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Infinitely many solutions for the $p(x)$-Laplacian equations without (AR)-type growth condition

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Abstract. Under no Ambrosetti–Rabinowitz-type growth condition, we study the existence of infinitely many solutions of the $p(x)$-Laplacian equations by applying the variant fountain theorems due to Zou [Manuscripta Math. 104 (2001), 343–358].

1. Introduction. Fountain theorems and their dual forms are effective tools in studying the existence of infinitely many large or small energy solutions (see [W]), and Palais–Smale condition ((P.S.) condition, for short) and its variants play an important role in these theorems and their applications. Moreover, we know that, in order to verify (P.S.) condition, the following Ambrosetti–Rabinowitz superquadraticity condition is often needed:

$$\exists\theta > 2,\ 0 < \theta F(x,u) \le uf(x,u),\ \forall u \in \mathbb{R}\setminus\{0\} \text{ and a.e. } x \in \Omega, \tag{1.1}$$

where f is the nonlinear term and F is a primitive function, and Ω is a bounded or unbounded domain. For the $p(x)$-Laplacian equations, we use the following condition which is a generalization of (1.1) to the variable exponent case:

$$\exists\theta > p^+,\ 0 < \theta F(x,u) \le uf(x,u),\ \forall u \in \mathbb{R}\setminus\{0\} \text{ and a.e. } x \in \Omega, \tag{1.2}$$

where $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$; (1.2) is called the *Ambrosetti–Rabinowitz-type growth condition* (*(AR)-type growth condition*, for short) and means that $\lim_{|u|\to\infty} F(x,u)/|u|^\theta = +\infty$, that is, f is superlinear. However, there are many functions which are superlinear but do not satisfy (1.2) for any $\theta > p^+$. For example, the function $f(x,t) = t^{\alpha(x)-1}(\alpha(x)\ln t + 1)$ (with $F(x,t) = t^{\alpha(x)}\ln t$), where $\alpha \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : h(x) > 1 \text{ for any } x \in \overline{\Omega}\}$, does not satisfy (1.2) if $2\alpha^- > p^+ > \alpha^+$, where $\alpha^- = \min_{x\in\bar\Omega}\alpha(x)$, $\alpha^+ = \max_{x\in\bar\Omega}\alpha(x)$.

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On the other hand, in order to verify (1.2), it is an annoying task to compute the primitive function of f and sometimes it is almost impossible: take for instance

$$f(x,u)=u^{\alpha(x)-2}(u+e^{\sin\sin\sin u}),\qquad u>0,$$

where $\alpha(\cdot)\in C_+(\overline{\Omega})$.

The purpose of the present paper is to study the existence of infinitely many solutions of $p(x)$-Laplacian equations by applying some variant fountain theorems; thus we can free ourselves of (1.2). Unlike the p-Laplace and Laplace equations, $p(x)$-Laplace equations are inhomogeneous, thus the problems involving them are more complicated. We refer to [R, ZH] for applied background, to [FZO, KR] for the variable exponent Lebesgue–Sobolev spaces and to [FH, FJ, FZN, J] for $p(x)$-Laplacian equations and the corresponding variational problems.

The paper is organized as follows. In Section 2 we present some preliminaries on variable exponent spaces and some variant fountain theorems due to Zou [ZO]. In Section 3, infinitely many large energy solutions for the symmetric $p(x)$-Laplacian Dirichlet problem are considered. In Section 4, we study infinitely many small energy solutions for the $p(x)$-Laplacian equation with concave and convex nonlinearities.

2. Preliminaries. Let $\Omega\subset\mathbb{R}^N$ $(N\ge 2)$ be a bounded domain with smooth boundary $\partial\Omega$, and throughout this paper, we always assume $p(\cdot)\in C_+(\overline{\Omega})$, c, c_i, C and C_i are positive constants which may vary from line to line. Set

$$L^{p(\cdot)}(\Omega)$$
$$=\Big\{u: u \text{ is a measurable real-valued function on } \Omega \text{ with} \int_\Omega |u|^{p(x)}\,dx<\infty\Big\},$$

with the norm

$$|u|_{L^{p(\cdot)}(\Omega)}=|u|_{p(\cdot)}=\inf\Big\{\lambda>0: \int_\Omega |u/\lambda|^{p(x)}\,dx\le 1\Big\}$$

and $(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)})$ becomes a Banach space, called a *generalized Lebesgue space*.

THEOREM 2.1 ([FZN]).

(i) $(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)})$ *is a separable, uniformly convex Banach space, and its conjugate space is* $L^{q(\cdot)}(\Omega)$ *where* $1/q(x)+1/p(x)=1$. *For any* $u\in L^{p(\cdot)}(\Omega)$ *and* $v\in L^{q(\cdot)}(\Omega)$, *we have*

$$\Big|\int_\Omega uv\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

(ii) *If* $p_1, p_2 \in C_+(\overline{\Omega})$ *and* $p_1(x) \le p_2(x)$ *for any* $x \in \overline{\Omega}$, *then* $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ *and the imbedding is continuous.*

THEOREM 2.2 ([FZO, FZN, KR]). *Let* $u, u_k \in L^{p(\cdot)}(\Omega)$, *and set* $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$.

(i) *For* $u \ne 0$, $|u|_{p(\cdot)} = \lambda \Leftrightarrow \rho(u/\lambda) = 1$.
(ii) $|u|_{p(\cdot)} < 1\ (=1; > 1) \Leftrightarrow \rho(u) < 1\ (=1; > 1)$.
(iii) *If* $|u|_{p(\cdot)} > 1$, *then* $|u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$.
(iv) *If* $|u|_{p(\cdot)} < 1$, *then* $|u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$.
(v) $\lim_{k\to\infty} |u_k|_{p(\cdot)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$.
(vi) $\lim_{k\to\infty} |u_k|_{p(\cdot)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty$.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \qquad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and set

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & p(x) < N,\\ \infty, & p(x) \ge N. \end{cases}$$

THEOREM 2.3 ([FZN]).

(i) $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable, reflexive Banach spaces.*
(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$, *then the imbedding of* $W^{1,p(\cdot)}(\Omega)$ *in* $L^{q(\cdot)}(\Omega)$ *is compact and continuous.*
(iii) *There is a constant* $C > 0$ *such that*

$$|u|_{p(\cdot)} \le C|\nabla u|_{p(\cdot)}, \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

By Theorem 2.3(iii), we know that $|\nabla u|_{p(\cdot)}$ and $\|u\|$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. We will use $|\nabla u|_{p(\cdot)}$ to replace $\|u\|$ in the following discussions.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$ and

$$B_k = \{u \in Y_k : \|u\| \le \rho_k\}, \qquad N_k = \{u \in Z_k : \|u\| = r_k\} \qquad \text{for } \rho_k > r_k > 0.$$

Consider the C^1 functional $\Psi_\lambda : X \to \mathbb{R}$ defined by

$$\Psi_\lambda(u) := A(u) - \lambda B(u), \qquad \lambda \in [1,2].$$

Assume that

(F_1) Ψ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Psi_\lambda(-u) = -\Psi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$.
(F_2) $B(u) \geq 0$ for all $u \in X$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$; or
(F_2') $B(u) \leq 0$ for all $u \in X$; $B(u) \to -\infty$ as $\|u\| \to \infty$.

Let, for $k \geq 2$,

$$\begin{aligned}
\Gamma_k &:= \{\gamma \in C(B_k, X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = \mathrm{id}\},\\
c_k(\lambda) &:= \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u)),\\
b_k(\lambda) &:= \inf_{u \in Z_k,\, \|u\| = r_k} I_\lambda(u),\\
a_k(\lambda) &:= \max_{u \in Y_k,\, \|u\| = \rho_k} I_\lambda(u).
\end{aligned}$$

The following are variant fountain theorems.

Theorem 2.4 ([ZO]). *Assume (F_1) and (F_2) (or (F_2')) hold. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1,2]$, then $c_k(\lambda) > b_k(\lambda)$ for all $\lambda \in [1,2]$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that*

$$\sup_n \|u_n^k(\lambda)\| < \infty, \quad \Psi_\lambda'(u_n^k(\lambda)) \to 0 \quad \text{and} \quad \Psi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \text{as } n \to \infty.$$

Theorem 2.5 ([ZO]). *Assume that the C^1 functional $\Psi_\lambda : X \to \mathbb{R}$ defined by*

$$\Psi_\lambda(u) := A(u) - \lambda B(u), \qquad \lambda \in [1,2],$$

satisfies

(T_1) *Ψ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Psi_\lambda(-u) = -\Psi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$.*
(T_2) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace of X.*
(T_3) *There exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u \in Z_k,\, \|u\| = \rho_k} \Psi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k,\, \|u\| = r_k} \Psi_\lambda(u)$$

for all $\lambda \in [1,2]$ and

$$d_k(\lambda) := \inf_{u \in Z_k,\, \|u\| \leq \rho_k} \Psi_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1$ with $u(\lambda_n) \in Y_n$ such that

$$\Psi_{\lambda_n}'|_{Y_n}(u(\lambda_n)) = 0, \qquad \Psi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \text{ as } n \to \infty.$$

In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then Ψ_1 has infinitely many nontrivial critical points $u_k \in X \setminus \{0\}$ satisfying $\Psi_1(u_k) \to 0^-$ as $k \to \infty$.

3. The symmetric $p(x)$-Laplacian Dirichlet problem. We first study the existence of infinitely many solutions of the equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. We assume that

(L_1) $f \in C(\overline{\Omega} \times \mathbb{R})$ and

$$|f(x,t)| \le C_1 + C_2|t|^{\alpha(x)-1}, \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$.

(L_2) $\liminf_{|u|\to\infty} f(x,u)u/|u|^\theta \ge c > 0$ uniformly in $x \in \Omega$, where $\theta > p^+$.

(L_3) $f(x,u)/u^{p^+-1}$ is increasing in u for u large enough.

(L_4) $f(x,u)u \ge 0$ and $f(x,-u) = -f(x,u)$ for $x \in \Omega$ and $u \in \mathbb{R}$.

For now on we write $X = W_0^{1,p(\cdot)}(\Omega)$. Define

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - \int_\Omega F(x,u)\,dx.$$

It is easy to see that $I \in C^1(X,\mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx - \int_\Omega f(x,u)v\,dx, \qquad \forall u,v \in X.$$

So the critical points of I are the weak solutions of (3.1).

THEOREM 3.1. *Assume that* (L_1)–(L_4) *hold. Then* (3.1) *has infinitely many solutions* $\{u_k\}$ *satisfying*

$$\int_\Omega \frac{1}{p(x)}|\nabla u_k|^{p(x)}\,dx - \int_\Omega F(x,u_k)\,dx \to \infty \qquad \text{as } k \to \infty.$$

REMARK 3.2. The result was first proved by Fan and Zhang [FZN] under the condition (1.2).

EXAMPLE 3.3. $f(x,u) = |u|^{\alpha(x)-3}u(|u| + e^{\cos\cos\cos u})$ is an example satisfying (L_1)–(L_4) if $\alpha^- > p^+$.

As X is a separable and reflexive Banach space, there exist (see [FZN]) $\{e_n\}_{n=1}^\infty \subset X$ and $\{f_n\}_{n=1}^\infty \subset X^*$ such that

$$f_n(e_m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases}$$

$$X = \overline{\operatorname{span}}\{e_n : n = 1,2,\ldots\}, \qquad X^* = \overline{\operatorname{span}}^{W^*}\{e_n : n = 1,2,\ldots\}.$$

For $k = 1, 2, \ldots,$ denote

$$X_k = \mathrm{span}\{e_k\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$$

and

$$B_k = \{u \in Y_k : \|u\| \le \rho_k\}, \qquad N_k = \{u \in Z_k : \|u\| = r_k\} \quad \text{for } \rho_k > r_k > 0.$$

For $k \ge 2$, let Γ_k, $c_k(\lambda)$, $b_k(\lambda)$ and $a_k(\lambda)$ be defined as in Section 2. Now consider

$$I_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega F(x,u)\, dx =: A(u) - \lambda B(u),$$

where $\lambda \in [1,2]$. By (L_4), it is easy to see that $B(u) \ge 0$ and $A(u) \to \infty$ as $\|u\| \to \infty$, $I_\lambda(-u) = -I_\lambda(u)$ for all $\lambda \in [1,2]$ and $u \in X$.

Lemma 3.4 ([FZN]). *If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k(\alpha(\cdot)) = \sup\{|u|_{\alpha(\cdot)} \mid \|u\| = 1,\, u \in Z_k\}.$$

Then $\lim_{k\to\infty} \beta_k(\alpha(\cdot)) = 0$.

Theorem 3.1 follows directly from the next lemmas.

Lemma 3.5. *Under the assumptions of Theorem* 3.1, *there exist $\lambda_n \to 1$ as $n \to \infty$, $\overline{c}_k > \overline{b}_k > 0$ and $\{z_n\}_{n=1}^{\infty} \subset X$ such that*

$$I'_{\lambda_n}(z_n) = 0, \qquad I_{\lambda_n}(z_n) \in [\overline{b}_k, \overline{c}_k].$$

Proof. By (L_1) and (L_2), for every $\epsilon > 0$, there exists C_ϵ such that

$$f(x,u)u \ge C_\epsilon |u|^\theta - \epsilon |u|^{p^+}, \qquad \forall u \in X.$$

Since all norms are equivalent in Y_k, it is easy to see, for some $\rho_k > 0$ large enough, that $a_k(\lambda) := \max_{u\in Y_k,\, \|u\|=\rho_k} I_\lambda(u) \le 0$ uniformly for $\lambda \in [1,2]$. On the other hand, from Lemma 3.4, $\beta_k(\alpha(\cdot)) \to 0$ as $k \to \infty$. Therefore, for $u \in Z_k$, $\|u\| = \gamma_k = (C\alpha^+ \beta_k^{\alpha^+})^{1/(p^- - \alpha^+)}$, by (L_1), we have

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega F(x,u)\, dx \\
&\ge \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx - c\lambda \int_\Omega |u|^{\alpha(x)}\, dx - c_1 \\
&\ge \|u\|^{p^-}/p^+ - c\lambda |u|_{\alpha(\cdot)}^{\alpha(\xi)} - c_2 \quad \text{for some } \xi \in \Omega \\
&= \begin{cases} \|u\|^{p^-}/p^+ - c_3 - c_2 & \text{if } |u|_{\alpha(\cdot)} \le 1, \\ \|u\|^{p^-}/p^+ - c_4 \beta_k^{\alpha_+} \|u\|^{\alpha_+} - c_2 & \text{if } |u|_{\alpha(\cdot)} > 1 \end{cases} \\
&\ge \|u\|^{p^-}/p^+ - c_3 \beta_k^{\alpha^+} \|u\|^{\alpha^+} - c_5
\end{aligned}$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\alpha^+}\right)(C\alpha^+\beta_k^{\alpha^+})^{\frac{p^-}{p^- - \alpha^+}} - c_5$$
$$\geq \frac{1}{2}\left(\frac{1}{p^+} - \frac{1}{\alpha^+}\right)(C\alpha^+\beta_k^{\alpha^+})^{\frac{p^-}{p^- - \alpha^+}} = \bar{b}_k \quad \text{if } k \text{ is large,}$$

which implies that $b_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\gamma_k} I_\lambda(u) \geq \bar{b}_k \to \infty$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$. Therefore, by Theorem 2.4, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sum_n \|u_n^k(\lambda)\| < \infty, \quad I'_\lambda(u_n^k(\lambda)) \to 0 \quad \text{and} \quad I_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \geq b_k(\lambda) \geq \bar{b}_k,$$

as $n \to \infty$. Moreover, since $c_k(\lambda) \leq \sup_{u\in B_k} I(u) =: \bar{c}_k$ and the embedding $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ is compact [FZN], the sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ has a convergent subsequence. Hence, there exists z_λ^k such that $I'_\lambda(z_\lambda^k) = 0$ and $I_\lambda(z_\lambda^k) \in [\bar{b}_k, \bar{c}_k]$. It is clear that we may find $\lambda_n \to 1$ and $\{z_n\}$ as desired. ∎

LEMMA 3.6. *The sequence $\{z_n\}_{n=1}^\infty$ is bounded.*

Proof. Assume that $\|z_n\| \to \infty$ as $n \to \infty$. Set $\omega_n := z_n/\|z_n\|$. Then, up to a subsequence,

$$\begin{aligned} \omega_n(x) &\rightharpoonup \omega(x) && \text{in } X,\ n \to \infty,\\ \omega_n(x) &\to \omega(x) && \text{in } L^{\alpha(\cdot)}(\Omega),\ n \to \infty,\\ \omega_n(x) &\to \omega(x) && \text{for a.e. } x \in \Omega,\ n \to \infty. \end{aligned}$$

CASE 1: $\omega \neq 0$ in X. Since $I'_{\lambda_n}(z_n) = 0$, we have

$$\int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^{p_+}}\,dx \leq c$$

if n is large. On the other hand, by Fatou's lemma and (L_2),

$$\int_\Omega \frac{f(x,z_n)z_n}{\|z_n\|^{p_+}}\,dx = \int_{\omega_n(x)\neq 0} |\omega_n(x)|^{p_+} \frac{f(x,z_n)z_n}{|z_n|^{p_+}}\,dx \to \infty$$

as $n \to \infty$. This is a contradiction.

CASE 2: $\omega = 0$ in X. We define

$$I_{\lambda_n}(t_n z_n) = \max_{t\in[0,1]} I_{\lambda_n}(t z_n).$$

For any $c > 1$ and $\bar{\omega}_n = (2p^+c)^{1/p^-}\omega_n$, we have, for n large enough,

$$I_{\lambda_n}(t_n z_n) \geq I_{\lambda_n}(\bar{\omega}_n) \geq 2c - \lambda_n \int_\Omega F(x,\bar{\omega}_n)\,dx \geq c,$$

which implies that $\lim_{n\to\infty} I_{\lambda_n}(t_n z_n) = \infty$. Furthermore, $\langle I'_{\lambda_n}(t_n z_n), t_n z_n\rangle = 0$, and it follows that

$$\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla t_n z_n|^{p(x)}\,dx + \lambda_n \int_\Omega \left(\frac{1}{p^+} t_n z_n f(x, t_n z_n) - F(x, t_n z_n)\right)dx \to \infty.$$

By condition (L_3), $h(t) = (1/p^+)t^{p^+} f(x,s)s - F(x,ts)$ is increasing in $t \in [0,1]$, hence $(1/p^+)f(x,s)s - F(x,s)$ is increasing in $s > 0$. Invoking the oddness of f, we have

$$\begin{aligned}&\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla z_n|^{p(x)}\,dx + \lambda_n \int_\Omega \left(\frac{1}{p^+} z_n f(x, z_n) - F(x, z_n)\right)dx\\ &\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla t_n z_n|^{p(x)}\,dx + \lambda_n \int_\Omega \left(\frac{1}{p^+} t_n z_n f(x, t_n z_n) - F(x, t_n z_n)\right)dx\\ &\to \infty.\end{aligned}$$

We get a contradiction since

$$\int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla z_n|^{p(x)}\,dx + \lambda_n \int_\Omega \left(\frac{1}{p^+} z_n f(x, z_n) - F(x, z_n)\right)dx = I_{\lambda_n}(z_n) - \frac{1}{p^+}\langle I'_{\lambda_n}(z_n), z_n\rangle = I_{\lambda_n}(z_n) \in [\bar{b}_k, \bar{c}_k]. \blacksquare$$

4. The $p(x)$-Laplacian equation with concave and convex nonlinearities. Now we consider the following quasilinear elliptic equation with concave and convex nonlinearities:

$$\text{(4.1)}\qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) + g(x,u) & \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. We want to find infinitely many small negative energy solutions. The following hypotheses are assumed for (4.1):

(S_1) $f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ are odd in u.

(S_2) There exist $s, s_1 \in C(\overline{\Omega})$ and $1 < s^- \leq s^+ < p^-$, $1 < s_1^- \leq s_1^+ < p^-$, $c_1, c_2, c_3 > 0$ such that
$$c_1|u|^{s(x)} \leq f(x,u)u \leq c_2|u|^{s(x)} + c_3|u|^{s_1(x)} \quad \text{for a.e. } x \in \Omega \text{ and } u \in \mathbb{R}.$$

(S_3) There exists $\alpha \in C(\bar{\Omega})$ with $p^+ < \alpha^- \leq \alpha^+$ and $\alpha(x) < p^*(x)$ such that
$$|g(x,u)| \leq C_1 + C_2|u|^{\alpha(x)-1}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$
Moreover, $\lim_{u\to 0} g(x,u)/u^{p^+-1} = 0$ uniformly for $x \in \Omega$.

(S_4) Assume one of the following conditions holds:

(1) $\lim_{|u|\to\infty} g(x,u)/|u|^{p^- -1} = 0$ uniformly for $x \in \Omega$.

(2) $\lim_{|u|\to\infty} g(x,u)/|u|^{p^- -1} = -\infty$ uniformly for $x \in \Omega$. Furthermore, $f(x,u)/u^{p^- -1}$ and $g(x,u)/u^{p^- -1}$ are increasing in u for u large enough.

(3) $\lim_{|u|\to\infty} g(x,u)/|u|^{p^- -1} = \infty$ uniformly for $x \in \Omega$, and $\frac{g(x,u)}{u^{p^- -1}}$ is increasing in u for u large enough. Moreover, there exists $\beta(\cdot) \in C_+(\overline{\Omega})$ with $\beta^- > \max\{p^+, s^+, s_1^+\}$ such that

$$\liminf_{|u|\to\infty} \frac{g(x,u)u - p^- G(x,u)}{|u|^{\beta(x)}} = \infty \quad \text{uniformly for } x \in \Omega.$$

Example 4.1.

$$f(x,u) = |u|^{s(x)-2} u \ln(2+|u|), \qquad g(x,u) = \mu |u|^{\alpha(x)-3} u \ln(1+|u|),$$

where $s \in C_+(\overline{\Omega})$ and $1 < s^- \le s^+ < p^-$ for a.e. $x \in \Omega$ and $u \in \mathbb{R}$, $\alpha \in C(\bar{\Omega})$ with $p^+ < \alpha^- \le \alpha^+$ and $\alpha(x) < p^*(x)$. Then (S_1), (S_2), (S_3) and $(S_4)(2)$ hold if $\mu < 0$; (S_1), (S_2), (S_3) and $(S_4)(3)$ hold if $\mu > 0$ and $\beta > \max\{p^+, s^+, s_1^+\}$ is a constant; if we assume $g(x,u) = |u|^{\alpha(x)-2}u, \alpha^- > 2$ for $|u| \le 1$, $g(x,u) = c|u|^{p^- - \alpha^- -1} u \ln(1+|u|), \alpha^- > 2$ for $|u| \ge 1$ (here $c = 1/\ln 2$), then (S_1), (S_2), (S_3) and $(S_4)(1)$ hold.

Define

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx - \int_\Omega F(x,u)\,dx - \int_\Omega G(x,u)\,dx.$$

It is clear that $\Phi \in C^1(X,\mathbb{R})$ and the critical points of Φ are the weak solutions of (4.1).

The following is the main result of this section.

Theorem 4.2. *Assume that* (D_1)–(D_4) *hold. Then* (4.1) *has infinitely many solutions* $\{u_k\}$ *satisfying*

$$\int_\Omega \frac{1}{p(x)} |\nabla u_k|^{p(x)}\,dx - \int_\Omega F(x,u_k)\,dx - \int_\Omega G(x,u_k)\,dx \to 0^- \quad \text{as } k \to \infty,$$

where F and G are the primitive functions of f and g respectively.

We consider

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx - \int_\Omega G(x,u)\,dx - \lambda \int_\Omega F(x,u)\,dx = A(u) - \lambda B(u),$$

where $\lambda \in [1,2]$. Then $B(u) \ge 0$ and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite-dimensional subspace. Let $n > k > 2$. By (S_3), for any $\epsilon > 0$, there exists C_ϵ such that

$$|G(x,u)| \le C_\epsilon |u|^{\alpha(x)} + \epsilon |u|^{p_+}, \qquad \forall u \in X.$$

Therefore, for $\|u\|$ small enough,

$$\Phi_\lambda(u) \ge \frac{\|u\|^{p^+}}{2p^+} - c_1|u|_{s(\cdot)}^{s^-} - c_2|u|_{s_1(\cdot)}^{s_1^-}.$$

Assume that $s^- \le s_1^-$ and let

$$\beta_k(s(\cdot)) = \sup_{u\in Z_k,\, \|u\|=1} |u|_{s(\cdot)}, \qquad \beta_k(s_1(\cdot)) = \sup_{u\in Z_k,\, \|u\|=1} |u|_{s_1(\cdot)}.$$

Then $\beta_k(s(\cdot)) \to 0$, $\beta_k(s_1(\cdot)) \to 0$ as $k \to \infty$. Then for $u \in Z_K$ and

$$\|u\| := \rho_k := \big(4cp^+\beta_k^{s^-}(s(\cdot)) + 4cp^+\beta_k^{s_1^-}(s_1(\cdot))\big)^{\frac{1}{p^+-s^-}},$$

we have $\Phi_\lambda(u) \ge \rho_k^{p^+}/(4p^+) > 0$. On the other hand, if $u \in Y_k$ with $\|u\|$ small enough, we have

$$\begin{aligned}\Phi_\lambda(u) &\le \frac{1}{p^-}\|u\|^{p^-} + C_\epsilon \int_\Omega |u|^{\alpha(x)}\,dx + \epsilon\int_\Omega |u|^{p^+}\,dx - \lambda c_1 \int_\Omega |u|^{s(x)}\,dx\\ &\le \frac{1}{p^-}\|u\|^{p^-} + C_\epsilon\|u\|^{\alpha^-} + \epsilon\|u\|^{p^+} - \lambda c_1\|u\|^{s^+} < 0.\end{aligned}$$

The above arguments imply that $b_k(\lambda) < 0 < a_k(\lambda)$ for $\lambda \in [1,2]$. Furthermore, for $u \in Z_k$ with $\|u\| \le \rho_k$, we see that $\Phi_\lambda(u) \ge -c_1\beta_k^{s^-}(s(\cdot))\rho_k^{s^-} - c_2\beta_k^{s_1^-}(s_1(\cdot))\rho_k^{s^-} \to 0$ as $k \to \infty$. So, $d_k(\lambda) \to 0$ as $k \to \infty$, and applying Theorem 2.5, we have the following lemma.

Lemma 4.3. *There exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \qquad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.$$

Theorem 4.2 is a consequence of Lemma 4.3 and Lemma 4.4 below.

Lemma 4.4. *The sequence $\{u(\lambda_n)\}$ is bounded in X.*

Proof. Since $\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, we have

$$\int_\Omega |\nabla u(\lambda_n)|^{p(x)}\,dx - \int_\Omega g(x,u(\lambda_n))u(\lambda_n)\,dx - \lambda_n\int_\Omega f(x,u(\lambda_n))u(\lambda_n)\,dx = 0.$$

If, up to a subsequence, $\|u(\lambda_n)\| \to \infty$ as $n \to \infty$, then

$$1 \le \int_\Omega \frac{g(x,u(\lambda_n))u(\lambda_n) - \lambda_n f(x,u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}\,dx.$$

By (S_2),

$$1 + o(1) \le \int_\Omega \frac{g(x,u(\lambda_n))u(\lambda_n)}{\|u(\lambda_n)\|^{p^-}}\,dx,$$

where $o(1) \to 0$ as $n \to \infty$. This is a contradiction if $(S_4)(1)$ holds.

Otherwise, set $\omega_n = u(\lambda_n)/\|u(\lambda_n)\|$. Then

$$\begin{aligned} \omega_n(x) &\rightharpoonup \omega(x) && \text{in } X,\ n \to \infty, \\ \omega_n(x) &\to \omega(x) && \text{in } L^{\alpha(\cdot)}(\Omega),\ n \to \infty, \\ \omega_n(x) &\to \omega(x) && \text{for a.e. } x \in \Omega,\ n \to \infty. \end{aligned}$$

If $\omega \neq 0$ in X and $\lim_{|u|\to\infty} g(x,u)/u^{p^- -1} = -\infty$ in $(S_4)(2)$, then, for n large enough, by Fatou's lemma, we have

$$\begin{aligned} -1 + o(1) &\geq \int_\Omega \frac{-g(x,u(\lambda_n))u(\lambda_n)}{|u(\lambda_n)|^{p^-}} |\omega_n|^{p^-}\,dx, \\ &\geq c + \int_{\{\omega\neq 0\}\cap\{|u(\lambda_n)|\geq c\}} \frac{-g(x,u(\lambda_n))u(\lambda_n)}{|u(\lambda_n)|^{p^-}} |\omega_n|^{p^-}\,dx \to \infty, \end{aligned}$$

a contradiction. Therefore $\omega = 0$ in X. Similar to the proof of Lemma 3.2, if we define

$$\Phi_{\lambda_n}(t_n u(\lambda_n)) = \max_{t\in[0,1]} \Phi_{\lambda_n}(tu(\lambda_n)),$$

then

$$\lim_{n\to\infty} \Phi_{\lambda_n}(t_n u(\lambda_n)) = \infty, \qquad \langle \Phi'_{\lambda_n}(t_n u(\lambda_n)), t_n u(\lambda_n)\rangle = 0.$$

It follows that

$$\begin{aligned} \infty &= \lim_{n\to\infty} \Phi_{\lambda_n}(t_n u(\lambda_n)) - \frac{1}{p^-}\langle \Phi'_{\lambda_n}(t_n u(\lambda_n)), t_n u(\lambda_n)\rangle \\ &\leq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^-}\right) |\nabla t_n u(\lambda_n)|^{p(x)}\,dx \\ &\quad + \lambda_n \int_\Omega \left(\frac{1}{p^-} t_n u(\lambda_n) f(x, t_n u(\lambda_n)) - F(x, t_n u(\lambda_n))\right) dx \\ &\quad + \int_\Omega \left(\frac{1}{p^-} t_n u(\lambda_n) g(x, t_n u(\lambda_n)) - G(x, t_n u(\lambda_n))\right) dx. \end{aligned}$$

If $(S_4)(2)$ holds, we have

$$\frac{1}{p^-} suf(x,su) - F(x,su) + \frac{1}{p^-} sug(x,su) - G(x,su) \leq c$$

for all $s > 0$ and $u \in \mathbb{R}$, a contradiction.

If $(S_4)(3)$ holds, then

$$\begin{aligned} \infty &\leq c_1 \int_\Omega |u(\lambda_n)|^{s(x)}\,dx + c_2 \int_\Omega |u(\lambda_n)|^{s_1(x)}\,dx - c_3 \int_\Omega |u(\lambda_n)|^{s(x)}\,dx \\ &\quad + \int_\Omega \left(\frac{1}{p^-} u(\lambda_n) g(x, u(\lambda_n)) - G(x, u(\lambda_n))\right) dx, \end{aligned}$$

which implies that

$$\int_\Omega \left(\frac{1}{p^-}u(\lambda_n)g(x,u(\lambda_n)) - G(x,u(\lambda_n))\right)dx \to \infty.$$

However, by the property of $u(\lambda_n)$, we have

$$\begin{aligned}
b_k(1) \geq {} & \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^-}\right)|\nabla u(\lambda_n)|^{p(x)}\,dx \\
& + \lambda_n \int_\Omega \left(\frac{1}{p^-}u(\lambda_n)f(x,u(\lambda_n)) - F(x,u(\lambda_n))\right)dx \\
& + \int_\Omega \left(\frac{1}{p^-}u(\lambda_n)g(x,u(\lambda_n)) - G(x,u(\lambda_n))\right)dx \\
\geq {} & \int_\Omega \left(\frac{1}{p^-}u(\lambda_n)g(x,u(\lambda_n)) - G(x,u(\lambda_n))\right)dx \\
& - \int_\Omega \left(\frac{1}{p^-} - \frac{1}{p(x)}\right)|\nabla u(\lambda_n)|^{p(x)}\,dx - c\|u\|^{s^+} - c\|u\|^{s_1^+}, \\
\to {} & \infty,
\end{aligned}$$

which contradicts the preceding estimate. So the sequence $\{u(\lambda_n)\}$ is bounded in X. ∎

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