A note on the plane Jacobian conjecture

by Nguyen Van Chau (Hanoi)

Abstract. It is shown that every polynomial function $P : \mathbb{C}^2 \to \mathbb{C}$ with irreducible fibres of the same genus must be a coordinate. Consequently, there do not exist counterexamples $F = (P, Q)$ to the Jacobian conjecture such that all fibres of P are irreducible curves with the same genus.

1. Concerning the plane Jacobian conjecture (JC2) ([10], [5]), Kaliman [9] observed that in order to prove (JC2), it is sufficient to consider only polynomial maps $F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2$ with non-zero constant Jacobian $J(P, Q) := P_x Q_y - P_y Q_x \equiv c \neq 0$ such that all fibres of P are irreducible curves. In 1979 Razar [16] proved the following:

THEOREM 1. *A non-zero constant Jacobian polynomial map $F = (P, Q)$ is invertible if all fibres of P are irreducible rational curves.*

In other words, there do not exist counterexamples $F = (P, Q)$ to (JC2) such that all fibres of P are irreducible rational curves. In an attempt to understand the nature of the plane Jacobian conjecture, Razar's result was reproved by Heitmann [8], Lê and Weber [12], Friedland [6], Nemethi and Sigray [14] in several different algebraic and algebro-geometric approaches.

Following [15], we shall call a polynomial $h \in \mathbb{C}[x, y]$ a *coordinate* if there is a polynomial diffeomorphism φ of $\mathbb{C}^2$ such that $h \circ \varphi(x, y) = x$. The following fact gives a natural explanation for Theorem 1.

THEOREM 2 ([18], [15]). *A polynomial function $P : \mathbb{C}^2 \to \mathbb{C}$ with irreducible rational fibres is a coordinate.*

This theorem was obtained by Vistoli [18] and Neumann and Norbury [15] and, as mentioned in [15], it is implicit in the earlier work [13, Lemma 1.7]. Razar [16] and Friedland [6] also proved that there do not exist coun-

---

[13]

terexamples $F = (P, Q)$ to (JC2) such that all fibres of P are irreducible and the generic fibres of P are elliptic curves.

As usual, by the genus of an irreducible plane curve we mean the genus of its smooth portion. In this short paper, we show that there do not exist counterexamples $F = (P, Q)$ to (JC2) such that all fibres of P are irreducible curves of the same genus. We will prove

THEOREM 3. *Suppose $P : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial function with irreducible fibres. If all the fibres of P have the same genus, then P must be a coordinate.*

In other words, the genus of the fibres of any polynomial with irreducible fibres, except for coordinate polynomials, must vary. Further investigations on how the genus of the fibres of polynomial functions in $\mathbb{C}^2$ varies should be useful in hunting for the solution of the plane Jacobian conjecture.

Note that Theorems 1–3 do not involve any assumption about the smoothness of the fibres of P. In Theorem 1, the Jacobian condition ensures that all fibres of P are smooth. In Theorem 3, the assumption that all fibres of P are irreducible curves with the same genus is so strong that it guarantees the smoothness of all fibres of P (see Lemma 1 below).

As will be seen in Section 2, in order to prove Theorem 3, we will show that if all the fibres of P are irreducible curves with same genus, then they are rational curves (Lemma 2). Then Theorem 3 will be deduced from Theorem 2. Note that the proofs of Theorem 2 in [13] and [18] use either the Abhyankar–Moh–Suzuki Embedding Theorem or Zariski's main theorem. In Section 3, we will give another elementary proof of Theorem 2 by applying the Newton–Puiseux Theorem and the basic properties of the standard resolution of singularities and of rational fibrations.

2. Let $P \in \mathbb{C}[x, y]$ be a given non-constant polynomial. Regard the plane $\mathbb{C}^2$ as a subset of the projective plane $\mathbb{P}^2$ and consider P as a rational morphism $P : \mathbb{P}^2 \to \mathbb{P}$, which is well defined everywhere on $\mathbb{P}^2$, except a finite number of points on the line at infinity $z = 0$. By blowing-up, we can remove the indeterminacy points of P and obtain a compactification X of $\mathbb{C}^2$ and a regular extension $p : X \to \mathbb{P}$ of P over X. The divisor $\mathcal{D} := X \setminus \mathbb{C}^2$ is a union of smooth rational curves with simple normal crossings. An irreducible component D of $\mathcal{D}$ is *horizontal* if $p_{|D}$ is a non-constant mapping. Note that the number of horizontal curves of P does not depend on the regular extension p. Such a regular extension $p : X \to \mathbb{P}$ is called *standard* if, among the components of $\mathcal{D}$, only the proper transform of the line at infinity $z = 0$ and the horizontal curves of P may have self-intersection -1. A standard extension of P over a compactification of $\mathbb{C}^2$ can be constructed by a blowing-up process in which we only blow up at the indeterminacy points of P and its blowing-up versions.

Now, suppose $p : X \to \mathbb{P}^1$ is a standard extension of P. Denote by C_s the fibre of p over $s \in \mathbb{P}^1$ and by C the generic fibre of p. Let $\mathcal{D}_s := \mathcal{D} \cap C_s$, the portion of C_s contained in $\mathcal{D}$. By definitions, the standard extension $p : X \to \mathbb{P}^1$ has the following useful property:

(∗) *Every irreducible component of $\mathcal{D}_s$, $s \in \mathbb{C}$, has self-intersection less than -1.*

We begin with the following observation.

LEMMA 1. *Assume that all fibres of P are irreducible curves with the same genus g. Then the fibres C_s, $s \in \mathbb{C}$, are smooth irreducible curves with the same genus g.*

Proof. As usual, we denote by K_X the canonical bundle of the surface X, by $\pi(V)$ the virtual genus of an algebraic curve V in X, and by $g(V)$ the genus of the desingularization of V whenever V is irreducible. By the adjunction formula

$$2\pi(V) - 2 = K_X.V + V.V,$$

and, for V irreducible, $\pi(V) = g(V)$ if and only if V is smooth. Furthermore, if V is a fibre of a fibration over X, then $V.V = 0$ and hence

$$2\pi(V) - 2 = K_X.V$$

(see, for example, [7]). For $s \in \mathbb{C}$ denote by F_s the closure in X of the curve $\{(x, y) \in \mathbb{C}^2 : P(x, y) = s\}$. By the assumptions the curves F_s are irreducible and

(1) $$g(F_s) \equiv g, \qquad s \in \mathbb{C}.$$

Now, let $s \in \mathbb{C}$ be given. By the adjunction formula (see [7])

(2) $$2g - 2 = K_X.C_s.$$

If C_s is irreducible, we have $C_s = F_s$ and $F_s.F_s = 0$. Again, using the adjunction formula, we get $2\pi(F_s) - 2 = K_X.F_s$. Therefore, by (1) and (2) we obtain $\pi(F_s) = g = g(F_s)$. Thus, C_s is a smooth irreducible curve of genus g.

So, to complete the proof we only need to show that C_s is irreducible. Indeed, assume it is not. Write

$$C_s = \sum_{i=1}^{k} m_i E_i + n F_s,$$

where E_i are irreducible components of $\mathcal{D}_s$ with multiplicity m_i and n is the multiplicity of F_s in C_s. The equality (2) becomes

(3) $$2g - 2 = \sum_{i=1}^{k} m_i K_X.E_i + n K_X.F_s.$$

Since E_i are smooth irreducible rational curves, $\pi(E_i) = 0$. Furthermore, $E_i.E_i < -1$ by property $(*)$ and $F_s < 0$ by Zariski's lemma (see, for example, [1]). Then, applying the adjunction formula to E_i and F_s we have

$$K_X.E_i = -(E_i.E_i + 2) \geq 0$$

and

$$K_X.F_s = 2\pi(F_s) - 2 - F_s.F_s > 2\pi(F_s) - 2.$$

From the above estimates and (3) it follows that $2g - 2 > 2\pi(F_s) - 2$. This is impossible, since $\pi(F_s) \geq g(F_s) = g$. Hence, C_s must be irreducible. ∎

LEMMA 2. *Let P be as in Lemma* 1. *Then $g = 0$ and P has only one horizontal curve. In particular, the fibres of P are irreducible rational curves.*

Proof. We will use Suzuki's formula [17]

$$\sum_{s \in \mathbb{P}^1} \chi(C_s) - \chi(C) = \chi(X) - 2\chi(C), \tag{4}$$

where $\chi(V)$ is the Euler–Poincaré characteristic of V. Denote by m the number of irreducible components of the divisor $\mathcal{D}$, by m_∞ the number of irreducible components of C_∞ and by h the number of horizontal curves of P. Note that $\chi(X) = 2 + m$ and $\chi(C_\infty) = 1 + m_\infty$. Furthermore, by Lemma 1, $\chi(C_s) = \chi(C) = 2 - 2g$ for all $s \in \mathbb{C}$, and $h = m - m_\infty$. Now, by the above estimates, we have

$$\sum_{s \in \mathbb{P}^1} \chi(C_s) - \chi(C) = \chi(C_\infty) - \chi(C) = 1 + m_\infty - (2 - 2g)$$

and

$$\chi(X) - 2\chi(C) = 2 + m - 2(2 - 2g).$$

Then, the equality (4) becomes

$$1 + m_\infty - (2 - 2g) = 2 + m - 2(2 - 2g),$$

or equivalently,

$$2g = 1 - (m - m_\infty) = 1 - h.$$

Since $g \geq 0$ and $h \geq 1$, it follows that $g = 0$ and $h = 1$. ∎

Proof of Theorem 3. Combine Lemma 2 with Theorem 2. ∎

3. The main arguments in the proofs of Theorem 2 in [18] and [13] lead to the fact that if the fibres of P are rational irreducible curves, then P has only one horizontal curve and its fibres are diffeomorphic to $\mathbb{C}$. Then the fact that P is a coordinate results from the Abhyankar–Moh–Suzuki Embedding Theorem, as in [13, Lemma 1.7], or from Zariski's main theorem, as in [18, Lemma 4.8]. However, Theorem 2 can also be proved by using the basic observations below.

(i) Let $H \in \mathbb{C}[x, y]$, $H(x, y) = \sum_{ij} c_{ij} x^i y^j$. Recall that the so-called *Newton polygon* N_H of H is the convex hull of the set $\{(0,0)\} \cup \{(i,j) : c_{ij} \neq 0\}$.

FACT 1. *Assume that* $\deg_x H > 0$ *and* $\deg_y H > 0$. *If* H *has only one horizontal curve, then*

(a) N_H *is a triangle with vertices* $(0,0)$, $(\deg_x H, 0)$ *and* $(0, \deg_y H)$,
(b) *the sum* H_E *of the monomials* $c_{ij} x^i y^j$ *in* H *with* (i, j) *lying on the edge joining* $(\deg_x H, 0)$ *and* $(0, \deg_y H)$ *is of the form*

$$H_E(x, y) = C(y^q - ax^p)^m,$$

where $C \neq 0$, $a \neq 0$, p, q, m *are natural numbers and* $\gcd(p, q) = 1$.

In fact, if H has only one horizontal curve, the branches at infinity of each generic fibre $H = c$ can be given by Newton–Puiseux expansions of the same form,

$$y = bx^{p/q} + \text{lower terms in } x, \quad p/q \leq 1,$$

or

$$x = by^{q/p} + \text{lower terms in } y, \quad q/p \leq 1,$$

where $b \neq 0$ and $\gcd(p, q) = 1$. Then the Newton polygon N_H and the face polynomial H_E can be detected by using the Newton–Puiseux Theorem and the basic properties of Newton–Puiseux expansions (see [4]).

(ii) Let $P \in \mathbb{C}[x, y]$ with $\deg P > 1$, $\deg_x P > 0$ and $\deg_y P > 0$. Assume that the fibres of P are irreducible rational curves, and hence P has only one horizontal curve. In view of Fact 1, we can assume that $P_E(x, y) = C(y^q - ax^p)^m$ with $C \neq 0$, $a \neq 0$ and $\gcd(p, q) = 1$.

FACT 2. $P_E(x, y)$ *has the form* $C(y^q - ax)^m$ *or* $C(y - ax^p)^m$, $p, q \geq 1$.

To see this, we only need to consider the case $p/q \neq 1$. Assume, for example, that $p/q < 1$. Let $p : X \to \mathbb{P}^1$ be a standard extension, which results from a blowing-up process $\pi : X \to \mathbb{P}^2$ that only blows up at the indeterminacy points of P and its blown-up versions. By Lemma 1 the divisor at infinity $\mathcal{D}$ is the union of the fibre C_∞ and the unique horizontal curve D, $\mathcal{D} = C_\infty \cup D$. The fibre C_∞ contains the proper transform D_0 of the line at infinity $z = 0$ of $\mathbb{C}^2$. Since $\deg P > 1$ and the morphism $p : X \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration, the fibre C_∞ is reducible and can be contracted to one of its components by blowing down components of self-intersection -1. Furthermore, the proper transform D_0 of the line at infinity $z = 0$ is the unique component of C_∞ having self-intersection -1. Then, one can see that the first Puiseux chain of the dual graph of $\mathcal{D}$ must be of the forms

$$\overset{D_0}{\underset{-1}{\bullet}} - \overset{e_1}{\underset{-2}{\circ}} - \cdots - \overset{e_k}{\underset{-2}{\circ}} - \overset{\vdots}{\underset{-2}{\bigcirc}} - \overset{f_1}{\underset{}{\circ}} - \cdots - \overset{f_l}{\underset{}{\circ}}$$

and

$$\overset{D_0}{\underset{-1}{\bullet}} - \overset{\vdots}{\underset{-2}{\bigcirc}} - \overset{f_1}{\underset{}{\circ}} - \cdots - \overset{f_l}{\underset{}{\circ}},$$

where the weights are the self-intersection numbers. This Puiseux chain coincides with the resolution graph of the germ curve at infinity γ, composed of the line at infinity $z = 0$ and a branch curve at infinity given by a Newton–Puiseux expansion of the form

$$y = bx^{p/q} + \text{lower terms in } x, \quad b \neq 0.$$

Note that the line at infinity $z = 0$ of $\mathbb{C}^2$ has self-intersection 1. Then, examining in detail the resolving singularities of γ, we can easily see that the conditions for self-intersection numbers are satisfied only when $p = 1$, $q > 1$ and $l = 1$.

Thus, once the fibres of P are irreducible rational curves, Fact 2 enables us to easily construct a polynomial automorphism Φ of $\mathbb{C}^2$ such that $P \circ \Phi(x, y) = x$.

4. Let us conclude with the following two remarks.

(i) It is not difficult to see that a non-zero constant Jacobian polynomial map $F = (P, Q)$ is invertible if P has only one horizontal curve. This is a key point in the geometric proof of Theorem 1 presented in [12]. In fact, let $p : X \to \mathbb{P}^1$ be a regular extension of P. We can consider Q as a rational morphism from X to $\mathbb{P}^1$. If P has only one horizontal curve, then by the Jacobian condition the restriction of Q to the unique horizontal curve of P must be a constant mapping with value ∞. So, the restriction of Q to any generic fibre $P = c$ is proper. Therefore, by the simple connectedness of $\mathbb{C}$ such a generic fibre $P = c$ is isomorphic to $\mathbb{C}$. Hence, (P, Q) is invertible. It seems very difficult to estimate the possible number of horizontal curves of polynomial components of non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$.

(ii) Boileau and Fourrier [3] presented a topological version of the Abhyankar–Moh–Suzuki Theorem: *if $P \in \mathbb{C}[x, y]$ is irreducible and the fibre $P = 0$ is diffeomorphic to $\mathbb{C}$, then the function $P : \mathbb{C}^2 \to \mathbb{C}$ gives a trivial fibration over $\mathbb{C}^2$ with fibre $\mathbb{C}$.* In particular, the fibres of such polynomials P are irreducible rational curves. Thus, a combination of this version with Facts 1 and 2 of the previous section enables us to give a geometric proof of the Abhyankar–Moh–Suzuki Theorem.

cussions and help. The author wishes to thank the referee for his helpful comments and suggestions.

References

[1] W. Barth, C. Peters, and A. van de Ven, *Compact Complex Surfaces*, Ergeb. Math. Grenzgeb. (3) 4, Springer, Berlin, 1984.

[2] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 287–330.

[3] M. Boileau and L. Fourrier, *Knot theory and plane algebraic curves*, in: Knot Theory and Its Applications, Chaos Solitons Fractals 9 (1998), 779–792.

[4] E. Brieskorn und H. Knörrer, *Ebene algebraische Kurven*, Birkhäuser, Basel, 1981.

[5] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Progr. Math. 190, Birkhäuser, Basel, 2000.

[6] Sh. Friedland, *Monodromy, differential equations and the Jacobian conjecture*, Ann. Polon. Math. 72 (1999), 219–249.

[7] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

[8] R. Heitmann, *On the Jacobian conjecture*, J. Pure Appl. Algebra 64 (1990), 36–72; Corrigendum, ibid. 90 (1993), 199–200.

[9] Sh. Kaliman, *On the Jacobian conjecture*, Proc. Amer. Math. Soc. 117 (1993), 45–51.

[10] O. H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. 47 (1939), 299–306.

[11] Lê Dũng Tráng, *Simple rational polynomials and the Jacobian conjecture*, Publ. RIMS Kyoto Univ. 44 (2008), 641–659.

[12] Lê Dũng Tráng and C. Weber, *Polynômes à fibres rationnelles et conjecture jacobienne à 2 variables*, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), 581–584.

[13] M. Miyanishi and T. Sugie, *Generically rational polynomials*, Osaka J. Math. 17 (1980), 339–362.

[14] A. Némethi and I. Sigray, *On the monodromy representation of polynomial maps in n variables*, Studia Sci. Math. Hungar. 39 (2002), 361–367.

[15] W. D. Neumann and P. Norbury, *Nontrivial rational polynomials in two variables have reducible fibres*, Bull. Austral. Math. Soc. 58 (1998), 501–503.

[16] M. Razar, *Polynomial maps with constant Jacobian*, Israel J. Math. 32 (1979), 97–106.

[17] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace* $\mathbb{C}^2$, J. Math. Soc. Japan 26 (1974), 241–257.

[18] A. Vistoli, *The number of reducible hypersurfaces in a pencil*, Invent. Math. l12 (1993), 247–262.

Nguyen Van Chau
Institute of Mathematics
Vietnam Academy of Science and Technology
18 Hoang Quoc Viet
10307 Hanoi,Vietnam
E-mail: nvchau@math.ac.vn