ANNALES
POLONICI MATHEMATICI
105.1 (2012)

Composition operators from weighted Bergman–Privalov spaces to Zygmund type spaces on the unit disk

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Abstract. We characterize the boundedness and compactness of composition operators from weighted Bergman–Privalov spaces to Zygmund type spaces on the unit disk.

1. Introduction. Let $H(\mathbb{D})$ be the space of all holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, $S(\mathbb{D})$ the class of all holomorphic self-maps of $\mathbb{D}$, and $\mathcal{Z}_\mu = \mathcal{Z}_\mu(\mathbb{D})$ the Zygmund type class, consisting of all $f \in H(\mathbb{D})$ such that

$$b_\mu(f) := \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty,$$

where μ is a nonnegative continuous function on $\mathbb{D}$ (*weight*). With the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + b_\mu(f),$$

the Zygmund type class becomes a Banach space, called the *Zygmund type space.*

The *little Zygmund type space*, denoted by $\mathcal{Z}_{\mu,0} = \mathcal{Z}_{\mu,0}(\mathbb{D})$, is the closed subspace of $\mathcal{Z}_\mu$ consisting of all functions f such that

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0.$$

For $\mu(z) = 1 - |z|^2$, we get the *Zygmund space* and the *little Zygmund space.*

Let $\alpha \in (-1,\infty)$ and $p \ge 1$. The *weighted Bergman–Privalov space* $\mathrm{AN}_{p,\alpha} = \mathrm{AN}_{p,\alpha}(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathrm{AN}_{p,\alpha}} = \int_{\mathbb{D}} \ln^p(1 + |f(z)|)\, dm_\alpha(z) < \infty,$$

2010 *Mathematics Subject Classification*: Primary 47B33; Secondary 46E10.
Key words and phrases: composition operator, boundedness, compactness, unit disk, weighted Bergman–Privalov space, Zygmund type space.

DOI: 10.4064/ap105-1-7

where

$$dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z), \qquad dm(z) = \frac{dxdy}{\pi}, \qquad z = x+iy.$$

It is easy to see that the function $\|\cdot\|_{\mathrm{AN}_{p,\alpha}}$ is not a norm on $\mathrm{AN}_{p,\alpha}$, however $d_{\mathrm{AN}_{p,\alpha}}(f,g) = \|f-g\|_{\mathrm{AN}_{p,\alpha}}$ defines a translation invariant metric on $\mathrm{AN}_{p,\alpha}$ and with this metric $\mathrm{AN}_{p,\alpha}$ is an F-space.

The following point evaluation estimate is well-known:

$$|f(z)| \le \exp\left\{\left(\frac{1+|z|}{1-|z|}\right)^{(\alpha+2)/p} \|f\|_{\mathrm{AN}_{p,\alpha}}\right\} - 1, \qquad z \in \mathbb{D}, \tag{1.1}$$

for every $f \in \mathrm{AN}_{p,\alpha}$, and each $p \ge 1$ and $\alpha \in (-1,\infty)$ (see, for example, [7]).

The composition operator C_φ induced by φ is defined by $C_\varphi f = f\circ\varphi$ for $f \in H(\mathbb{D})$. The study of composition operators lies at the interface of the theory of analytic functions and operator theory. Recently, there has been some interest in studying composition and related operators from a particular domain space of holomorphic functions into Zygmund type spaces (see, for example, [3, 4, 5, 12, 20, 21]). In this paper, we study the boundedness and compactness of the operator $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) continuing the line of research in [14, 16, 17, 22].

2. Boundedness and compactness of $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$). Recall that a linear map $T : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is *bounded* if $T(E) \subseteq \mathcal{Z}_\mu$ is bounded for every bounded subset E of $\mathrm{AN}_{p,\alpha}$. The map is *compact* if $T(E) \subseteq \mathcal{Z}_\mu$ is relatively compact for every bounded set $E \subset \mathrm{AN}_{p,\alpha}$.

The following criterion for compactness follows from standard arguments similar to those in [9] (see also [6]).

LEMMA 2.1. *Let $\alpha > -1$, $p \ge 1$, μ be a weight and $\varphi \in S(\mathbb{D})$. Then $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{AN}_{p,\alpha}$ converging to zero on compact subsets of $\mathbb{D}$, we have $\lim_{n\to\infty} \|C_\varphi f_n\|_{\mathcal{Z}_\mu} = 0$.*

THEOREM 2.2. *Let $\alpha > -1$, $p \ge 1$, μ be a weight and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) *$C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is bounded.*
(ii) *$C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is compact.*
(iii) *For each $c > 0$,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0, \tag{2.1}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0, \tag{2.2}$$

$\varphi \in \mathcal{Z}_\mu$ *and*

$$B_{\mu,\varphi} := \sup_{z\in\mathbb{D}} \mu(z)|\varphi'(z)|^2 < \infty. \tag{2.3}$$

Proof. (ii)⇒(i). This implication is clear.

(i)⇒(iii). By taking the test function $f(z) = z \in \mathrm{AN}_{p,\alpha}$, we get $\varphi \in \mathcal{Z}_\mu$. By taking $f_1(z) = z^2/2 \in \mathrm{AN}_{p,\alpha}$, we get

$$\|C_\varphi f_1\|_{\mathcal{Z}_\mu} = \sup_{z\in\mathbb{D}} \mu(z)|(\varphi'(z))^2 + \varphi(z)\varphi''(z)| < \infty.$$

Since $\varphi \in \mathcal{Z}_\mu$ and $|\varphi(z)| < 1$, $z \in \mathbb{D}$, we get (2.3).

Let $\zeta \in \mathbb{D}$ and $c > 0$. Consider the function

$$f_\zeta(z) = \exp\left\{6c\left(\frac{1}{2}\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p}} - \frac{1}{3}\frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p}}\right)\right\}. \tag{2.4}$$

We have

$$\begin{aligned}|f_\zeta(z)| &\le \exp\left\{3c\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} + \frac{2c(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-|\varphi(\zeta)||z|)^{(\alpha+2)/p}|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}}\right\}\\ &\le \exp\left\{c(3+2^{(\alpha+2)/p+1})\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}}\right\},\end{aligned}$$

from which along with the inequality $e^x + 1 \le e^{x+1}$, $x \ge 0$, we get

$$\ln(1+|f_\zeta(z)|) \le 1 + c(3+2^{(\alpha+2)/p+1})\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}},$$

and so $\|f_\zeta\|_{\mathrm{AN}_{p,\alpha}} \le C_1(c,p,\alpha) < \infty$, for every $\zeta \in \mathbb{D}$ (see e.g. [8]).

Further, we have

$$\begin{aligned}f'_\zeta(z) &= 6c\frac{\alpha+2}{p}\overline{\varphi(\zeta)}\left(\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p+1}} - \frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p+1}}\right)\\ &\quad\times \exp\left\{6c\left(\frac{1}{2}\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p}} - \frac{1}{3}\frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p}}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}f''_\zeta(z) &= \left\{\left[6c\frac{\alpha+2}{p}\overline{\varphi(\zeta)}\left(\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p+1}} - \frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p+1}}\right)\right]^2\right.\\ &\quad + \frac{6c(\alpha+2)[\overline{\varphi(\zeta)}]^2}{p}\\ &\quad \left.\times\left[\frac{(2\alpha+4+p)(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{p(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p+2}} - \frac{(3\alpha+6+p)(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{p(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p+2}}\right]\right\}\\ &\quad\times\exp\left\{6c\left(\frac{1}{2}\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{2(\alpha+2)/p}} - \frac{1}{3}\frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{(1-\overline{\varphi(\zeta)}z)^{3(\alpha+2)/p}}\right)\right\}.\end{aligned}$$

Thus, $f'_\zeta(\varphi(\zeta))=0$, and

$$f''_\zeta(\varphi(\zeta)) = -\frac{6cp^{-2}(\alpha+2)^2[\overline{\varphi(\zeta)}]^2}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p+2}}\exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\}. \tag{2.5}$$

Since $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is bounded, we can find a constant $M_1 > 0$ such that

$$\begin{aligned} M_1 &\geq \mu(\zeta)|\varphi''(\zeta)f'_\zeta(\varphi(\zeta)) + (\varphi'(\zeta))^2 f''_\zeta(\varphi(\zeta))| \\ &= \frac{\mu(\zeta)|\varphi'(\zeta)|^2 6cp^{-2}(\alpha+2)^2|\varphi(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p+2}}\exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\}, \qquad \zeta\in\mathbb{D}, \end{aligned}$$

from which it follows that

$$\frac{\mu(\zeta)|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^2}\exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\} \leq \frac{M(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|\varphi(\zeta)|^2}, \tag{2.6}$$

where $M = M_1p^2/(6c(\alpha+2)^2)$. By letting $|\varphi(\zeta)|\to 1$ in (2.6), we get (2.2).

Now consider the function

$$g_\zeta(z) = (z-\varphi(\zeta))\left(\frac{1-|\varphi(\zeta)|^2}{(1-\overline{\varphi(\zeta)}z)^2}\right)^{(\alpha+2)/p+1} f_\zeta(z).$$

Then

$$\begin{aligned} |g_\zeta(z)| &\leq 4\left(\frac{1-|\varphi(\zeta)|^2}{|1-\overline{\varphi(\zeta)}z|^2}\right)^{(\alpha+2)/p} \\ &\quad\times \exp\left\{6c\left(\frac{1}{2}\,\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} + \frac{1}{3}\,\frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{3(\alpha+2)/p}}\right)\right\}. \end{aligned}$$

Using the inequalities $\ln(1+xy)\leq\ln(1+x)+\ln(1+y)$, $\ln(1+sx)\leq s\ln(1+x)$ and $\ln(1+e^x)\leq x+1$, which hold for $x\geq 0$, $y\geq 0$ and $s\geq 1$, we have

$$\begin{aligned} &\ln(1+|g_\zeta(z)|) \\ &\quad\leq \left\{4\ln\left(1+\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}}\right)\right. \\ &\qquad\left. + \ln\left(1+\exp\left(3c\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} + 2c\frac{(1-|\varphi(\zeta)|^2)^{2(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{3(\alpha+2)/p}}\right)\right)\right\} \\ &\quad\leq 1 + 4\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} + 3c\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} \\ &\qquad + 2^{(\alpha+2)/p+1}c\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}} \\ &\quad= 1 + (4+3c+c2^{(\alpha+2)/p+1})\frac{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}{|1-\overline{\varphi(\zeta)}z|^{2(\alpha+2)/p}}, \end{aligned}$$

and so $\|g_\zeta\|_{\mathrm{AN}_{p,\alpha}} \leq C_2(c,p,\alpha) < \infty$ for every $\zeta\in\mathbb{D}$.

Further, notice that $g_\zeta(\varphi(\zeta)) = 0$. Also, by some calculation, using the equality $f'_\zeta(\varphi(\zeta)) = 0$ and (2.5) one can easily check that

$$g'_\zeta(\varphi(\zeta)) = \frac{1}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p+1}} \exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\}$$

$$g''_\zeta(\varphi(\zeta)) = \frac{4((\alpha+2)p^{-1}+1)\overline{\varphi(\zeta)}}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p+2}} \exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\}.$$

Since $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is bounded, there is a constant $M_2 > 0$ such that

$$M_2 \geq \mu(\zeta)|\varphi''(\zeta)g'_\zeta(\varphi(\zeta)) + (\varphi'(\zeta))^2 g''_\zeta(\varphi(\zeta))|, \qquad \zeta \in \mathbb{D},$$

from which it follows that

$$\begin{aligned}\text{(2.7)}\qquad & \frac{\mu(\zeta))|\varphi''(\zeta)|}{1-|\varphi(\zeta)|^2} \exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\} \\ &\leq M_2(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p} \\ &\qquad + \frac{4((\alpha+2)/p+1)\mu(\zeta)|\varphi'(\zeta)|^2}{(1-|\varphi(\zeta)|^2)^2} \exp\left\{\frac{c}{(1-|\varphi(\zeta)|^2)^{(\alpha+2)/p}}\right\}.\end{aligned}$$

Letting $|\varphi(\zeta)| \to 1$ in (2.7), and using (2.2), we get (2.1).

(iii)$\Rightarrow$(ii). From (2.1) and (2.2) we deduce that for each $c > 0$ and every $\varepsilon > 0$ there is an $r_0 \in (0,1)$ such that, for $r_0 < |\varphi(z)| < 1$,

$$\text{(2.8)}\qquad \frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} < \varepsilon,$$

$$\text{(2.9)}\qquad \frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} < \varepsilon.$$

Note that if $f \in \mathrm{AN}_{p,\alpha}$, then by (1.1) and the Cauchy integral formula for derivatives, we have

$$\begin{aligned}\text{(2.10)}\qquad (1-|z|^2)|f'(z)| &\leq \frac{4}{\pi} \int\limits_{|\zeta-z|=(1-|z|)/2} \frac{|f(\zeta)|}{1-|z|}\,|d\zeta| \\ &\leq 4\exp\left\{\frac{8^{(\alpha+2)/p}\|f\|_{\mathrm{AN}_{p,\alpha}}}{(1-|z|^2)^{(\alpha+2)/p}}\right\},\end{aligned}$$

$$\begin{aligned}\text{(2.11)}\qquad (1-|z|^2)^2|f''(z)| &\leq \frac{32}{\pi} \int\limits_{|\zeta-z|=(1-|z|)/2} \frac{|f(\zeta)|}{1-|z|}\,|d\zeta| \\ &\leq 32\exp\left\{\frac{8^{(\alpha+2)/p}\|f\|_{\mathrm{AN}_{p,\alpha}}}{(1-|z|^2)^{(\alpha+2)/p}}\right\}.\end{aligned}$$

Choose a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{AN}_{p,\alpha}$, say by L, converging to zero on compact subsets of $\mathbb{D}$ as $n\to\infty$. Then for each $r\in(0,1)$, we have

$$\begin{aligned}(2.12)\qquad &\sup_{|\varphi(z)|\le r}\mu(z)|(C_\varphi f_n)''(z)|\\ &\le \sup_{|\varphi(z)|\le r}(\mu(z)|\varphi''(z)f_n'(\varphi(z))+(\varphi'(z))^2f_n''(\varphi(z))|)\\ &\le \|\varphi\|_{\mathcal{Z}_\mu}\sup_{|\varphi(z)|\le r}|f_n'(\varphi(z))|+B_{\mu,\varphi}\sup_{|\varphi(z)|\le r}|f_n''(\varphi(z))|\to 0\end{aligned}$$

as $n\to\infty$, since by the Weierstrass inequality, for each $k\in\mathbb{N}$, also $f_n^{(k)}$ tends to zero on compact subsets of $\mathbb{D}$ as $n\to\infty$.

On the other hand, for each $r\in(r_0,1)$, applying (2.8) and (2.9) with $c=8^{(\alpha+2)/p}L$, we deduce for $r<|\varphi(z)|<1$ that

$$\begin{aligned}(2.13)\qquad &\mu(z)|(C_\varphi f_n)''(z)|\\ &\le \sup_{|\varphi(z)|>r}4\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{8^{(\alpha+2)/p}L}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\}\\ &\quad+\sup_{|\varphi(z)|>r}32\frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\left\{\frac{8^{(\alpha+2)/p}L}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\}<36\varepsilon.\end{aligned}$$

Using (2.12) and (2.13) along with the fact that $|f_n(\varphi(0))|$ and $|f_n'(\varphi(0))\varphi'(0)|$ tend to 0 as $n\to\infty$, we have

$$\lim_{n\to\infty}\|C_\varphi f_n\|_{\mathcal{Z}_\mu}=0.$$

Hence by Lemma 2.1, the operator $C_\varphi:\mathrm{AN}_{p,\alpha}\to\mathcal{Z}_\mu$ is compact, as claimed. ∎

Lemma 2.3. *Let $\alpha>-1$, $p\ge 1$, μ be a weight and $\varphi\in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) *For each $c>0$,*

$$(2.14)\qquad \lim_{|z|\to 1}\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\}=0.$$

(ii) *For each $c>0$,*

$$(2.15)\qquad \lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\}=0$$

and $\varphi\in\mathcal{Z}_{\mu,0}$.

Proof. (i)⇒(ii). From (i) and since the function

$$f_c(x)=\frac{1}{1-x^2}\exp\frac{c}{(1-x^2)^{(\alpha+2)/p}}$$

is bounded below on $[0,1)$ by e^c, we have

$$\mu(z)|\varphi''(z)| \le e^{-c}\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2}\exp\bigg\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} \to 0$$

as $|z| \to 1$. Hence $\varphi \in \mathcal{Z}_{\mu,0}$. If $|\varphi(z)| \to 1$, then $|z| \to 1$, from which (2.15) follows.

(ii)⇒(i). Suppose that the conditions in (ii) hold, but (2.14) is not true for some $c > 0$. Then there are $c_0, \epsilon_0 > 0$ and a sequence $(z_n)_{n\in\mathbb{N}}$ tending to $\partial\mathbb{D}$ such that for every $n \in \mathbb{N}$,

$$\frac{\mu(z_n)|\varphi''(z_n)|}{1-|\varphi(z_n)|^2}\exp\bigg\{\frac{c_0}{(1-|\varphi(z_n)|^2)^{(\alpha+2)/p}}\bigg\} \ge \epsilon_0. \tag{2.16}$$

Using the fact that $f_{c_0}(x) \to +\infty$ as $x \to 1^-$, and the assumption $\varphi \in \mathcal{Z}_{\mu,0}$, along with (2.16), it follows that $(z_n)_{n\in\mathbb{N}}$ has a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ such that $|\varphi(z_{n_k})| \to 1$ as $k \to \infty$. On the other hand, (2.15) implies

$$\lim_{k\to\infty}\frac{\mu(z_{n_k})|\varphi''(z_{n_k})|}{1-|\varphi(z_{n_k})|^2}\exp\bigg\{\frac{c}{(1-|\varphi(z_{n_k})|^2)^{(\alpha+2)/p}}\bigg\} = 0,$$

which contradicts (2.16). Hence (2.14) holds, as desired. ∎

A similar proof gives the following lemma.

Lemma 2.4. *Let $\alpha > -1$, $p \ge 1$, μ be a weight and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) *For each $c > 0$,*

$$\lim_{|z|\to1}\frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\bigg\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} = 0.$$

(ii) *For each $c > 0$,*

$$\lim_{|\varphi(z)|\to1}\frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\bigg\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} = 0.$$

and $\lim_{|z|\to1}\mu(z)|\varphi'(z)|^2 = 0$.

Theorem 2.5. *Let $\alpha > -1$, $p \ge 1$, μ be a weight and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$ *is bounded.*
(ii) $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$ *is compact.*
(iii) *For each $c > 0$,*

$$\lim_{|z|\to1}\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2}\exp\bigg\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} = 0, \tag{2.17}$$

$$\lim_{|z|\to1}\frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\bigg\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} = 0. \tag{2.18}$$

Proof. (ii)⇒(i). This implication is obvious.

(i)⇒(iii). Since the boundedness of $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$ implies the boundedness of $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$, by Theorem 2.2 we see that (2.1) and (2.2) hold. By using $f(z) = z \in \mathrm{AN}_{p,\alpha}$ we get $\varphi \in \mathcal{Z}_{\mu,0}$. From this, (2.1) and Lemma 2.3, we get (2.17). Employing the test function $f(z) = z^2/2 \in \mathrm{AN}_{p,\alpha}$, as in Theorem 2.2, we get

$$\lim_{|z|\to 1} \mu(z)|\varphi'(z)|^2 = 0, \tag{2.19}$$

which along with (2.2) and Lemma 2.4 implies (2.18).

(iii)⇒(ii). First note that (2.17) and (2.18) imply respectively (2.1) and (2.2). Now we show that (2.17) also implies $\varphi \in \mathcal{Z}_{\mu,0}$. Assume to the contrary that $\varphi \notin \mathcal{Z}_{\mu,0}$. Then there would be a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\mu(z_n)|\varphi''(z_n)| \geq \delta > 0, \quad n \in \mathbb{N}. \tag{2.20}$$

We may also assume that $(\varphi(z_n))_{n\in\mathbb{N}}$ is a convergent sequence. From this, (2.20), and the boundedness below of $f_c(x)$, we would infer that (2.17) does not hold, which would be a contradiction. Similarly, we can show that (2.18) implies (2.19). Thus $\varphi \in \mathcal{Z}_\mu$ and $B_{\mu,\varphi} < \infty$. Hence by Theorem 2.2, $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ is bounded.

By (2.10), (2.11), (2.17) and (2.18), for every $f \in \mathrm{AN}_{p,\alpha}$ we have

$$\begin{aligned}\mu(z)|(C_\varphi f)''(z)| \leq \bigg(4\frac{\mu(z)|\varphi''(z)|}{1-|\varphi(z)|^2} &+ 32\frac{\mu(z)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\bigg) \\ &\times \exp\bigg\{\frac{8^{(\alpha+2)/p}\|f\|_{\mathrm{AN}_{p,\alpha}}}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\bigg\} \to 0\end{aligned}$$

as $|z| \to 1$, that is, $f \in \mathcal{Z}_{\mu,0}$. Thus $C_\varphi(\mathrm{AN}_{p,\alpha}) \subseteq \mathcal{Z}_{\mu,0}$, which implies the boundedness of $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$. ∎

The *generalized composition operator*

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)\, d\zeta,$$

where $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, was introduced in [3]. For related product type operators, including composition operators, see, for example, [1, 2, 4, 5, 10, 11, 13, 15, 18, 19] and the references therein.

Since

$$(C_\varphi^g f(z))' = f'(\varphi(z))g(z),$$

we see that the operator $C_\varphi^g : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) can be treated similarly to the composition operator $C_\varphi : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$). Note that

$$(C_\varphi f(z))' = f'(\varphi(z))\varphi'(z).$$

Hence the only difference is that the function φ', in the case of the generalized composition operator, is replaced by the function g. In light of this observation, from the proofs of Theorems 2.2 and 2.5 we see that the following results hold.

THEOREM 2.6. *Let $\alpha > -1$, $p \geq 1$, μ be a weight, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) $C_\varphi^g : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ *is bounded.*
(ii) $C_\varphi^g : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_\mu$ *is compact.*
(iii) *For each $c > 0$,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0,$$

and $\sup_{z\in\mathbb{D}} \mu(z)|g'(z)| < \infty$, *and* $\sup_{z\in\mathbb{D}} \mu(z)|\varphi'(z)g(z)| < \infty$.

THEOREM 2.7. *Let $\alpha > -1$, $p \geq 1$, μ be a weight, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent:*

(i) $C_\varphi^g : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$ *is bounded.*
(ii) $C_\varphi^g : \mathrm{AN}_{p,\alpha} \to \mathcal{Z}_{\mu,0}$ *is compact.*
(iii) *For each $c > 0$,*

$$\lim_{|z|\to 1} \frac{\mu(z)|g'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0,$$

$$\lim_{|z|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{(1-|\varphi(z)|^2)^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{(\alpha+2)/p}}\right\} = 0.$$

Acknowledgements. This research was partly supported by National Board of Higher Mathematics (NBHM)/DAE, India (grant no. 48/4/2009/R&D-II/426).

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Received 29.8.2011
and in final form 10.12.2011 (2527)