3}$$

Proof. Since $x \in \overline{K}_b$, we can verify that

$$|x^{(n-1)}(t)| \le b \qquad \text{for every } t \in J_a, \tag{2.4}$$

$$|x^{(i)}(0)| \le b \qquad \text{for every } i \in \{0, 1, \ldots, n-2\}. \tag{2.5}$$

Also, we observe that inequality (2.3) is equivalent to

$$|x^{(n-i)}| \le \sum_{k=0}^{i-1} P(k, b), \qquad i \in \{1, \ldots, n\}. \tag{2.6}$$

We will prove (2.6) by induction on i. From (2.4), we see that (2.6) holds for $i = 1$. Suppose that (2.6) holds for a fixed $i \in \{1, \ldots, n-1\}$. We will prove that it also holds for $i + 1$. Indeed,

$$-\sum_{k=0}^{i-1} P(k, b) \le x^{(n-i)} \le \sum_{k=0}^{i-1} P(k, b),$$

so for $t \in J_a$ we have

$$-\sum_{k=0}^{i-1}\Big(\int_0^t P(k, b)(s)\, ds\Big) \le \int_0^t x^{(n-i)}(s)\, ds \le \sum_{k=0}^{i-1}\Big(\int_0^t P(k, b)(s)\, ds\Big),$$

therefore using (2.5) we conclude that

$$-\sum_{k=0}^{i} P(k, b) \le x^{(n-i-1)} \le \sum_{k=0}^{i} P(k, b). \ \blacksquare$$

Let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{1, \ldots, n-1\}$, and for $x \in \overline{K}_b$ define the functions

$$N_{x;i} = \begin{cases} -I(0, f, \sigma, x) & \text{if } i = 0, \\ -I(i, f, \sigma, x) \\ \quad + \sum_{k=1}^{i} P(k, h_{n-i+k-1}(N_{x;i-k}, \xi_{n-i+k-1})) & \text{if } i \in \{1, \ldots, n-1\}. \end{cases}$$

Lemma 2.3. *Every solution x of the problem* (2.1)–(2.2) *satisfies*

$$\begin{aligned} x^{(i)} &= -I(n-i-1, f, \sigma, x) \\ &\quad + \sum_{k=0}^{n-i-1} P(k, h_{i+k}(N_{x;n-i-k-1}, \xi_{i+k})) \quad \textit{for } i \in \{0, 1, \ldots, n-1\}, \end{aligned} \tag{2.7}$$

where $h_j \in \mathcal{A}_{\alpha_j}$, $j \in \{0, 1, \ldots, n-1\}$.

Proof. We will use reverse induction on $\{0, 1, \ldots, n-1\}$.

For $i = n-1$, equation (2.7) takes the form

$$x^{(n-1)} = -I(0, f, \sigma, x) + P(0, h_{n-1}(-I(0, f, \sigma, x), \xi_{n-1})). \tag{2.8}$$

Since x satisfies (2.1), for any $t \in J_a$ we have

$$x^{(n)}(t) = -f(t, x),$$

therefore

$$x^{(n-1)}(t) = x^{(n-1)}(0) - \int_0^t f(\sigma, x)\, d\sigma,$$

so, taking into account (2.2), we conclude that there exists $h_{n-1} \in \mathcal{A}_{\alpha_{n-1}}$ such that

$$x^{(n-1)}(0) = h_{n-1}(-I(0, f, \sigma, x), \xi_{n-1}).$$

It is now easy to see that (2.8) is true.

Now, assume that (2.7) is true for some $i \in \{1, \ldots, n-1\}$. We will show that it is also true for $i-1$. Indeed, from

$$x^{(i)} = -I(n-i-1, f, \sigma, x) + \sum_{k=0}^{n-i-1} P(k, h_{i+k}(N_{x;n-i-k-1}, \xi_{i+k}))$$

we get

$$x^{(i-1)} = P(0, x^{(i-1)}(0)) - I(n-i, f, \sigma, x) + \sum_{k=1}^{n-i} P(k, h_{i-1+k}(N_{x;n-i-k}, \xi_{i-1+k})).$$

Since $\xi_{i-1} = \alpha_{i-1}(x^{(i-1)})$, there exists $h_{i-1} \in \mathcal{A}_{\alpha_{i-1}}$ such that

$$P(0, x^{(i-1)}(0)) = P(0, h_{i-1}(N_{x;n-i}, \xi_{i-1})),$$

therefore

$$\begin{aligned} x^{(i-1)} &= -I(n-i, f, \sigma, x) + P(0, h_{i-1}(N_{x;n-i}, \xi_{i-1})) \\ &\quad + \sum_{k=1}^{n-i} P(k, h_{i-1+k}(N_{x;n-i-k}, \xi_{i-1+k})) \\ &= -I(n-i, f, \sigma, x) + \sum_{k=0}^{n-i} P(k, h_{i-1+k}(N_{x;n-i-k}, \xi_{i-1+k})). \ \blacksquare \end{aligned}$$

Let $h_i \in \mathcal{A}_{\alpha_i}$ for $i = \{0, 1, \ldots, n-1\}$, and define the operator

$$T_{h_0 \ldots h_{n-1}} : \overline{K}_b \to C^{n-1}(J_a, \mathbb{R})$$

by

$$T_{h_0 \ldots h_{n-1}} x = -I(n-1, f, \sigma, x) + \sum_{k=0}^{n-1} P(k, h_k(N_{x;n-k-1}, \xi_k)). \tag{2.9}$$

Lemma 2.4. *For every $i \in \{0, 1, \ldots, n-1\}$ we have*

$$T^{(i)}_{h_0 \ldots h_{n-1}} x = -I(n-i-1, f, \sigma, x) + \sum_{k=0}^{n-i-1} P(k, h_{i+k}(N_{x;n-i-k-1}, \xi_{i+k})). \tag{2.10}$$

Proof. Formula (2.10) can be derived from (2.9) by successive differentiations. ■

Lemma 2.5. *Let $h_i \in \mathcal{A}_{\alpha_i}$ for $i \in \{0, 1, \ldots, n-1\}$. Then $T_{h_0 \ldots h_{n-1}}(\overline{K}_b) \subseteq E$.*

Proof. Let $x \in \overline{K}_b$. Using Lemma 2.4, we get

$$T^{(n-1)}_{h_0 \ldots h_{n-1}} x = -I(0, f, \sigma, x) + P(0, h_{n-1}(-I(0, f, \sigma, x), \xi_{n-1})).$$

Also, for $t \in J_a$, we have

$$\int_0^t f(\sigma,x)\,d\sigma \le \int_0^t g(\sigma,x)\,d\sigma \le \int_0^t g\Big(\sigma, \sum_{k=0}^{n-1} P(k,b)\Big)\,d\sigma$$
$$\le \int_0^a g\Big(\sigma, \sum_{k=0}^{n-1} P(k,b)\Big)\,d\sigma \le w(b). \ \blacksquare$$

LEMMA 2.6. *A function $x \in \overline{K}_b$ is a solution of the problem* (2.1)–(2.2) *if and only if x is a fixed point of the operator $T_{h_0\dots h_{n-1}} : \overline{K}_b \to E$ defined by* (2.9) *for some $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\dots,n-1\}$.*

Proof. Assume that x is a solution of the problem (2.1)–(2.2) in $\overline{K}_b$, and $t \in J_a$. Then, in view of Lemma 2.3, x is a fixed point of $T_{h_0\dots h_{n-1}}$.

Conversely, let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\dots,n-1\}$, and let $x \in \overline{K}_b$ be a fixed point of $T_{h_0\dots h_{n-1}}$. Then, by Lemma 2.4, x satisfies (2.1). Let i be in $\{0,1,\dots,n-1\}$, and $\zeta_i \in \mathbb{R}$ be such that $\alpha_i(x^{(i)}) = \zeta_i$. Then there exists $\hat{h}_i \in \mathcal{A}_{\alpha_i}$ with

$$h_i(N_{x;n-i-1},\xi_i) = \hat{h}_i(N_{x;n-i-1},\zeta_i),$$

so $\xi_i = \zeta_i$, and therefore $\alpha_i(x^{(i)}) = \xi_i$. ∎

The main result of this paper is based on the following theorem, which is known as the *Krasnosel'skiĭ fixed point theorem.*

THEOREM 2.7 ([6, 10]). *Let B be a Banach space and let K be a cone in B. Assume that Ω_1, Ω_2 are open, bounded subsets of B with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

$$\begin{cases} \|Tu\| \le \|u\| & \text{for every } u \in K \cap \partial\Omega_1, \\ \|Tu\| \ge \|u\| & \text{for every } u \in K \cap \partial\Omega_2, \end{cases}$$

or

$$\begin{cases} \|Tu\| \ge \|u\| & \text{for every } u \in K \cap \partial\Omega_1, \\ \|Tu\| \le \|u\| & \text{for every } u \in K \cap \partial\Omega_2. \end{cases}$$

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The main result

LEMMA 3.1. *For every $x \in \overline{K}_b$ we have*

$$P(i,-w(b)) \le -I(i,f,\sigma,x) \le P(i,w(b)) \quad \text{for } i \in \{0,1,\dots,n-1\}. \tag{3.1}$$

Proof. We will use induction on $\{0,1,\dots,n-1\}$.

For $i = 0$, inequality (3.1) takes the form

$$P(0,-w(b)) \le -I(0,f,\sigma,x) \le P(0,w(b)).$$

Since $x \in \overline{K}_b$, we have

$$-I(0,f,\sigma,x) \le I(0,g,\sigma,x) \le I\Big(0,g,\sigma,\sum_{k=0}^{n-1} P(k,b)\Big) \le P(0,w(b)),$$

$$-I(0,f,\sigma,x) \ge -I(0,g,\sigma,x) \ge -I\Big(0,g,\sigma,\sum_{k=0}^{n-1} P(k,b)\Big) \ge P(0,-w(b)).$$

Assume that (3.1) is true for some $i \in \{0,1,\ldots,n-2\}$. Then, by integration, we can show that it is also true for $i+1$. ■

Let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\ldots,n-1\}$, and for $t \in J_a$ define the functions

$$V_{t;i} = \begin{cases} P(0,w(t)) & \text{if } i = 0, \\ P(i,w(t)) & \\ \quad + \sum_{k=1}^{i} P(k, h_{n-i+k-1}(U_{t;i-k}, \xi_{n-i+k-1})) & \text{if } i \in \{1,\ldots,n-1\}, \end{cases}$$

$$U_{t;i} = \begin{cases} P(0,-w(t)) & \text{if } i = 0, \\ P(i,-w(t)) & \\ \quad + \sum_{k=1}^{i} P(k, h_{n-i+k-1}(V_{t;i-k}, \xi_{n-i+k-1})) & \text{if } i \in \{1,\ldots,n-1\}. \end{cases}$$

Lemma 3.2. *For every $x \in \overline{K}_b$ we have*

$$U_{b;i} \le N_{x;i} \le V_{b;i} \quad \textit{for } i \in \{0,1,\ldots,n-1\}. \tag{3.2}$$

Proof. This can be deduced from Lemma 3.1 and the definition of $N_{x;i}$. ■

Lemma 3.3. *Let $x \in \overline{K}_b$ and $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\ldots,n-1\}$. Then*

$$T^{(i)}_{h_0\ldots h_{n-1}} x \le P(n-i-1, w(b)) + \sum_{k=0}^{n-i-1} P(k, h_{i+k}(U_{b;n-i-k-1}, \xi_{i+k})),$$

$$T^{(i)}_{h_0\ldots h_{n-1}} x \ge P(n-i-1, -w(b)) + \sum_{k=0}^{n-i-1} P(k, h_{i+k}(V_{b;n-i-k-1}, \xi_{i+k})).$$

Proof. This can be deduced from Lemmas 3.1 and 3.2. ■

Lemma 3.4. *Let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\ldots,n-1\}$. Suppose that there exists $M \in (0,\infty)$ such that*

$$P(n-1,-w(M)) + \sum_{k=0}^{n-1} P(k, h_k(V_{M;n-k-1}, \xi_k)) \ge 0. \tag{3.3}$$

Then $T_{h_0\ldots h_{n-1}}(\overline{K}_M) \subseteq K$.

Proof. This can be deduced from Lemma 2.5 and (3.3). ■

LEMMA 3.5. *Let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\dots,n-1\}$. Suppose that there exists $M \in (0,\infty)$ such that*

$$h_i(V_{M;n-i-1},\xi_i) \geq 0 \quad \text{for every } i \in \{0,1,\dots,n-2\}, \tag{3.4}$$

$$h_{n-1}(V_{M;0},\xi_{n-1}) - w(M) \geq 0. \tag{3.5}$$

Then $T_{h_0\dots h_{n-1}}(\overline{K}_M) \subseteq K$.

Proof. This can be deduced from Lemma 3.4 and the observation that

$$\begin{aligned} P(n-1,-w(M)) &+ \sum_{k=0}^{n-1} P(k,h_k(V_{M;n-k-1},\xi_k)) \\ &= \sum_{k=0}^{n-2} P(k,h_k(V_{M;n-k-1},\xi_k)) + P(n-1,h_{n-1}(V_{M;0},\xi_{n-1}) - w(M_2)). \ \blacksquare \end{aligned}$$

LEMMA 3.6. *Let $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\dots,n-1\}$. Suppose that there exist $M_2 > M_1 > 0$ such that both inequalities* (3.4) *and* (3.5) *hold for $M = M_2$. Then the operator $T_{h_0\dots h_{n-1}} : \overline{K}_{M_2} \setminus K_{M_1} \to K$ is completely continuous.*

Proof. Let $S \subseteq \overline{K}_{M_2}$, $x \in S$, and $t \in J_a$. Then, using Lemma 3.3, we see that $T_{h_0\dots h_{n-1}}(S)$ is uniformly bounded.

Additionally, for $t_1, t_2 \in J_a$ with $0 \leq t_1 \leq t_2$ we have

$$\begin{aligned} |T^{(n-1)}_{h_0\dots h_{n-1}}x(t_2) - T^{(n-1)}_{h_0\dots h_{n-1}}x(t_1)| &= \Big|\int_{t_1}^{t_2} f(\sigma,x)\,d\sigma\Big| \leq \int_{t_1}^{t_2} |f(\sigma,x)|\,d\sigma \\ &\leq \int_{t_1}^{t_2} g(\sigma,x)\,d\sigma \leq \int_{t_1}^{t_2} g\Big(\sigma,\sum_{k=0}^{n-1} P(k,b)\Big)\,d\sigma. \end{aligned}$$

So $T_{h_0\dots h_{n-1}}(S)$ is equicontinuous.

The result follows by applying the well-known Arzelà–Ascoli theorem. ■

The following theorem is the main result of this paper.

THEOREM 3.7. *Let $M_1, M_2 \in (0,\infty)$. Suppose that for some $h_i \in \mathcal{A}_{\alpha_i}$, $i \in \{0,1,\dots,n-1\}$, we have*

$$h_i(V_{\max\{M_1,M_2\};n-i-1},\xi_i) \geq 0 \quad \text{for every } i \in \{0,1,\dots,n-2\}, \tag{3.6}$$

$$h_{n-1}(V_{\max\{M_1,M_2\};0},\xi_{n-1}) - w(\max\{M_1,M_2\}) \geq 0, \tag{3.7}$$

$$h_{n-1}(P(0,w(M_1)),\xi_{n-1}) \geq M_1, \tag{3.8}$$

$$w(M_2) + h_{n-1}(U_{M_1;0},\xi_{n-1}) \leq M_2, \tag{3.9}$$

$$-w(M_2) + h_{n-1}(V_{M_1;0},\xi_{n-1}) \geq -M_2. \tag{3.10}$$

Then the problem (2.1)–(2.2) *has a positive solution* y *such that*

$$\min\{M_1, M_2\} \le \sup_{t\in J_a} |y^{(n-1)}(t)| \le \max\{M_1, M_2\}. \tag{3.11}$$

Proof. From Lemma 3.5 and assumptions (3.6) and (3.7), we get

$$T_{h_0\dots h_{n-1}}(\overline{K}_{\max\{M_1,M_2\}} \setminus K_{\min\{M_1,M_2\}}) \subseteq K.$$

Also, from Lemma 3.6 and assumptions (3.6) and (3.7), we find that

$$T_{h_0\dots h_{n-1}} : \overline{K}_{\max\{M_1,M_2\}} \setminus K_{\min\{M_1,M_2\}} \to K$$

is completely continuous.

For any $x \in K$ with $\|x\| = M_1$ we have

$$T^{(n-1)}_{h_0\dots h_{n-1}} x(0) = h_{n-1}(-I(0, f, \sigma, x), \xi_{n-1}). \tag{3.12}$$

Also, it is a matter of simple calculations to see that $-I(0, f, \sigma, x) \le w(M_1)$, therefore

$$h_{n-1}(-I(0, f, \sigma, x), \xi_{n-1}) \ge h_{n-1}(P(0, w(M_1), \xi_{n-1}). \tag{3.13}$$

Combining assumption (3.8) with (3.12) and (3.13), we see that $T^{(n-1)}_{h_0\dots h_{n-1}} x(0) \ge M_1$, therefore

$$\|T_{h_0\dots h_{n-1}} x\| \ge \|x\| \quad \text{for any } x \in K \text{ with } \|x\| = M_1.$$

Additionally, from Lemma 3.3 and assumptions (3.9) and (3.10), we get

$$\|T_{h_0\dots h_{n-1}} x\| \le \|x\| \quad \text{for any } x \in K \text{ with } \|x\| = M_2.$$

At this point we can apply Theorem 2.7 to get the result. ■

4. An application. Let $n \in \mathbb{N}$, $\lambda \in (0, \infty)$, $\psi \in C(J_a, \mathbb{R})$ with

$$0 \le \psi(t) \le \frac{\lambda}{1+t^2} \quad \text{for } t \in J_a,$$

and $\phi_i \in C(J_a, \mathbb{R})$, $i \in \{0, 1, \dots, n-1\}$, with

$$0 \le \phi_i(t) \le \frac{\lambda}{1+t^{n+1-i}} \quad \text{for } t \in J_a.$$

For $x \in \overline{K}_b$, consider the nth order differential equation

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \phi_k(t) x^{(k)}(t) + \psi(t) = 0, \quad t \in J_a, \tag{4.1}$$

along with the conditions

$$(x^{(i)}(0))^2 + x^{(i)}(0) = \xi_i, \quad i \in \{0, 1, \dots, n-1\} \tag{4.2}$$

where $\xi_i \geq 0$ for every $i \in \{0, 1, \dots, n-1\}$. In this case,

$$f(t,x) = \psi(t) + \sum_{k=0}^{n-1} \phi_k(t) x^{(k)}(t),$$

$$g(t,x) = \psi(t) + \sum_{k=0}^{n-1} \phi_k(t) |x^{(k)}(t)|,$$

$$w(t) = \int_0^a \frac{\lambda}{1+s^2}\, ds + t \int_0^a \Big(\sum_{k=0}^{n-1} \frac{\lambda}{1+s^{n+1-k}} \sum_{j=k}^{n-1} \frac{s^{j-k}}{(j-k)!} \Big)\, ds.$$

Additionally, for $i \in \{0, 1, \dots, n-1\}$, we have

$$\alpha_i(y) = y^2(0) + y(0), \qquad y \in C(J_a, \mathbb{R}),$$

therefore $\mathcal{R}(\alpha_i) = [-0.25, \infty)$ and $\mathcal{A}_{\alpha_i} = \{h, \hat{h}\}$ where

$$h(y,\zeta) = 0.5(-2y(0) - 1 + \sqrt{|1+4\zeta|}), \qquad (y,\zeta) \in C(J_a, \mathbb{R}) \times \mathbb{R},$$
$$\hat{h}(y,\zeta) = 0.5(-2y(0) - 1 - \sqrt{|1+4\zeta|}), \qquad (y,\zeta) \in C(J_a, \mathbb{R}) \times \mathbb{R}.$$

We can verify that all the assumptions on f and α_i, $i \in \{0, 1, \dots, n-1\}$, are satisfied. Although the sets $\mathcal{A}_{\alpha_i}$ in this particular case are equal to each other for all $i \in \{0, 1, \dots, n-1\}$, which is convenient for the purposes of this application, this is not required in general.

For $i \in \{0, 1, \dots, n-1\}$ let

$$h_i(y,\zeta) = 0.5(-2y(0) - 1 + \sqrt{|1+4\zeta|}), \qquad (y,\zeta) \in C(J_a, \mathbb{R}) \times \mathbb{R}.$$

Then

$$V_{t;i}(s) = \begin{cases} w(t) \quad \text{if } i = 0, \\ w(t)\frac{s^i}{i!} + \sum_{k=1}^{i} 0.5(-2U_{t;i-k}(0) - 1 + \sqrt{1+4\xi_{n-i+k-1}})\frac{s^k}{k!} \\ \hfill \text{if } i \in \{1, \dots, n-1\} \end{cases}$$

and

$$U_{t;i}(s) = \begin{cases} -w(t) \quad \text{if } i = 0, \\ -w(t)\frac{s^i}{i!} + \sum_{k=1}^{i} 0.5(-2V_{t;i-k}(0) - 1 + \sqrt{1+4\xi_{n-i+k-1}})\frac{s^k}{k!} \\ \hfill \text{if } i \in \{1, \dots, n-1\}. \end{cases}$$

It is easy to see that

$$V_{t;i}(0) = 0 \quad \text{for } i \in \{1, \dots, n-1\},$$
$$U_{t;i}(0) = 0 \quad \text{for } i \in \{1, \dots, n-1\}.$$

Therefore

$$V_{t;i}(s) = \begin{cases} w(t) & \text{if } i = 0, \\ 0.5(4w(t) - 1 + \sqrt{1+4\xi_{n-1}})\frac{s^i}{i!} & \\ \quad + \sum_{k=1}^{i-1} 0.5(-1 + \sqrt{1+4\xi_{n-i+k-1}})\frac{s^k}{k!} & \text{if } i \in \{1, \dots, n-1\} \end{cases}$$

and

$$U_{t;i}(s)=\begin{cases}-w(t) & \text{if } i=0,\\ 0.5(-4w(t)-1+\sqrt{1+4\xi_{n-i+k-1}})\frac{s^i}{i!} & \\ \quad+\sum_{k=1}^{i-1}0.5(-1+\sqrt{1+4\xi_{n-i+k-1}})\frac{s^k}{k!} & \text{if } i\in\{1,\dots,n-1\}.\end{cases}$$

Inequalities (3.6)–(3.10) take the following form:

$$\sqrt{1+4\xi_i}\geq 1 \quad \text{for every } i\in\{0,1,\dots,n-2\}, \tag{4.3}$$

$$\sqrt{1+4\xi_{n-1}}\geq 1+4w(M_1), \tag{4.4}$$

$$\sqrt{1+4\xi_{n-1}}\geq 1+2w(M_1)+2M_1, \tag{4.5}$$

$$\sqrt{1+4\xi_{n-1}}\leq 1-4w(M_2)+2M_2, \tag{4.6}$$

$$\sqrt{1+4\xi_{n-1}}\geq 1+4w(M_2)-2M_2. \tag{4.7}$$

Since $\xi_i\geq 0$ for every $i\in\{0,1,\dots,n-1\}$, we conclude that (4.3) is always satisfied. Therefore, if

$$\max\left\{\begin{matrix}1+4w(M_1)\\ 1+2w(M_1)+2M_1\\ 1+4w(M_2)-2M_2\end{matrix}\right\}\leq\sqrt{1+4\xi_{n-1}}\leq 1-4w(M_2)+2M_2 \tag{4.8}$$

then the problem (2.1)–(2.2) has a positive solution such that (3.11) holds.

For convenience, let

$$A=\int_0^a\frac{1}{1+s^2}\,ds,\qquad B=\int_0^a\left(\sum_{k=0}^{n-1}\left(\frac{1}{1+s^{n+1-k}}\sum_{i=k}^{n-1}\frac{s^{i-k}}{(i-k)!}\right)\right)ds.$$

Then $w(t)=\lambda A+\lambda Bt$, therefore equation (4.8) takes the form

$$\max\left\{\begin{matrix}1+4\lambda A+4\lambda BM_1\\ 1+2\lambda A+2(\lambda B+1)M_1\\ 1+4\lambda A-2(1-2\lambda B)M_2\end{matrix}\right\}\leq\sqrt{1+4\xi_{n-1}} \tag{4.9}$$
$$\leq 1-4\lambda A+2(1-2\lambda B)M_2.$$

Let $\lambda\in(0,0.25B^{-1})$. Then $1-2\lambda B>0$, so

$$1+4\lambda A+4\lambda BM_1\geq 1+4\lambda A-2(1-2\lambda B)M_2.$$

Therefore (4.9) holds if

$$\max\left\{\begin{matrix}1+4\lambda A+4\lambda BM_1\\ 1+2\lambda A+2(\lambda B+1)M_1\end{matrix}\right\}\leq\sqrt{1+4\xi_{n-1}} \tag{4.10}$$
$$\leq 1-4\lambda A+2(1-2\lambda B)M_2.$$

We can see that the inequality

$$1+4\lambda A+4\lambda BM_1\leq 1-4\lambda A+2(1-2\lambda B)M_2$$

holds if and only if

$$(1 - 2\lambda B)M_2 \geq (2\lambda B)M_1 + 4\lambda A, \tag{4.11}$$

and the inequality

$$1 + 2\lambda A + 2(\lambda B + 1)M_1 \leq 1 - 4\lambda A + 2(1 - 2\lambda B)M_2$$

holds if and only if

$$(1 - 2\lambda B)M_2 \geq (1 + \lambda B)M_1 + 3\lambda A. \tag{4.12}$$

Let $\omega_1, \omega_2 : [0, \infty) \to \mathbb{R}$ with

$$\begin{aligned} \omega_1(t) &= (1 - 2\lambda B)t, & t \in [0, \infty), \\ \omega_2(t) &= (2\lambda B)t + 4\lambda A, & t \in [0, \infty). \end{aligned}$$

We can verify that

$$\omega_1(t) = \omega_2(t) \quad \text{if and only if} \quad t = \frac{4\lambda A}{1 - 4\lambda B}$$

provided that $1 - 4\lambda B \neq 0$. Suppose that $1 - 4\lambda B > 0$. Then $1 - 2\lambda B > 2\lambda B$, so

$$\begin{aligned} \omega_1(t) \geq \omega_2(t) \quad &\text{for } t \in \left(\frac{4\lambda A}{1 - 4\lambda B}, \infty\right), \\ \omega_1(t) \leq \omega_2(t) \quad &\text{for } t \in \left[0, \frac{4\lambda A}{1 - 4\lambda B}\right]. \end{aligned}$$

Therefore, if

$$1 - 4\lambda B > 0, \tag{4.13}$$

$$M_2 \geq \frac{4\lambda A}{1 - 4\lambda B}, \tag{4.14}$$

$$M_1 \leq M_2, \tag{4.15}$$

then inequality (4.11) is true.

Similarly, let $\omega_3 : [0, \infty) \to \mathbb{R}$ with

$$\omega_3(t) = (1 + \lambda B)t + 3\lambda A, \quad t \in [0, \infty).$$

We can see that

$$\omega_1(t) = \omega_3(t) \quad \text{if and only if} \quad t = -A/B,$$

so $\omega_3(t) \geq \omega_1(t)$ for $t \in [0, \infty)$, therefore we cannot use an approach similar to the one used for ω_1 and ω_2.

If

$$M_2 > \frac{3\lambda A}{1 - 2\lambda B}, \tag{4.16}$$

$$M_1 \leq \frac{(1 - 2\lambda B)M_2 - 3\lambda A}{1 + \lambda B}, \tag{4.17}$$

then inequality (4.12) is true. Observe that in view of (4.16), the numerator of the fraction on the right side of (4.17) is strictly positive. Also, from (4.17) we get $M_1 \leq M_2$.

Based on inequality (4.13), we have

$$\frac{4\lambda A}{1-4\lambda B} > \frac{3\lambda A}{1-2\lambda B},$$

therefore (4.14) implies (4.16). Also, (4.17) implies (4.15). Consequently, we have the following theorem.

THEOREM 4.1. *If*

$$1-4\lambda B > 0, \tag{4.18}$$

$$M_2 \geq \frac{4\lambda A}{1-4\lambda B}, \tag{4.19}$$

$$M_1 \leq \frac{(1-2\lambda B)M_2 - 3\lambda A}{1+\lambda B}, \tag{4.20}$$

then the problem (4.1)–(4.2) *has a positive solution such that* (3.11) *holds.*

COROLLARY 4.2. *If*

$$1-4\lambda B > 0, \tag{4.21}$$

$$M_2 \geq \frac{4\lambda A}{1-4\lambda B}, \tag{4.22}$$

$$M_1 \leq \frac{\lambda A + 4\lambda^2 AB}{(1+\lambda B)(1-4\lambda B)}, \tag{4.23}$$

then the problem (4.1)– (4.2) *has a positive solution such that* (3.11) *holds.*

Proof. From (4.22), we get

$$\frac{(1-2\lambda B)M_2 - 3\lambda A}{1+\lambda B} \geq \frac{\lambda A + 4\lambda^2 AB}{(1+\lambda B)(1-4\lambda B)},$$

and the result follows by applying Theorem 4.1. ∎

Specifically, for $a = 100$ and $n = 4$, we have

$$A = \int_0^{100} \frac{1}{1+s^2}\, ds \simeq 1.56,$$

$$B = \int_0^{100} \left(\sum_{k=0}^{3} \left(\frac{1}{1+s^{5-k}} \sum_{i=k}^{3} \frac{s^{i-k}}{(i-k)!} \right) \right) ds \simeq 8.65.$$

It is a matter of simple calculations to verify that for $\lambda = 0.02$, $M_2 = 0.44$

and $M_1 = 0.12$ assumptions (4.21)–(4.23) are satisfied, so the problem

$$x^{(4)}(t) + \sum_{k=0}^{3} \phi_k(t) x^{(k)}(t) + \psi(t) = 0, \qquad t \in [0,1], \tag{4.24}$$

$$(x^{(i)}(0))^2 + x^{(i)}(0) = \xi_i, \qquad i \in \{0,1,2,3\}, \tag{4.25}$$

where $\xi_i \geq 0$ for every $i \in \{0,1,2,3\}$, $\psi \in C([0,1],\mathbb{R})$ with

$$0 \leq \psi(t) \leq \frac{0.02}{1+t^2} \qquad \text{for } t \in [0,1]$$

and $\phi_i \in C([0,1],\mathbb{R})$, $i \in \{0,1,2,3\}$, with

$$0 \leq \phi_i(t) \leq \frac{0.02}{1+t^{5-i}} \qquad \text{for } t \in [0,1],$$

has a positive solution y such that

$$0.12 \leq \sup_{t\in[0,1]} |y^{(3)}(t)| \leq 0.44. \tag{4.26}$$

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