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# Probability distribution solutions of a general linear equation of infinite order, II

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**Abstract.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tau\colon \mathbb{R}\times\Omega\to\mathbb{R}$ be a mapping strictly increasing and continuous with respect to the first variable, and $\mathcal{A}$-measurable with respect to the second variable. We discuss the problem of existence of probability distribution solutions of the general linear equation

$$F(x) = \int_\Omega F(\tau(x,\omega))\,P(d\omega).$$

We extend our uniqueness-type theorems obtained in Ann. Polon. Math. 95 (2009), 103–114.

**1. Introduction.** Throughout the paper, $(\Omega, \mathcal{A}, P)$ is a probability space and $\tau\colon \mathbb{R}\times\Omega\to\mathbb{R}$ is a mapping such that for every $x\in\mathbb{R}$ the function $\tau(x,\cdot)$ is $\mathcal{A}$-measurable, and for every $\omega\in\Omega$ the function $\tau(\cdot,\omega)$ is strictly increasing and continuous.

We investigate the set of probability distribution (p.d.) solutions of the linear functional equation

$$F(x) = \int_\Omega F(\tau(x,\omega))\,P(d\omega) \tag{1.1}$$

extending the results obtained in [KM]; for the background of equation (1.1) see the references therein.

As explained in detail in [KM, §2], we may restrict our considerations to the case where

$$\{x\in\mathbb{R} : \tau(x,\omega)=x \text{ for almost all } \omega\in\Omega\}=\emptyset. \tag{1.2}$$

This follows from [MR, Theorem 2]; also by that theorem, we know that (1.2) forces every p.d. solution $F$ of (1.1) to be automatically continuous.

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From now on we assume (1.2). For any interval $J \subset \mathbb{R}$ we define

$$\mathcal{C}(J) = \{F\colon \mathbb{R} \to [0,1] \mid F \text{ is a weakly increasing (and continuous) solution of (1) such that } F(\inf J) = 0 \text{ and } F(\sup J) = 1\},$$

with the notation $F(-\infty) = \lim_{x\to-\infty} F(x)$ and $F(+\infty) = \lim_{x\to+\infty} F(x)$.

We say that a set $S \subset \mathbb{R}$ is *$\tau$-invariant* if $S \neq \emptyset$ and for every $x \in S$ we have $\tau(x,\omega) \in S$ for almost all $\omega \in \Omega$. Put

$$\mathcal{S}_1 = \{I \subset \mathbb{R} : I \text{ is a minimal compact } \tau\text{-invariant interval}\},$$

$$\mathcal{S}_2 = \Big\{I \subset \mathbb{R} \setminus \bigcup \mathcal{S}_1 : I \text{ is a maximal } \tau\text{-invariant half-line}\Big\},$$

and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. In view of the definition and our assumption (1.2), the family $\mathcal{S}$ consists of pairwise disjoint non-degenerate closed proper subintervals of $\mathbb{R}$. Therefore, since $\bigcup \mathcal{S}_1$ is closed (cf. the proof of [KM, Claim 7]), $\bigcup \mathcal{S}$ is closed as well. This implies that $\mathbb{R} \setminus \bigcup \mathcal{S}$ is a non-empty open set. Indeed, $\bigcup \mathcal{S} = \mathbb{R}$ would imply that the set of all end-points of the intervals from $\mathcal{S}$ is a countable perfect set, which is impossible.

By virtue of [KM, Corollary 2, Remarks 1 and 2], we find that:

(i) Every p.d. solution $F$ of (1.1) is constant on each member of $\mathcal{S}$.

(ii) For each open component $J$ of the set $\mathbb{R} \setminus \bigcup \mathcal{S}$ the class $\mathcal{C}(J)$ has at most one element.

(iii) If $F$ is a p.d. solution of (1.1) and $J$ is an open component of $\mathbb{R} \setminus \bigcup \mathcal{S}$ with $\mathcal{C}(J) = \{G\}$, then

$$G = \frac{F - F(\inf J)}{F(\sup J) - F(\inf J)}.$$

(iv) The existence of any p.d. solution of (1.1) is equivalent to $\mathcal{C}(J) \neq \emptyset$ for at least one open component $J$ of $\mathbb{R} \setminus \bigcup \mathcal{S}$.

These four statements show that in order to describe every p.d. solution $F$ of equation (1.1) we should be able to decide whether $\mathcal{C}(J) \neq \emptyset$ and, if so, to describe the unique member of $\mathcal{C}(J)$, for every open component $J$ of $\mathbb{R} \setminus \bigcup \mathcal{S}$. This is the aim of the present paper.

**2. Some lemmas.** We start with two auxiliary lemmas which yield certain connections between solutions of any of the two inequalities:

$$F_0(x) \geq \int_\Omega F_0(\tau(x,\omega))\, P(d\omega), \tag{2.1}$$

$$F_0(x) \leq \int_\Omega F_0(\tau(x,\omega))\, P(d\omega) \tag{2.2}$$

and solutions of equation (1.1).

LEMMA 2.1. *If* $F_0 \colon \mathbb{R} \to [0,1]$ *is an increasing solution of* (2.1) (*or* (2.2)), *then the sequence* $(F_n)_{n\in\mathbb{N}}$ *of functions* $F_n \colon \mathbb{R} \to [0,1]$ *defined by the formula*

$$F_n(x) = \int_{\Omega} F_{n-1}(\tau(x,\omega))\, P(d\omega) \quad \text{for } n \in \mathbb{N},\ x \in \mathbb{R} \tag{2.3}$$

*is decreasing (respectively increasing), hence it is pointwise convergent to a certain* $F \colon \mathbb{R} \to [0,1]$.

*Moreover, the function* $F$ *is either constant or*

$$\frac{F - F(-\infty)}{F(+\infty) - F(-\infty)} \in \mathcal{C}(\mathbb{R}). \tag{2.4}$$

*Proof.* If $F_0$ satisfies (2.1), then by the definition, $F_1 \le F_0$. In particular, $F_1(\tau(x,\omega)) \le F_0(\tau(x,\omega))$ for all $x \in \mathbb{R}$ and $\omega \in \Omega$. After integration we get $F_2 \le F_1$ and, by induction, $F_n \le F_{n-1}$ for every $n \in \mathbb{N}$. Analogously, if $F_0$ satisfies (2.2), then $F_{n-1} \le F_n$ for every $n \in \mathbb{N}$. Let $F = \lim_{n\to\infty} F_n$. Since $F_0$ is increasing, each $F_n$ and $F$ itself are increasing as well. Moreover, $F$ satisfies (1.1). Thus, in view of [MR, Theorem 2], either $F$ is constant or (2.4) holds. ∎

In the following we consider the product space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ and the iterates $\tau^n \colon \mathbb{R} \times \Omega^\infty \to \mathbb{R}$ defined (cf. [BJ], [BK], [D]) by putting

$$\begin{aligned} \tau^1(x,\omega_1,\ldots) &= \tau(x,\omega_1),\\ \tau^{n+1}(x,\omega_1,\ldots) &= \tau(\tau^n(x,\omega_1,\ldots),\omega_{n+1}) \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

It is easily seen that for each $n \in \mathbb{N}$ we have

$$\tau^{n+1}(x,\omega_1,\ldots) = \tau^n(\tau(x,\omega_1),\omega_2,\ldots)$$

and the $n$th iterate $\tau^n(x,\omega)$ depends only on the first $n$ coordinates of $\omega$. Hence it is justified to write $\tau^n(x,\omega_1,\ldots,\omega_m)$ instead of $\tau^n(x,\omega_1,\ldots)$, if $m \ge n$.

LEMMA 2.2. *Assume* $x_0, y_0 \in \mathbb{R} \cup \{-\infty,+\infty\}$.

(i) *If* $x_0 \in \mathbb{R}$ *and* $(F_n)_{n\in\mathbb{N}}$ *is given by* (2.3) *with* $F_0 = \chi_{[x_0,+\infty)}$, *then*

$$F_n(x) = P^\infty(\tau^n(x,\omega) \ge x_0) \quad \text{for } n \in \mathbb{N},\ x \in \mathbb{R}.$$

*If* $(x_0,y_0)$ *is a component of* $\mathbb{R} \setminus \bigcup \mathcal{S}$, *then* $F_0$ *satisfies* (2.1).

(ii) *If* $y_0 \in \mathbb{R}$ *and* $(F_n)_{n\in\mathbb{N}}$ *is given by* (2.3) *with* $F_0 = \chi_{[y_0,+\infty)}$, *then*

$$F_n(x) = P^\infty(\tau^n(x,\omega) \ge y_0) \quad \text{for } n \in \mathbb{N},\ x \in \mathbb{R}.$$

*If* $(x_0,y_0)$ *is a component of* $\mathbb{R} \setminus \bigcup \mathcal{S}$, *then* $F_0$ *satisfies* (2.2).

*Proof.* Since the proofs of both assertions (i) and (ii) are similar, we show only the first one.

For any $x \in \mathbb{R}$ we have

$$\begin{aligned} F_1(x) &= \int_{\Omega} \chi_{[x_0,+\infty)}(\tau(x,\omega_1))\, P(d\omega_1) \\ &= P(\tau(x,\omega_1) \geq x_0) = P^\infty(\tau^1(x,\omega_1,\ldots) \geq x_0). \end{aligned}$$

Assuming that the desired formula holds true for a fixed $n \in \mathbb{N}$ and every $x \in \mathbb{R}$, we get

$$\begin{aligned} F_{n+1}(x) &= \int_{\Omega} F_n(\tau(x,\omega_1))\, P(d\omega_1) \\ &= \int_{\Omega} P^n(\tau^n(\tau(x,\omega_1),\omega_2,\ldots,\omega_{n+1}) \geq x_0)\, P(d\omega_1) \\ &= \int_{\Omega} P^n(\tau^{n+1}(x,\omega_1,\ldots,\omega_{n+1}) \geq x_0)\, P(d\omega_1) \\ &= P^{n+1}(\tau^{n+1}(x,\omega_1,\ldots,\omega_{n+1}) \geq x_0) \\ &= P^\infty(\tau^{n+1}(x,\omega_1,\ldots) \geq x_0). \end{aligned}$$

Now, suppose $(x_0, y_0)$ is a component of $\mathbb{R} \setminus \bigcup \mathcal{S}$. Then $x_0$ is a right end-point of some $\tau$-invariant interval. Hence $\tau(x_0,\omega) \leq x_0$ for almost all $\omega \in \Omega$. Therefore, if $x < x_0$ then $\tau(x,\omega) < \tau(x_0,\omega) \leq x_0$ for almost all $\omega \in \Omega$, hence $F_0(\tau(x,\omega)) = 0$ for almost all $\omega \in \Omega$ and (2.1) holds; if $x \geq x_0$ then $F_0(x) = 1$ and again (2.1) holds. ∎

**3. The case $\mathcal{S} \neq \emptyset$.** Obviously, recursion (2.3) may produce a non-trivial solution of equation (1.1) only if the initial function $F_0$ is non-constant. Lemma 2.2 guarantees that in the case $\mathcal{S} \neq \emptyset$ it is always possible to find a suitable solution of (2.1) or (2.2). We will exploit this fact in the next lemma.

Throughout this section we assume $\mathcal{S} \neq \emptyset$.

LEMMA 3.1. *Assume $J = (x_0, y_0)$ is a component of $\mathbb{R} \setminus \bigcup \mathcal{S}$.*

(i) *If $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$, $(F_n)_{n\in\mathbb{N}}$ is given by (2.3) with $F_0 = \chi_{[x_0,+\infty)}$ and $F = \lim_{n\to\infty} F_n$, then $\mathcal{C}(J) = \{F\}$.*

(ii) *If $x_0 \in \mathbb{R}$, $y_0 = +\infty$, $(F_n)_{n\in\mathbb{N}}$ is given by (2.3) with $F_0 = \chi_{[x_0,+\infty)}$ and $F = \lim_{n\to\infty} F_n$, then $F \neq 0$ implies $\mathcal{C}(J) = \{F\}$, whereas $F = 0$ implies $\mathcal{C}(J) = \emptyset$.*

(iii) *If $x_0 = -\infty$, $y_0 \in \mathbb{R}$, $(F_n)_{n\in\mathbb{N}}$ is given by (2.3) with $F_0 = \chi_{[y_0,+\infty)}$ and $F = \lim_{n\to\infty} F_n$, then $F \neq 1$ implies $\mathcal{C}(J) = \{F\}$, whereas $F = 1$ implies $\mathcal{C}(J) = \emptyset$.*

*Proof.* (i) Since $J$ is a bounded component of $\mathbb{R} \setminus \bigcup \mathcal{S}$, we infer that $x_0$ and $y_0$ are (respectively right and left) end-points of some $\tau$-invariant intervals. Consequently,

$$P(\tau(x,\omega) \le x_0) = 1, \quad P(\tau(y,\omega) \ge y_0) = 1 \quad \text{for } x \le x_0,\ y \ge y_0.$$

A simple induction yields

$$P^\infty(\tau^n(x,\omega) \le x_0) = 1,\ P^\infty(\tau^n(y,\omega) \ge y_0) = 1 \quad \text{for } n \in \mathbb{N},\ x \le x_0,\ y \ge y_0,$$

which, in the light of Lemma 2.2, means nothing else than

$$F_n|_{(-\infty,x_0]} = 0 \quad \text{and} \quad F_n|_{[y_0,+\infty)} = 1 \quad \text{for } n \in \mathbb{N}.$$

By Lemma 2.1, $F \in \mathcal{C}(J)$.

(ii) Analogously to the proof of (i) we can show that $F|_{(-\infty,x_0]} = 0$ and either $F$ is constant or the function $F/F(+\infty)$ belongs to $\mathcal{C}(J)$. Now it is enough to prove that $\mathcal{C}(J) \ne \emptyset$ implies $F(+\infty) = 1$.

Let $G \in \mathcal{C}(J)$. Obviously, $G \le F_0$. Therefore

$$G(x) = \int_\Omega G(\tau(x,\omega))\, P(d\omega) \le \int_\Omega F(\tau(x,\omega))\, P(d\omega) = F_1(x)$$

for $x \in \mathbb{R}$ and further, by induction, $G \le F_n$ for every $n \in \mathbb{N}$. Hence $G \le F$, which implies $F(+\infty) = 1$.

(iii) The proof runs analogously to the proof of (ii). ∎

As a consequence of Lemmas 2.2 and 3.1 we obtain the following theorem.

THEOREM 3.2. *Assume $\mathcal{S} \ne \emptyset$ and $J = (x_0, y_0)$ is a component of $\mathbb{R} \setminus \bigcup \mathcal{S}$.*

(i) *If $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$, then $\mathcal{C}(J) \ne \emptyset$. The unique member of $\mathcal{C}(J)$ is given by $F = \lim_{n\to\infty} F_n$, where $(F_n)_{n\in\mathbb{N}}$ is defined by* (2.3) *with $F_0 = \chi_{[x_0,+\infty)}$.*

(ii) *If $x_0 \in \mathbb{R}$ and $y_0 = +\infty$, then $\mathcal{C}(J) \ne \emptyset$ if and only if*

$$\lim_{x\to+\infty} \lim_{n\to\infty} P^\infty(\tau^n(x,\omega) \ge x_0) > 0.$$

*In that case the unique member of $\mathcal{C}(J)$ is given by $F = \lim_{n\to\infty} F_n$, where $(F_n)_{n\in\mathbb{N}}$ is defined by* (2.3) *with $F_0 = \chi_{[x_0,+\infty)}$.*

(iii) *If $x_0 = -\infty$, $y_0 \in \mathbb{R}$, then $\mathcal{C}(J) \ne \emptyset$ if and only if*

$$\lim_{y\to-\infty} \lim_{n\to\infty} P^\infty(\tau^n(y,\omega) < y_0) > 0.$$

*In that case the unique member of $\mathcal{C}(J)$ is given by $F = \lim_{n\to\infty} F_n$, where $(F_n)_{n\in\mathbb{N}}$ is defined by* (2.3) *with $F_0 = \chi_{[y_0,+\infty)}$.*

Unfortunately, the two necessary and sufficient conditions appearing in assertions (ii) and (iii) are hard to verify in concrete situations. Nevertheless, there are clear conditions which are necessary for $\mathcal{C}(J) \ne \emptyset$.

THEOREM 3.3. *Assume $J = (x_0, y_0)$ is a component of $\mathbb{R} \setminus \bigcup \mathcal{S}$.*

(i) *If $x_0 \in \mathbb{R}$, $y_0 = +\infty$ and $\mathcal{C}(J) \neq \emptyset$, then almost all functions $\tau(\cdot, \omega)$ are unbounded from above.*

(ii) *If $x_0 = -\infty$, $y_0 \in \mathbb{R}$ and $\mathcal{C}(J) \neq \emptyset$, then almost all functions $\tau(\cdot, \omega)$ are unbounded from below.*

*Proof.* Since the proofs of (i) and (ii) are similar, we only show (i).

Suppose that there exists $M \in \mathbb{R}$ such that $\alpha = P(\tau(\cdot, \omega) \le M) > 0$. Define a sequence $(\xi_n)_{n\ge 0}$ by the formula

$$\xi_n = \inf\{x \in \mathbb{R} : F_n(x) = 1\}, \tag{3.1}$$

where $(F_n)_{n\in\mathbb{N}}$ is given by (2.3) with $F_0 = \chi_{[x_0,+\infty)}$ (we put $\xi_n = +\infty$ if the underlying set is empty). By Lemma 2.2(i), $F_0$ satisfies (2.1), hence Lemma 2.1 implies that the sequence $(F_n)_{n\in\mathbb{N}}$ is decreasing. Therefore $(\xi_n)_{n\ge 0}$ is increasing, and since $\xi_0 = x_0$, it follows that $x_0 \le \xi_{n-1} \le \xi_n$ for $n \in \mathbb{N}$.

We will show that

$$\xi_n = \inf\{x \in \mathbb{R} : P(\tau(x, \omega) \ge \xi_{n-1}) = 1\} \quad \text{for } n \in \mathbb{N}. \tag{3.2}$$

The case $\xi_n = +\infty$ is trivial, since then formula (2.3) implies that there is no $x \in \mathbb{R}$ with $P(\tau(x, \omega) \ge \xi_{n-1}) = 1$. Thus we may assume $\xi_n < +\infty$, and consequently $\xi_{n-1} < +\infty$. Denote the right-hand side of (3.2) by $\eta_n$. The equality

$$F_n(x) = \int_\Omega F_{n-1}(\tau(x, \omega))\, P(d\omega) = 1, \tag{3.3}$$

jointly with (3.1), implies that $P(\tau(x, \omega) \ge \xi_{n-1}) = 1$ for every $x > \xi_n$. Therefore $P(\tau(\xi_n, \omega) \ge \xi_{n-1}) = 1$, i.e. $\xi_n \ge \eta_n$. For the converse inequality fix any $x \in \mathbb{R}$ with $P(\tau(x, \omega) \ge \xi_{n-1}) = 1$. Then (3.3) yields $F_n(x) = 1$, hence $\xi_n \le x$. This shows that $\xi_n \le \eta_n$.

Let $\xi = \lim_{n\to\infty} \xi_n$. If $\xi$ were finite then, in view of (3.2), we would have $\tau(\xi, \omega) \ge \xi$ for almost all $\omega \in \Omega$, which means that $\xi$ is a left end-point of some $\tau$-invariant interval. This, however, is impossible, since $\xi \ge x_0$ and $(x_0, +\infty) \subset \mathbb{R} \setminus \bigcup \mathcal{S}$. Therefore $\xi = +\infty$, hence there is $n \in \mathbb{N}$ such that $\xi_n > M$. In view of (3.1), we have $F_n(M) < 1$. For every $x \in \mathbb{R}$ we thus obtain

$$\begin{aligned} F_{n+1}(x) = \int_\Omega F_n(\tau(x, \omega))\, P(d\omega) &= \int_{\tau(x,\omega)\le M} + \int_{\tau(x,\omega)>M} \\ &\le \alpha F_n(M) + (1-\alpha) < 1. \end{aligned}$$

Consequently, $F_{n+1}(+\infty) < 1$. Hence we also have

$$\lim_{x\to+\infty} \lim_{n\to\infty} F_n(x) < 1,$$

which in the light of Lemma 2.2(i) and Theorem 3.2(ii) implies $\mathcal{C}(J) = \emptyset$. Thus the proof has been completed. ∎

**4. The case $\mathcal{S}=\emptyset$.** Throughout this section we assume $\mathcal{S}=\emptyset$.

LEMMA 4.1. *If $F\in\mathcal{C}(\mathbb{R})$, then $F(\mathbb{R})\subset(0,1)$.*

*Proof.* Put $x_0=\inf\{x\in\mathbb{R}:F(x)=1\}$ and suppose $x_0\in\mathbb{R}$. Then it follows from

$$1=F(x_0)=\int_{\Omega}F(\tau(x_0,\omega))\,P(d\omega)$$

that $F(\tau(x_0,\omega))=1$ for almost all $\omega\in\Omega$, thus $\tau(x_0,\omega)\ge x_0$ for almost all $\omega\in\Omega$, which contradicts $\mathcal{S}=\emptyset$. Similarly we can prove that $F(x)>0$ for every $x\in\mathbb{R}$. ∎

LEMMA 4.2. *If $x_0\in\mathbb{R}$, $F\in\mathcal{C}(\mathbb{R})$ and $(F_n)_{n\in\mathbb{N}}$ is given by* (2.3) *with $F_0=\chi_{[x_0,+\infty)}$, then*

$$\frac{F(x)-F(x_0)}{1-F(x_0)}\le F_n(x)\le\frac{F(x)}{F(x_0)}\quad\textit{for } n\in\mathbb{N},\ x\in\mathbb{R}. \tag{4.1}$$

*Proof.* Iterating equation (1.1) and using Lemma 2.2(i) we get

$$\begin{aligned}F(x)&=\int_{\Omega^\infty}F(\tau^n(x,\omega))P^\infty(d\omega)=\int_{\tau^n(x_0,\omega)\ge x_0}+\int_{\tau^n(x_0,\omega)<x_0}\\&\le P^\infty(\tau^n(x,\omega)\ge x_0)+F(x_0)P^\infty(\tau^n(x,\omega)<x_0)\\&=F_n(x)+F(x_0)(1-F_n(x))\end{aligned}$$

for $n\in\mathbb{N}$, $x\in\mathbb{R}$. By Lemma 4.1 we obtain the first estimate in (4.1). To show the second one we write

$$\begin{aligned}F(x)&\ge\int_{\tau^n(x_0,\omega)\ge x_0}F(\tau^n(x,\omega))\,P^\infty(d\omega)\\&\ge F(x_0)P^\infty(\tau^n(x,\omega)\ge x_0)=F(x_0)F_n(x)\end{aligned}$$

for $n\in\mathbb{N}$, $x\in\mathbb{R}$. ∎

LEMMA 4.3. *If $x_0\in\mathbb{R}$ and $(F_n)_{n\in\mathbb{N}}$ is given by* (2.3) *with $F_0=\chi_{[x_0,+\infty)}$, then the function $\underline{F}\colon\mathbb{R}\to[0,1]$ defined by*

$$\underline{F}(x)=\liminf_{n\to\infty}F_n(x)$$

*is increasing and satisfies*

$$\underline{F}(x)\ge\int_{\Omega}\underline{F}(\tau(x,\omega))\,P(d\omega), \tag{4.2}$$

*whereas the function $\overline{F}\colon\mathbb{R}\to[0,1]$ defined by*

$$\overline{F}(x)=\limsup_{n\to\infty}F_n(x)$$

*is increasing and satisfies*

$$\overline{F}(x)\le\int_{\Omega}\overline{F}(\tau(x,\omega))\,P(d\omega). \tag{4.3}$$

*Proof.* It is obvious that $\underline{F}$ and $\overline{F}$ are increasing. Inequalities (4.2) and (4.3) immediately follow from the Fatou lemma applied to the sequences $(F_n)_{n\in\mathbb{N}}$ and $(1-F_n)_{n\in\mathbb{N}}$, respectively. ∎

From now on $x_0 \in \mathbb{R}$ is fixed and $\underline{F}$, $\overline{F}$ stand for the two functions defined in Lemma 4.3.

LEMMA 4.4. *In each of the following cases:*

(a) $(F_n)_{n\in\mathbb{N}}$ *is given by* (2.3) *with* $F_0 = \underline{F}$ *and* $F = \lim_{n\to\infty} F_n$*;*
(b) $(F_n)_{n\in\mathbb{N}}$ *is given by* (2.3) *with* $F_0 = \overline{F}$ *and* $F = \lim_{n\to\infty} F_n$*,*

*we have:*

(i) *If* $F$ *is non-constant, then* $\mathcal{C}(\mathbb{R}) = \{F\}$*.*
(ii) *If* $F$ *is constant, then* $\mathcal{C}(\mathbb{R}) = \emptyset$*.*

*Proof.* Both in case (a) and (b), Lemma 4.3, jointly with Lemma 2.1, implies that $F\colon \mathbb{R} \to [0,1]$ is a well-defined function such that (2.4) holds provided $F$ is non-constant. Now it is enough to show that $\mathcal{C}(\mathbb{R}) \neq \emptyset$ implies both $F(-\infty) = 0$ and $F(+\infty) = 1$.

Let $G \in \mathcal{C}(\mathbb{R})$. By Lemma 4.2,

$$\frac{G(x) - G(x_0)}{1 - G(x_0)} \le \underline{F}(x) \le \overline{F}(x) \le \frac{G(x)}{G(x_0)} \quad \text{for } x \in \mathbb{R}.$$

Substituting $\tau(x,\omega)$ for $x$, integrating both sides and applying a simple induction we arrive at the inequalities

$$\frac{G(x) - G(x_0)}{1 - G(x_0)} \le F_n(x) \le \frac{G(x)}{G(x_0)} \quad \text{for } x \in \mathbb{R},$$

where $F_n$ may be defined either as in case (a) or as in case (b). In both cases we may pass to the limits as $n \to \infty$ and then $x \to \pm\infty$ to obtain $F(-\infty) = 0$ and $F(+\infty) = 1$. ∎

As a consequence of Lemma 4.4 we obtain the following theorem.

THEOREM 4.5. *We have* $\mathcal{C}(\mathbb{R}) \neq \emptyset$ *if and only if the limit*

$$F(x) = \lim_{n\to\infty} P^{\infty}(\tau^n(x,\omega) > x_0) \tag{4.4}$$

*exists for every* $x \in \mathbb{R}$ *and the function* $F$ *is a probability distribution. In that case* $\mathcal{C}(J) = \{F\}$*.*

*Proof.* Assume first that formula (4.4) defines a p.d. function $F$ : $\mathbb{R} \to [0,1]$. Then, by virtue of Lemma 2.2(i), we infer that $F = \lim_{n\to\infty} F_n$, where $(F_n)_{n\in\mathbb{N}}$ is given by (2.3) with $F_0 = \chi_{[x_0,+\infty)}$. Thus it follows immediately from (2.3) that $F$ is a solution of (1.1). Hence $\mathcal{C}(\mathbb{R}) = \{F\}$.

Now, assume that there exists a function $G \in \mathcal{C}(\mathbb{R})$ and let us distinguish cases (a) and (b) of Lemma 4.4. By using Lemmas 4.3 and 2.2(i), we obtain what follows.

CASE (a). For any $x \in \mathbb{R}$ we have

$$F_1(x) = \int_{\Omega} \underline{F}(\tau(x,\omega))\, P(d\omega) \le \underline{F}(x) = \liminf_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0).$$

Since Lemmas 2.1 and 4.3 imply that the sequence $(F_n)_{n\in\mathbb{N}}$ is decreasing, we infer that

$$F_m(x) \le \liminf_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0) \quad \text{for } m \in \mathbb{N},\ x \in \mathbb{R}. \tag{4.5}$$

By Lemma 4.4, $\lim_{m\to\infty} F_m = G$ and hence (4.5) yields

$$G(x) \le \liminf_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0) \quad \text{for } x \in \mathbb{R}. \tag{4.6}$$

CASE (b). For any $x \in \mathbb{R}$ we have

$$F_1(x) = \int_{\Omega} \overline{F}(\tau(x,\omega))\, P(d\omega) \ge \overline{F}(x) = \limsup_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0).$$

Since Lemmas 2.1 and 4.3 imply that the sequence $(F_n)_{n\in\mathbb{N}}$ is increasing, we infer that

$$F_m(x) \ge \limsup_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0) \quad \text{for } m \in \mathbb{N}, x \in \mathbb{R}. \tag{4.7}$$

By Lemma 4.4, $\lim_{m\to\infty} F_m = G$ and hence (4.7) yields

$$G(x) \ge \limsup_{n\to\infty} P^{\infty}(\tau^n(x,\omega) \ge x_0) \quad \text{for } x \in \mathbb{R}. \tag{4.8}$$

Inequalities (4.6) and (4.8) show that the limit $F(x)$ given by (4.4) exists and for every $x \in \mathbb{R}$ we have $F(x) = G(x)$, which completes the proof. ∎

The last result, which is analogous to Theorem 3.3, gives a necessary condition for $\mathcal{C}(\mathbb{R}) \ne \emptyset$.

THEOREM 4.6. *If $\mathcal{C}(\mathbb{R}) \ne \emptyset$, then almost all functions $\tau(\cdot,\omega)$ are unbounded from below and from above.*

*Proof.* Suppose first that there exists $M \in \mathbb{R}$ such that $P(\tau(\cdot,\omega) \le M) > 0$. Let $(F_n)_{n\in\mathbb{N}}$ and $F$ be as in Lemma 4.4(a). Define a sequence $(\xi_n)_{n\ge0}$ by formula (3.1). If $\xi_0 = -\infty$ then obviously we have $\underline{F} = F = 1$, hence Lemma 4.4(ii) implies $\mathcal{C}(\mathbb{R}) = \emptyset$. Thus we may assume $\xi_0 > -\infty$. In view of inequality (4.2) and Lemma 2.1, the sequence $(F_n)_{n\in\mathbb{N}}$ is decreasing. As in the proof of Theorem 3.3, we deduce that

$$\xi_n = \inf\{x \in \mathbb{R} : P(\tau(x,\omega) \ge \xi_{n-1}) = 1\} \ge \xi_{n-1} \quad \text{for } n \in \mathbb{N},$$

and we may consider $\xi = \lim_{n\to\infty} \xi_n$. If $\xi$ were finite then we would have $\tau(\xi,\omega) \ge \xi$ for almost all $\omega \in \Omega$, which contradicts the fact that $\mathcal{S} = \emptyset$. Therefore $\xi = +\infty$ and, by the argument of the proof of Theorem 3.3, we infer that $F(+\infty) < 1$. Hence, in view of Lemma 4.4, we must have $\mathcal{C}(\mathbb{R}) = \emptyset$.

Now, suppose that for some $m \in \mathbb{R}$ we have $\alpha = P(\tau(\cdot,\omega) \ge m) > 0$. Let $(F_n)_{n\in\mathbb{N}}$ and $F$ be as in Lemma 4.4(b). Define a sequence $(\nu_n)_{n\ge0}$ by the formula

$$\nu_n = \sup\{x \in \mathbb{R} : F_n(x) = 0\} \tag{4.9}$$

($\nu_n = -\infty$ if the underlying set is empty). If $\nu_0 = +\infty$ then obviously $\overline{F} = F = 0$, hence Lemma 4.4(ii) implies $\mathcal{C}(\mathbb{R}) = \emptyset$. Thus we may assume $\nu_0 < +\infty$.

In view of inequality (4.3) and Lemma 2.1, the sequence $(F_n)_{n\in\mathbb{N}}$ is increasing. Therefore $(\nu_n)_{n\geq 0}$ is decreasing: $\nu_n \leq \nu_{n-1} \leq \nu_0 < +\infty$ for $n \in \mathbb{N}$. Just as above we conclude that $\lim_{n\to\infty} \nu_n = -\infty$, hence $\nu_n < m$ for some $n \in \mathbb{N}$. By (4.9), we have $F_n(m) > 0$. For every $x \in \mathbb{R}$ we thus obtain

$$F_{n+1}(x) = \int_{\Omega} F_n(\tau(x,\omega))\, P(d\omega) \geq \int_{\tau(x,\omega)\geq m} \geq \alpha F_n(m) > 0.$$

Consequently, $F_n(-\infty) > 0$ and also $F(-\infty) > 0$. In view of Lemma 4.4, we infer that $\mathcal{C}(\mathbb{R}) = \emptyset$. ∎

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