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Non-trivial solutions for a class of $(p_1,\dots,p_n)$-biharmonic systems with Navier boundary conditions

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Abstract. Using a recent critical point theorem due to Bonanno, the existence of a non-trivial solution for a class of systems of n fourth-order partial differential equations with Navier boundary conditions is established.

1. Introduction. This paper deals with the existence of at least one non-trivial solution for the following nonlinear elliptic system of n fourth-order partial differential equations under Navier boundary conditions

$$(1.1)\qquad \begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i)=\lambda F_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega,\\ u_i=\Delta u_i=0 & \text{on } \partial\Omega \end{cases}$$

for $1\le i\le n$, where $n\ge 1$ is an integer, $p_i>\max\{1,N/2\}$ for $1\le i\le n$, $\Omega\subset\mathbb{R}^N$ $(N\ge 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $\lambda>0$, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that $F(\cdot,t_1,\dots,t_n)$ is measurable in Ω for all $(t_1,\dots,t_n)\in\mathbb{R}^n$, $F(x,\cdot,\dots,\cdot)$ is C^1 in $\mathbb{R}^n$ for every $x\in\Omega$ and for every $\varrho>0$,

$$\sup_{|(t_1,\dots,t_n)|\le\varrho}\sum_{i=1}^n |F_{t_i}(x,t_1,\dots,t_n)|\in L^1(\Omega),$$

and F_{u_i} denotes the partial derivative of F with respect to u_i for $1\le i\le n$. The system (1.1) is called $(p_1,\dots,p_n)$-*biharmonic.*

Fourth-order nonlinear equations furnish a model to study traveling waves in suspension bridges, so they are important to physics. Due to this, many researchers have discussed the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree

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theory, and variational methods; we refer the reader to [1, 2, 4–20, 24] and references therein.

As an example, we point out the following special case of our main results.

THEOREM 1.1. *Let $p > \max\{1, N/2\}$. Let $h : \Omega \to \mathbb{R}$ be a positive and essentially bounded function and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative function such that $\lim_{t\to 0^+} g(t)/t^{p-1} = +\infty$. Then for each λ in*

$$\left]0, \frac{1}{pk^p \int_\Omega h(x)\,dx} \sup_{\nu>0} \frac{\nu}{\int_0^{\sqrt[p]{\nu}} g(\xi)\,d\xi}\right[$$

where

$$k := \sup_{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{(\int_\Omega |\Delta u_i(x)|^p\,dx)^{1/p}},$$

the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda h(x) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial weak solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

In the present paper, our motivation comes from the recent paper [7].

2. Main results. First, for the reader's convenience we recall Theorem 2.5 of [21] as given in [3, Theorem 5.1] (see also [3, Proposition 2.1] for related results), which is our main tool to transfer the existence of a weak solution of (1.1) into the existence of a critical point of the Euler functional:

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define

$$\beta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

THEOREM 2.1 ([3, Theorem 5.1]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then for each $\lambda \in]1/\rho(r_1,r_2), 1/\beta(r_1,r_2)[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ *such that* $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r_1,r_2[)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

From now on, X will denote the Cartesian product of n Sobolev spaces $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$ for $i = 1,\dots,n$, i.e., $X = (W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)) \times \dots \times (W^{2,p_n}(\Omega) \cap W_0^{1,p_n}(\Omega))$ endowed with the norm

$$\|(u_1,\dots,u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i}$$

where

$$\|u_i\|_{p_i} = \Big(\int_\Omega |\Delta u_i(x)|^{p_i}\,dx\Big)^{1/p_i}$$

for $1 \le i \le n$.

We say that $u = (u_1,\dots,u_n)$ is a *weak solution* to the system (1.1) if $u = (u_1,\dots,u_n) \in X$ and

$$\int_\Omega \sum_{i=1}^n |\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x)\,dx - \lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx = 0$$

for every $(v_1,\dots,v_n) \in X$.

For all $\gamma > 0$ we define

$$K(\gamma) = \Big\{t = (t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \gamma\Big\}. \tag{2.1}$$

Put

$$k := \max\Big\{\sup_{u_i \in W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} : 1 \le i \le n\Big\}. \tag{2.2}$$

If $p_i > \max\{1, N/2\}$ for $1 \le i \le n$, since the embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $1 \le i \le n$ is compact, one has $k < \infty$.

Fix $x^0 \in \Omega$ and pick s_1, s_2 with $0 < s_1 < s_2$ such that

$$B(x^0,s_1) \subset B(x^0,s_2) \subseteq \Omega$$

where $B(x^0,s_i)$ denotes the (open) ball with center at x^0 and radius of s_i for $i = 1,2$.

Put

$$\sigma_i = \sigma_i(N,p_i,s_1,s_2) := \frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\Big(\frac{k\pi^{N/2}(s_2^N - s_1^N)}{\Gamma(1+N/2)}\Big)^{1/p_i} \tag{2.3}$$

for $1 \le i \le n$ and

$$(2.4)\qquad \theta_i = \theta_i(N, p_i, s_1, s_2) :=$$
$$\begin{cases} \dfrac{3N}{(s_2-s_1)(s_1+s_2)}\left(\dfrac{k\pi^{N/2}((s_1+s_2)^N-(2s_1)^N)}{2^N\Gamma(1+N/2)}\right)^{1/p_i} & \text{if } N < \dfrac{4s_1}{s_2-s_1}, \\ \dfrac{12s_1}{(s_2-s_1)^2(s_1+s_2)}\left(\dfrac{k\pi^{N/2}((s_1+s_2)^N-(2s_1)^N)}{2^N\Gamma(1+N/2)}\right)^{1/p_i} & \text{if } N \ge \dfrac{4s_1}{s_2-s_1}, \end{cases}$$

for $1 \le i \le n$, where Γ denotes the Gamma function. For given two positive constants ν and τ with $\nu \ne \sum_{i=1}^n \prod_{j=1,\, j\ne i}^n p_j(\tau\sigma_i)^{p_i}$, put

$$a_\tau(\nu) := \frac{\int_\Omega \sup_{t\in K(\nu/\prod_{i=1}^n p_i)} F(x,t)\,dx - \int_{B(x^0,s_1)} F(x,\tau,\dots,\tau)\,dx}{\nu - \sum_{i=1}^n \prod_{j=1,\, j\ne i}^n p_j(\tau\sigma_i)^{p_i}}.$$

THEOREM 2.2. *Assume that there exist a non-negative constant* ν_1 *and two positive constants* ν_2 *and* τ *with*

$$\nu_1 < \sum_{i=1}^n \prod_{j=1,\, j\ne i}^n p_j(\tau\theta_i)^{p_i} \quad \text{and} \quad \sum_{i=1}^n \prod_{j=1,\, j\ne i}^n p_j(\tau\sigma_i)^{p_i} < \nu_2$$

such that

(A1) $F(x,t) \ge 0$ *for each* $(x,t) \in (\overline{\Omega} \setminus B(x^0,s_1)) \times [0,\tau]^n$;
(A2) $a_\tau(\nu_2) < a_\tau(\nu_1)$.

Then for each λ *in*

$$\left]\frac{1}{k\prod_{i=1}^n p_i}\,\frac{1}{a_\tau(\nu_1)}, \frac{1}{k\prod_{i=1}^n p_i}\,\frac{1}{a_\tau(\nu_2)}\right[$$

the system (1.1) *admits a non-trivial weak solution* $u_0 = (u_{01},\dots,u_{0n}) \in X$ *such that*

$$\nu_1/k < \sum_{i=1}^n \prod_{j=1,\, j\ne i}^n p_j\|u_{0i}\|_{p_i}^{p_i} < \nu_2/k.$$

Proof. To apply Theorem 2.1, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for $u = (u_1,\dots,u_n) \in X$ as follows:

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \qquad \Psi(u) = \int_\Omega F(x, u_1(x),\dots,u_n(x))\,dx.$$

It is well known that Φ and Ψ are well defined and continuously differentiable with derivatives at $u=(u_1,\dots,u_n)\in X$ given by

$$\Phi'(u)(v)=\int_\Omega\sum_{i=1}^n|\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x)\,dx,$$
$$\Psi'(u)(v)=\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx,$$

for every $v=(v_1,\dots,v_n)\in X$; moreover Ψ is sequentially weakly upper semicontinuous. Lemma 2.1 of [11] shows that Φ' admits a continuous inverse on X^*, and since Φ' is monotone, we infer that Φ is sequentially weakly lower semicontinuous (see [23, Proposition 25.20(d)]). Furthermore, $\Psi':X\to X^*$ is a compact operator (for details, see [14]).

Set $w(x)=(w_1(x),\dots,w_n(x))$ where for $1\le i\le n$,

$$w_i(x)=\begin{cases}0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s_2),\\ \dfrac{\tau(3(l^4-s_2^4)-4(s_1+s_2)(l^3-s_2^3)+6s_1s_2(l^2-s_2^2))}{(s_2-s_1)^3(s_1+s_2)} & \\ & \text{if } x\in B(x^0,s_2)\setminus B(x^0,s_1),\\ \tau & \text{if } x\in B(x^0,s_1),\end{cases}$$

where $l=\operatorname{dist}(x,x^0)=\sqrt{\sum_{i=1}^N(x_i-x_i^0)^2}$, and define

$$r_1=\frac{\nu_1}{k\prod_{i=1}^n p_i}\quad\text{and}\quad r_2=\frac{\nu_2}{k\prod_{i=1}^n p_i}.$$

We have

$$\frac{\partial w_i(x)}{\partial x_i}=\begin{cases}0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s_2)\cup B(x^0,s_1),\\ \dfrac{12\tau(l^2(x_i-x_i^0)-(s_1+s_2)l(x_i-x_i^0)+s_1s_2(x_i-x_i^0))}{(s_2-s_1)^3(s_1+s_2)} & \\ & \text{if } x\in B(x^0,s_2)\setminus B(x^0,s_1),\end{cases}$$

$$\frac{\partial^2 w_i(x)}{\partial x_i^2}=\begin{cases}0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s_2)\cup B(x^0,s_1),\\ \dfrac{12\tau(s_1s_2+(2l-s_1-s_2)(x_i-x_i^0)^2/l-(s_2+s_1-l)l)}{(s_2-s_1)^3(s_1+s_2)} & \\ & \text{if } x\in B(x^0,r_2)\setminus B(x^0,r_1),\end{cases}$$

and

$$\sum_{i=1}^N\frac{\partial^2 w_i(x)}{\partial x_i^2}=\begin{cases}0 & \text{if } x\in\overline{\Omega}\setminus B(x^0,s_2)\cup B(x^0,s_1),\\ \dfrac{12\tau((N+2)l^2-(N+1)(s_1+s_2)l+Ns_1s_2)}{(s_2-s_1)^3(s_1+s_2)} & \\ & \text{if } x\in B(x^0,s_2)\setminus B(x^0,s_1).\end{cases}$$

It is easy to see that $w = (w_1, \dots, w_n) \in X$ and, in particular,

$$\|w_i\|_{p_i}^{p_i} = \frac{(12\tau)^{p_i} 2\pi^{N/2}}{(s_2 - s_1)^{3p_i}(s_1 + s_2)^{p_i}\Gamma(N/2)} \times \int_{s_1}^{s_2} |(N+2)\xi^2 - (N+1)(s_1+s_2)\xi + Ns_1s_2|^{p_i}\xi^{N-1}\,d\xi \tag{2.5}$$

for $1 \le i \le n$. Hence, from (2.3)–(2.5) we get

$$(\tau\theta_i)^{p_i}/k < \|w_i\|_{p_i}^{p_i} < (\tau\sigma_i)^{p_i}/k \tag{2.6}$$

for $1 \le i \le n$. However, bearing in mind the assumptions on ν_1, ν_2 and τ, we see that

$$r_1 < \Phi(w) < r_2.$$

From (2.2) for each $(u_1, \dots, u_n) \in X$,

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \le k\|u_i\|_{p_i}^{p_i}$$

for $i = 1, \dots, n$, so

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le k \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{2.7}$$

for each $u = (u_1, \dots, u_n) \in X$, and so using (2.7), we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_2[) &= \left\{(u_1, \dots, u_n) \in X : \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_2\right\} \\ &\subseteq \left\{(u_1, \dots, u_n) \in X : \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} < kr_2 \text{ for all } x \in \Omega\right\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int_\Omega F(x, u(x))\,dx \\ &\le \int_\Omega \sup_{t\in K(kr_2)} F(x,t)\,dx.\end{aligned}$$

Since for $1 \le i \le n$, $0 \le w_i(x) \le \tau$ for each $x \in \Omega$, condition (A1) ensures that

$$\int_{\overline{\Omega}\setminus B(x^0,s_2)} F(x, w(x))\,dx + \int_{B(x^0,s_2)\setminus B(x^0,s_1)} F(x, w(x))\,dx \ge 0.$$

Therefore, taking (2.6) into account, one has

$$\begin{aligned}\beta(r_1, r_2) &\le \frac{\sup_{u\in\Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_\Omega \sup_{t\in K(kr_2)} F(x,t)\,dx - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \Big(k\prod_{i=1}^n p_i\Big) \frac{\int_\Omega \sup_{t\in K(kr_2)} F(x,t)\,dx - \int_{B(x^0,s_1)} F(x,\tau,\dots,\tau)\,dx}{\nu_2 - \sum_{i=1}^n \prod_{j=1,\,j\ne i}^n p_j(\tau\sigma_i)^{p_i}} \\ &= \Big(k\prod_{i=1}^n p_i\Big) a_\tau(\nu_2).\end{aligned}$$

On the other hand, by a similar reasoning,

$$\begin{aligned}\rho(r_1, r_2) &\ge \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\Psi(w) - \int_\Omega \sup_{t\in K(kr_2)} F(x,t)\,dx}{\Phi(w) - r_1} \\ &\ge \Big(k\prod_{i=1}^n p_i\Big) \frac{\int_{B(x^0,s_1)} F(x,\tau,\dots,\tau)\,dx - \int_\Omega \sup_{t\in K(kr_1)} F(x,t)\,dx}{\sum_{i=1}^n \prod_{j=1,\,j\ne i}^n p_j(\tau\sigma_i)^{p_i} - \nu_1} \\ &= \Big(k\prod_{i=1}^n p_i\Big) a_\tau(\nu_1).\end{aligned}$$

Hence, from assumption (A2), one has $\beta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, applying Theorem 2.1, taking into account that the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$, we obtain the conclusion. ∎

Now we point out the following consequence of Theorem 2.2.

Theorem 2.3. *Assume that assumption* (A1) *in Theorem* 2.2 *holds. Suppose that there exist two positive constants ν and τ with*

$$\sum_{i=1}^n \prod_{j=1,\,j\ne i}^n p_j(\tau\sigma_i)^{p_i} < \nu$$

such that

(A3) $\dfrac{\int_\Omega \sup_{t\in K(\nu/\prod_{i=1}^n p_i)} F(x,t)\,dx}{\nu} < \dfrac{\int_{B(x^0,s_1)} F(x,\tau,\dots,\tau)\,dx}{\sum_{i=1}^n \prod_{j=1,\,j\ne i}^n p_j(\tau\sigma_i)^{p_i}}$;

(A4) $F(x, 0, \dots, 0) = 0$ *for every* $x \in \Omega$.

Then for $r := \nu/k\prod_{i=1}^n p_i$ *and each* λ *in*

$$\left]\frac{1}{k\prod_{i=1}^n p_i}\frac{\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}}{\int_{B(x^0,s_1)}F(x,\tau,\dots,\tau)\,dx},\ \frac{1}{k\prod_{i=1}^n p_i}\frac{\nu}{\int_\Omega\sup_{t\in K(kr)}F(x,t)\,dx}\right[$$

the system (1.1) *admits a non-trivial weak solution* $u_0=(u_{01},\dots,u_{0n})\in X$ *such that* $\sup_{x\in\Omega}\sum_{i=1}^n|u_i(x)|^{p_i}/p_i<kr$.

Proof. Applying Theorem 2.2 we have the conclusion, by picking $\nu_1=0$ and $\nu_2=\nu$. Indeed, owing to our assumptions, one has

$$\begin{aligned}a_\tau(\nu)&<\frac{\left(1-\frac{\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}}{\nu}\right)\int_\Omega\sup_{t\in K(kr)}F(x,t)\,dx}{\nu-\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}}\\&=\frac{\int_\Omega\sup_{t\in K(kr)}F(x,t)\,dx}{\nu}\\&<\frac{\int_{B(x^0,s_1)}F(x,\tau,\dots,\tau)\,dx}{\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}}=a_\tau(0).\end{aligned}$$

In particular,

$$a_\tau(\nu)<\frac{\int_\Omega\sup_{t\in K(kr)}F(x,t)\,dx}{\nu}.$$

Hence, Theorem 2.2, taking (2.3) into account, yields the result. ■

We now point out the following special case of the previous results when F does not depend on $x\in\Omega$. To be precise, let $F:\mathbb{R}^n\to\mathbb{R}$ be a C^1 function in $\mathbb{R}^n$ such that $F(0,\dots,0)=0$.

Consider the following nonlinear elliptic system of n fourth-order partial differential equations under Navier boundary conditions:

$$\begin{cases}\Delta(|\Delta u_i|^{p_i-2}\Delta u_i)=\lambda F_{u_i}(u_1,\dots,u_n) & \text{in } \Omega,\\ u_i=\Delta u_i=0 & \text{on } \partial\Omega\end{cases}\tag{2.8}$$

for $1\le i\le n$. Given two positive constants ν and τ with

$$\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}\neq\nu,$$

put

$$b_\tau(\nu):=\frac{m(\Omega)\max_{t\in K(\nu/\prod_{i=1}^n p_i)}F(t)-s_1^N\pi^{N/2}/\Gamma(1+N/2)F(\tau,\dots,\tau)}{\nu-\sum_{i=1}^n\prod_{j=1,\,j\neq i}^n p_j(\tau\sigma_i)^{p_i}}.$$

We have the following existence results.

COROLLARY 2.4. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with with $\nu_1 < \sum_{i=1}^n \prod_{j=1,\, j\neq i}^n p_j(\tau\theta_i)^{p_i}$ and $\sum_{i=1}^n \prod_{j=1,\, j\neq i}^n p_j(\tau\sigma_i)^{p_i} < \nu_2$ such that*

(B1) *$F(t) \geq 0$ for each $t \in [0,\tau]^n$;*
(B2) *$b_\tau(\nu_2) < b_\tau(\nu_1)$.*

Then for each λ in

$$\left]\frac{1}{k\prod_{i=1}^n p_i}\,\frac{1}{b_\tau(\nu_1)},\ \frac{1}{k\prod_{i=1}^n p_i}\,\frac{1}{b_\tau(\nu_2)}\right[$$

the system (2.4) admits a non-trivial weak solution $u_0 = (u_{01}, \dots, u_{0n}) \in X$ such that $\nu_1/k < \sum_{i=1}^n \prod_{j=1,\, j\neq i}^n p_j \|u_{0i}\|_{p_i}^{p_i} < \nu_2/k$.

Proof. Set $F(x,t) = F(t)$ for all $x \in \Omega$ and $t_i \in \mathbb{R}$ for $1 \leq i \leq n$. Since $m(B(x^0,s_1)) = s_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}$, Theorem 2.2 yields the conclusion. ∎

We point out the following consequence of Theorem 2.2 when $n = 1$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function. Let F be defined by $F(x,t) = \int_0^t f(x,s)\,ds$ for each $(x,t) \in \Omega \times \mathbb{R}$. Put

$$\sigma = \sigma(N,p,s_1,s_2) := \frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\left(\frac{k\pi^{N/2}(s_2^N - s_1^N)}{\Gamma(1+N/2)}\right)^{1/p} \tag{2.9}$$

and

$$\theta = \theta(N,p,s_1,s_2) := \tag{2.10}$$
$$\begin{cases} \dfrac{3N}{(s_2-s_1)(s_1+s_2)}\left(\dfrac{k\pi^{N/2}((s_1+s_2)^N - (2s_1)^N)}{2^N\Gamma(1+N/2)}\right)^{1/p} & \text{if } N < \dfrac{4s_1}{s_2-s_1}, \\ \dfrac{12s_1}{(s_2-s_1)^2(s_1+s_2)}\left(\dfrac{k\pi^{N/2}((s_1+s_2)^N - (2s_1)^N)}{2^N\Gamma(1+N/2)}\right)^{1/p} & \text{if } N \geq \dfrac{4s_1}{s_2-s_1}, \end{cases}$$

where

$$k = \sup_{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|}{(\int_\Omega |\Delta u_i(x)|^p\,dx)^{1/p}}.$$

Given two positive constants ν and τ with $\nu \neq (\sigma\tau)^p$, put

$$c_\tau(\nu) := \frac{\int_\Omega \sup_{|t|\leq\nu} F(x,t)\,dx - \int_{B(x^0,s_1)} F(x,\tau)\,dx}{\nu - (\tau\sigma)^p}.$$

THEOREM 2.5. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\nu_1 < (\tau\theta)^p$ and $(\tau\sigma)^p < \nu_2$ such that*

(C1) *$F(x,t) \geq 0$ for each $(x,t) \in (\overline{\Omega} \setminus B(x^0,s_1)) \times [0,\tau]$;*
(C2) *$c_\tau(\nu_2) < c_\tau(\nu_1)$.*

Then for each $\lambda \in \left]\frac{1}{pk^p}\frac{1}{c_\tau(\nu_1)}, \frac{1}{pk^p}\frac{1}{c_\tau(\nu_2)}\right[$ *the problem*

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.11}$$

has a non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that*

$$\nu_1/k < \int_\Omega |\Delta u_0(x)|^p\,dx < \nu_2/k.$$

The last result gives the existence of a non-trivial weak solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the problem (2.11) in the autonomous case.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and put $F(t) = \int_0^t f(\xi)\,d\xi$ for all $t \in \mathbb{R}$. The following result is a direct consequence of Theorem 2.5.

Theorem 2.6. *Assume that there exist a non-negative constant* ν_1 *and two positive constants* ν_2 *and* τ *with* $\nu_1 < (\tau\theta)^p$ *and* $(\tau\sigma)^p < \nu_2$ *such that*

(D1) $f(t) \geq 0$ *for each* $t \in [-\nu_2, \max\{\nu_2, \tau\}]$;

(D2) $$\frac{m(\Omega)F(\nu_2) - s_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}F(\tau)}{\nu_2 - (\tau\sigma)^p} < \frac{m(\Omega)F(\nu_1) - s_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}F(\tau)}{\nu_1 - (\tau\sigma)^p}.$$

Then for each λ *in*

$$\left]\frac{1}{pk^p}\frac{\nu_1 - (\tau\sigma)^p}{m(\Omega)F(\nu_1) - s_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}F(\tau)}, \frac{1}{pk^p}\frac{\nu_2 - (\tau\sigma)^p}{m(\Omega)F(\nu_2) - s_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}F(\tau)}\right[$$

the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that*

$$\nu_1/k < \int_\Omega |\Delta u_0(x)|^p\,dx < \nu_2/k.$$

Finally, we prove the theorem in the introduction.

Proof of Theorem 1.1. For fixed λ as in the conclusion, there exists a positive constant ν such that

$$\lambda < \frac{1}{pk^p\int_\Omega h(x)\,dx}\frac{\nu}{\int_0^{\sqrt[p]{\nu}} g(\xi)\,d\xi}.$$

Moreover, $\lim_{t\to 0^+} g(t)/t^{p-1} = +\infty$ implies $\lim_{t\to 0^+}\int_0^t g(\xi)\,d\xi/t^p = +\infty$. Therefore, one can choose a positive constant τ satisfying $\tau < \sqrt[p]{\nu}/\sigma$ and such that

$$\frac{\sigma^p}{\lambda pk^p}\frac{1}{\int_{B(x^0,s_1)} h(x)\,dx} < \frac{\int_0^\tau g(\xi)\,d\xi}{\tau^p}.$$

Hence, the conclusion follows from Theorem 2.5 with $\nu_1 = 0$, $\nu_2 = \nu$ and $f(x,t) = h(x)g(t)$ for every $(x,t) \in \Omega \times \mathbb{R}$. ∎

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