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Region of variability for functions with positive real part

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Abstract. For $\gamma \in \mathbb{C}$ such that $|\gamma| < \pi/2$ and $0 \le \beta < 1$, let $\mathcal{P}_{\gamma,\beta}$ denote the class of all analytic functions P in the unit disk $\mathbb{D}$ with $P(0) = 1$ and

$$\mathrm{Re}(e^{i\gamma}P(z)) > \beta\cos\gamma \quad \text{in } \mathbb{D}.$$

For any fixed $z_0 \in \mathbb{D}$ and $\lambda \in \overline{\mathbb{D}}$, we shall determine the region of variability $V_{\mathcal{P}}(z_0,\lambda)$ for $\int_0^{z_0} P(\zeta)\,d\zeta$ when P ranges over the class

$$\mathcal{P}(\lambda) = \{P \in \mathcal{P}_{\gamma,\beta} : P'(0) = 2(1-\beta)\lambda e^{-i\gamma}\cos\gamma\}.$$

As a consequence, we present the region of variability for some subclasses of univalent functions. We also graphically illustrate the region of variability for several sets of parameters.

1. Introduction. We denote by $\mathcal{H}$ the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. We consider the subclass of functions $\phi \in \mathcal{H}$ with $\phi(0) = 0 = \phi'(0) - 1$ such that ϕ maps $\mathbb{D}$ univalently onto a domain that is starlike (with respect to the origin). That is, $t\phi(z) \in \phi(\mathbb{D})$ for each $t \in [0,1]$. We denote the class of such functions by $\mathcal{S}^*$. Analytically, each $\phi \in \mathcal{S}^*$ is characterized by the condition

$$\mathrm{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Functions in $\mathcal{S}^*$ are referred to as *starlike functions.* A function $\phi \in \mathcal{H}$ with $\phi(0) = 0 = \phi'(0) - 1$ is said to belong to $\mathcal{C}$ if and only if $\phi(\mathbb{D})$ is a convex domain. It is well-known that $\phi \in \mathcal{C}$ if and only if $z\phi' \in \mathcal{S}^*$. Functions in $\mathcal{C}$ are referred to as *convex functions.*

Let $\mathcal{P}_{\gamma,\beta}$ denote the class of functions $P \in \mathcal{H}$ with $P(0) = 1$ and

$$\mathrm{Re}(e^{i\gamma}P(z)) > \beta\cos\gamma \quad \text{in } \mathbb{D}$$

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for some β with $\beta < 1$ and $\gamma \in \mathbb{C}$ with $|\gamma| < \pi/2$. Let $\mathcal{A}$ denote the class of functions f in $\mathcal{H}$ such that $f(0) = 0 = f'(0) - 1$. When $P(z) = zf'(z)/f(z)$ and $\beta = 0$, the class $\mathcal{P}_{\gamma,\beta}$ becomes

$$\mathcal{S}^{\gamma}(0) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(e^{i\gamma} \frac{zf'(z)}{f(z)} \right) > 0 \text{ in } \mathbb{D} \right\}$$

for some γ with $|\gamma| < \pi/2$. Functions in $\mathcal{S}^{\gamma}(0)$ are known to be univalent in $\mathbb{D}$ and $\mathcal{S}^{0}(0) \equiv \mathcal{S}^{*}$. Functions in $\mathcal{S}^{\gamma}(0)$ are called *spirallike functions* (see [S]).

2. Preliminary investigation about the class $\mathcal{P}_{\gamma,\beta}$. Herglotz representation for analytic functions with positive real part in $\mathbb{D}$ shows that if $P \in \mathcal{P}_{\gamma,\beta}$, then there exists a unique positive unit measure μ on $(-\pi, \pi]$ such that

$$P(z) = \int_{-\pi}^{\pi} \frac{1 + [1 - 2\beta e^{-i\gamma} \cos\gamma] z e^{-it}}{1 - z e^{-it}} \, d\mu(t).$$

Let $\mathcal{B}_0$ be the class of analytic functions ω in $\mathbb{D}$ such that $|\omega(z)| < 1$ in $\mathbb{D}$ and $\omega(0) = 0$. Then it is a simple exercise to see that for each $P \in \mathcal{P}_{\gamma,\beta}$ there exists an $\omega_P \in \mathcal{B}_0$ such that

$$\omega_P(z) = \frac{e^{i\gamma} P(z) - e^{i\gamma}}{e^{i\gamma} P(z) - (2\beta \cos\gamma - e^{-i\gamma})}, \qquad z \in \mathbb{D}, \tag{2.1}$$

and conversely. Clearly, we have

$$P'(0) = 2e^{-i\gamma} \omega_P'(0)(1 - \beta) \cos\gamma.$$

Suppose that $P \in \mathcal{P}_{\gamma,\beta}$. Then, because $|\omega_P'(0)| \le 1$, by the classical Schwarz lemma (see for example [Di, Du, Po2, PS]) we may let

$$P'(0) = 2\lambda e^{-i\gamma} (1 - \beta) \cos\gamma$$

for some $\lambda \in \overline{\mathbb{D}}$, with $\omega_P'(0) = \lambda$. Using (2.1), one can compute

$$\frac{\omega_P''(0)}{2} = \frac{e^{i\gamma} P''(0)}{4(1 - \beta)\cos\gamma} - \lambda^2. \tag{2.2}$$

Also if we let

$$g(z) = \begin{cases} \dfrac{\omega_P(z)/z - \lambda}{1 - \overline{\lambda}\omega_P(z)/z} & \text{for } |\lambda| < 1, \\ 0 & \text{for } |\lambda| = 1, \end{cases}$$

then we see that

$$g'(0) = \begin{cases} \dfrac{1}{1 - |\lambda|^2} \left(\dfrac{\omega_P(z)}{z} \right)' \Bigg|_{z=0} = \dfrac{1}{1 - |\lambda|^2} \dfrac{\omega_P''(0)}{2} & \text{for } |\lambda| < 1, \\ 0 & \text{for } |\lambda| = 1. \end{cases}$$

By the Schwarz lemma, $|g(z)| \le |z|$ and $|g'(0)| \le 1$. Equality holds in both the cases if and only if $g(z) = e^{i\alpha} z$ for some $\alpha \in \mathbb{R}$. The condition $|g'(0)| \le 1$ shows that there exists an $a \in \overline{\mathbb{D}}$ such that $g'(0) = a$.

In view of (2.2) we may represent $P''(0)$ as

$$P''(0) = 4(1-\beta)[(1-|\lambda|^2)a + \lambda^2]e^{-i\gamma}\cos\gamma \tag{2.3}$$

for some $a \in \overline{\mathbb{D}}$. Consequently, for $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ and $z_0 \in \mathbb{D}$ fixed, it is natural to introduce (for convenience with the notation $\mathcal{P}(\lambda)$ instead of $\mathcal{P}_{\gamma,\beta}(\lambda)$)

$$\mathcal{P}(\lambda) = \{P \in \mathcal{P}_{\gamma,\beta} : P'(0) = 2(1-\beta)e^{-i\gamma}\lambda\cos\gamma\},$$

$$V_{\mathcal{P}}(z_0, \lambda) = \Big\{ \int_0^{z_0} P(\zeta)\, d\zeta : P \in \mathcal{P}(\lambda) \Big\}.$$

Obviously, each $f \in \mathcal{P}(\lambda)$ has to satisfy condition (2.3) for some $a \in \overline{\mathbb{D}}$ and so we do not need to include it in the definition of $\mathcal{P}(\lambda)$.

For each fixed $z_0 \in \mathbb{D}$, using extreme function theory, it has been shown by Grunsky [Du, Theorem 10.6] that the region of variability of

$$V(z_0) = \left\{ \log \frac{f(z_0)}{z_0} : f \in \mathcal{S} \right\}$$

is precisely a closed disk, where $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$. It is also well-known that the region of variability

$$V(z_0) = \{\log \phi'(z_0) : \phi \in \mathcal{C}\}$$

is the set $\{\log(1-z)^{-2} : |z| \le |z_0|\}$. Several authors have studied region of variability problems for various subclasses of univalent functions in $\mathcal{H}$; see [Pa, Pi, PV, PVV1, PVV2, PVY1, PVY2, Y1, Y2].

The main aim of this paper is to determine the region of variability of $V_{\mathcal{P}}(z_0, \lambda)$ for $\int_0^{z_0} P(\zeta)\, d\zeta$ when P ranges over the class $\mathcal{P}(\lambda)$. In Section 3, we present some basic properties of $V_{\mathcal{P}}(z_0, \lambda)$, whereas in Section 4, we investigate the growth condition for functions in $\mathcal{P}(\lambda)$. The precise geometric description of the set $V_{\mathcal{P}}(z_0, \lambda)$ is established in Theorem 5.1 in Section 5. Two interesting special cases are presented in Section 6. Finally, in Section 7, we graphically represent the region of variability for several sets of parameters.

3. Basic properties of $V_{\mathcal{P}}(z_0, \lambda)$. For a positive integer p, let $(\mathcal{S}^*)^p = \{f = f_0^p : f_0 \in \mathcal{S}^*\}$. A sufficient condition (see [Y1]) for an analytic function f in $\mathbb{D}$ with $f(z) = z^p + \cdots$ to be in $(\mathcal{S}^*)^p$ is

$$\operatorname{Re}\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, \qquad z \in \mathbb{D}. \tag{3.1}$$

This fact will be used in the following result.

Proposition 3.1.

(1) $V_{\mathcal{P}}(z_0,\lambda)$ *is a compact subset of* $\mathbb{C}$.
(2) $V_{\mathcal{P}}(z_0,\lambda)$ *is a convex subset of* $\mathbb{C}$.
(3) *For* $|\lambda|=1$ *or* $z_0=0$,

$$V_{\mathcal{P}}(z_0,\lambda)=\left\{z_0-2(1-\beta)e^{-i\gamma}(\cos\gamma)\left(z_0+\frac{1}{\lambda}\log(1-\lambda z_0)\right)\right\}. \tag{3.2}$$

(4) *For* $|\lambda|<1$ *and* $z_0\in\mathbb{D}\setminus\{0\}$, $V_{\mathcal{P}}(z_0,\lambda)$ *has*

$$z_0-2(1-\beta)e^{-i\gamma}(\cos\gamma)\left(z_0+\frac{1}{\lambda}\log(1-\lambda z_0)\right)$$

as an interior point.

Proof. (1) Since $\mathcal{P}(\lambda)$ is a compact subset of $\mathcal{H}$, it follows that $V_{\mathcal{P}}(z_0,\lambda)$ is also compact.

(2) If $p_1,p_2\in\mathcal{P}(\lambda)$ and $0\le t\le 1$, then the function

$$P_t(z)=(1-t)p_1(z)+tp_2(z)$$

is evidently in $\mathcal{P}(\lambda)$. Also, because of the representation of P_t, we easily see that the set $V_{\mathcal{P}}(z_0,\lambda)$ is convex.

(3) If $z_0=0$, (3.2) trivially holds. If $|\lambda|=1$, then from our earlier observation $\omega_P(z)=\lambda z$, and so $P\in\mathcal{P}(\lambda)$ defined by (2.1) takes the form

$$P(z)=\frac{1+\lambda z[2(1-\beta)e^{-i\gamma}\cos\gamma-1]}{1-\lambda z},$$

or equivalently,

$$P(z)=1-2(1-\beta)e^{-i\gamma}(\cos\gamma)\left(1-\frac{1}{1-\lambda z}\right).$$

Consequently,

$$V_{\mathcal{P}}(z_0,\lambda)=\left\{z_0-2(1-\beta)e^{-i\gamma}(\cos\gamma)\left(z_0+\frac{1}{\lambda}\log(1-\lambda z_0)\right)\right\}.$$

(4) For $|\lambda|<1$ and $a\in\overline{\mathbb{D}}$, we let

$$\delta(z,\lambda)=\frac{z+\lambda}{1+\overline{\lambda}z}, \tag{3.3}$$

and in order to get the extremal function in $\mathcal{P}(\lambda)$, we define

$$H_{a,\lambda}(z)=1+2(1-\beta)e^{-i\gamma}\cos\gamma\,\frac{\delta(az,\lambda)z}{1-\delta(az,\lambda)z}. \tag{3.4}$$

Clearly $H_{a,\lambda}(0)=1$. Since $\delta(az,\lambda)$ lies in the unit disk $\mathbb{D}$ and $\varphi(w)=w/(1-w)$ maps $|w|<1$ onto $\operatorname{Re}\varphi(w)>-1/2$, we obtain

$$\operatorname{Re}(e^{i\gamma}H_{a,\lambda}(z))>\beta\cos\gamma \quad \text{in } \mathbb{D}.$$

Also, from (3.4), we have the normalization condition

$$H'_{a,\lambda}(0) = 2(1-\beta)e^{-i\gamma}\lambda\cos\gamma.$$

Thus, $H_{a,\lambda} \in \mathcal{P}(\lambda)$. We observe that

$$\omega_{H_{a,\lambda}}(z) = z\delta(az,\lambda). \tag{3.5}$$

We claim that the mapping

$$\mathbb{D} \ni a \mapsto \int_0^{z_0} H_{a,\lambda}(\zeta)\, d\zeta$$

is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{D}\backslash\{0\}$ and $\lambda \in \mathbb{D}$. To see this, we introduce

$$h(z) = \frac{3e^{i\gamma}}{2(1-\beta)(1-|\lambda|^2)\cos\gamma}\,\frac{\partial}{\partial a}\left\{\int_0^z H_{a,\lambda}(\zeta)\, d\zeta\right\}\Bigg|_{a=0}$$

so that

$$h(z) = \frac{3}{1-|\lambda|^2}\,\frac{\partial}{\partial a}\left\{\int_0^z \frac{\delta(a\zeta,\lambda)\zeta}{1-\delta(a\zeta,\lambda)\zeta}\, d\zeta\right\}\Bigg|_{a=0}.$$

A computation gives

$$h(z) = 3\frac{\partial}{\partial a}\left\{\int_0^z \frac{\zeta^2}{(1-\lambda\zeta)^2}\,\frac{d\zeta}{(1-a\delta(a\zeta,\lambda)\zeta)^2}\right\}\Bigg|_{a=0},$$

which clearly implies

$$h(z) = 3\int_0^z \frac{\zeta^2}{(1-\lambda\zeta)^2}\, d\zeta = z^3 + \cdots,$$

from which it is easy to see that

$$\mathrm{Re}\left\{\frac{zh''(z)}{h'(z)}\right\} = 2\,\mathrm{Re}\left\{\frac{1}{1-\lambda z}\right\} > \frac{2}{1+|\lambda|} \geq 1, \qquad z \in \mathbb{D}.$$

By (3.1), there exists a function $h_0 \in \mathcal{S}^*$ with $h = h_0^3$. The univalence of h_0 together with the condition $h_0(0) = 0$ implies that $h(z_0) \neq 0$ for $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto \int_0^{z_0} H_{a,\lambda}(\zeta)\, d\zeta$ is a non-constant analytic function of a, and hence it is an open mapping. Thus, $V_{\mathcal{P}}(z_0,\lambda)$ contains the open set

$$\left\{\int_0^{z_0} H_{a,\lambda}(\zeta)\, d\zeta : |a| < 1\right\}.$$

In particular,

$$\int_0^{z_0} H_{0,\lambda}(\zeta)\, d\zeta = z_0 - 2(1-\beta)e^{-i\gamma}(\cos\gamma)\left(z_0 + \frac{1}{\lambda}\log(1-\lambda z_0)\right)$$

is an interior point of

$$\left\{ \int_0^{z_0} H_{a,\lambda}(\zeta)\, d\zeta : a \in \mathbb{D} \right\} \subset V_{\mathcal{P}}(z_0, \lambda). \blacksquare$$

We remark that, since $V_{\mathcal{P}}(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V_{\mathcal{P}}(z_0, \lambda)$ is a Jordan curve and $V_{\mathcal{P}}(z_0, \lambda)$ is the union of $\partial V_{\mathcal{P}}(z_0, \lambda)$ and its inner domain.

4. Growth condition for functions in $\mathcal{P}(\lambda)$

PROPOSITION 4.1. *For $P \in \mathcal{P}(\lambda)$ with $\lambda \in \mathbb{D}$, we have*

$$|P(z) - c(z, \lambda)| \le r(z, \lambda), \qquad z \in \mathbb{D}, \tag{4.1}$$

where

$$c(z, \lambda) = \frac{(1 + \lambda z(e^{-i\gamma} - 2\beta \cos \gamma)e^{-i\gamma})(1 - \overline{\lambda}\overline{z})}{(1 - |z|^2)(1 + |z|^2 - 2\,\mathrm{Re}(\lambda z))} + \frac{|z|^2(\overline{z} - \lambda)(\overline{\lambda} + z(e^{-i\gamma} - 2\beta \cos \gamma)e^{-i\gamma})}{(1 - |z|^2)(1 + |z|^2 - 2\,\mathrm{Re}(\lambda z))},$$

$$r(z, \lambda) = \frac{2(1 - |\lambda|^2)(1 - \beta)|z|^2 \cos \gamma}{(1 - |z|^2)(1 + |z|^2 - 2\,\mathrm{Re}(\lambda z))}.$$

For each $z \in \mathbb{D} \backslash \{0\}$, equality holds if and only if $P = H_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$.

Proof. Let $P \in \mathcal{P}(\lambda)$. Then there exists $\omega_P \in \mathcal{B}_0$ satisfying (2.1). As observed in Section 2 ($|g(z)| \le |z|$), we have

$$\left| \frac{\omega_P(z)/z - \lambda}{1 - \overline{\lambda}\omega_P(z)/z} \right| \le |z|, \qquad z \in \mathbb{D}. \tag{4.2}$$

From (2.1) this is equivalent to

$$\left| \frac{P(z) - A(z, \lambda)}{P(z) + B(z, \lambda)} \right| \le |z|\, |\tau(z, \lambda)|, \tag{4.3}$$

where

$$\begin{cases} A(z, \lambda) = \dfrac{1 + e^{-i\gamma}\lambda z(e^{-i\gamma} - 2\beta \cos \gamma)}{1 - \lambda z}, \\[2ex] B(z, \lambda) = \dfrac{\overline{\lambda} + e^{-i\gamma} z(e^{-i\gamma} - 2\beta \cos \gamma)}{z - \overline{\lambda}}, \\[2ex] \tau(z, \lambda) = \dfrac{z - \overline{\lambda}}{1 - \lambda z}. \end{cases} \tag{4.4}$$

A simple calculation shows that the inequality (4.3) is equivalent to

$$\left|P(z)-\frac{A(z,\lambda)+|z|^2\,|\tau(z,\lambda)|^2B(z,\lambda)}{1-|z|^2\,|\tau(z,\lambda)|^2}\right| \le \frac{|z|\,|\tau(z,\lambda)|\,|A(z,\lambda)+B(z,\lambda)|}{1-|z|^2\,|\tau(z,\lambda)|^2}. \tag{4.5}$$

Using (4.4) we can easily see that

$$1-|z|^2\,|\tau(z,\lambda)|^2=\frac{(1-|z|^2)(1+|z|^2-2\operatorname{Re}(\lambda z))}{|1-\lambda z|^2},$$

$$A(z,\lambda)+B(z,\lambda)=\frac{2(1-|\lambda|^2)(1-\beta)(\cos\gamma)e^{-i\gamma}z}{(1-\lambda z)(z-\overline{\lambda})}$$

and

$$\begin{aligned}A(z,\lambda)&+|z|^2|\tau(z,\lambda)|^2B(z,\lambda)\\ &=\frac{(1+\lambda z(e^{-i\gamma}-2\beta\cos\gamma)e^{-i\gamma})(1-\overline{\lambda}\overline{z})}{|1-\lambda z|^2}\\ &\quad+\frac{|z|^2(\overline{z}-\lambda)(\overline{\lambda}+z(e^{-i\gamma}-2\beta\cos\gamma)e^{-i\gamma})}{|1-\lambda z|^2}.\end{aligned}$$

Thus, by a simple computation, we see that

$$\frac{A(z,\lambda)+|z|^2|\tau(z,\lambda)|^2B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}=c(z,\lambda),$$

$$\frac{|z|\,|\tau(z,\lambda)|\,|A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}=r(z,\lambda).$$

Now the inequality (4.1) follows from these equalities and (4.5).

It is easy to see that equality occurs in (4.1) for a $z\in\mathbb{D}$ when $P=H_{e^{i\theta},\lambda}$ for some $\theta\in\mathbb{R}$. Conversely if equality occurs in (4.1) for some $z\in\mathbb{D}\setminus\{0\}$, then equality must hold in (4.2). Thus from the Schwarz lemma there exists a $\theta\in\mathbb{R}$ such that $\omega_P(z)=z\delta(e^{i\theta}z,\lambda)$ for all $z\in\mathbb{D}$. This implies $P=H_{e^{i\theta},\lambda}$. ■

The choice of $\lambda=0$ gives the following result which may deserve a special mention.

Corollary 4.2. *For $P\in\mathcal{P}(0)$ we have*

$$\left|P(z)-\frac{1+(1-2\beta)|z|^4}{1-|z|^4}\right|\le\frac{2(1-\beta)|z|^2}{1-|z|^4},\qquad z\in\mathbb{D}. \tag{4.6}$$

For each $z\in\mathbb{D}\setminus\{0\}$, equality holds if and only if $P=H_{e^{i\theta},0}$ for some $\theta\in\mathbb{R}$.

Corollary 4.3. *Let $\gamma : z(t)$, $0 \le t \le 1$, be a C^1-curve in $\mathbb{D}$ with $z(0) = 0$ and $z(1) = z_0$. Then*

$$V_{\mathcal{P}}(z_0, \lambda) \subset \{w \in \mathbb{C} : |w - C(\lambda, \gamma)| \le R(\lambda, \gamma)\},$$

where

$$C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda) z'(t)\, dt \quad \text{and} \quad R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda)|z'(t)|\, dt.$$

Proof. The proof follows as in [PVV2]. ∎

For the proof of our next result, we need the following lemma.

Lemma 4.4. *For $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{D}$, the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta^2}{\{1 + (\overline{\lambda} e^{i\theta} - \lambda)\zeta - e^{i\theta}\zeta^2\}^2}\, d\zeta, \qquad z \in \mathbb{D},$$

has a zero of order three at the origin and no zeros elsewhere in $\mathbb{D}$. Furthermore there exists a starlike univalent function G_0 in $\mathbb{D}$ such that $G = \frac{1}{3} e^{i\theta} G_0^3$ and $G_0(0) = G_0'(0) - 1 = 0$.

Proof. For a proof, we refer to [PVV2, Lemma 3.4] with $\beta = 1$ there. ∎

Proposition 4.5. *Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have*

$$\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta \in \partial V_{\mathcal{P}}(z_0, \lambda).$$

Furthermore if $\int_0^{z_0} P(\zeta)\, d\zeta = \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta$ for some $P \in \mathcal{P}(\lambda)$ and $\theta \in (-\pi, \pi]$, then $P = H_{e^{i\theta},\lambda}$.

Proof. From (3.4) we have

$$\begin{aligned} H_{a,\lambda}(z) &= \frac{1 + [2(1-\beta)(\cos\gamma)e^{-i\gamma} - 1]\delta(az, \lambda)z}{1 - \delta(az, \lambda)z} \\ &= \frac{1 + \overline{\lambda} az + (\lambda z + az^2)(2(1-\beta)(\cos\gamma)e^{-i\gamma} - 1)}{1 + (\overline{\lambda} a - \lambda)z - az^2}. \end{aligned}$$

Using (4.4) we compute

$$H_{a,\lambda}(z) - A(z, \lambda) = \frac{2(1-\beta)(1-|\lambda|^2)(\cos\gamma)e^{-i\gamma} az^2}{(1 - \lambda z)(1 + (\overline{\lambda} a - \lambda)z - az^2)},$$

$$H_{a,\lambda}(z) + B(z, \lambda) = \frac{2(1-\beta)(1-|\lambda|^2)(\cos\gamma)e^{-i\gamma} z}{(z - \overline{\lambda})(1 + (\overline{\lambda} a - \lambda)z - az^2)}$$

and hence

$$
\begin{aligned}
H_{a,\lambda}(z) - c(z,\lambda) &= H_{a,\lambda}(z) - \frac{A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1 - |z|^2|\tau(z,\lambda)|^2} \\
&= \frac{1}{1 - |z|^2|\tau(z,\lambda)|^2}\Big\{ H_{a,\lambda}(z) - A(z,\lambda) \\
&\qquad\qquad - |z|^2|\tau(z,\lambda)|^2 (H_{a,\lambda}(z) + B(z,\lambda)) \Big\} \\
&= \frac{2(1-\beta)(1-|\lambda|^2)(\cos\gamma) e^{-i\gamma} a z^2 \overline{[1 + (\overline{\lambda} a - \lambda) z - a z^2]}}{(1-|z|^2)(1+|z|^2 - 2\,\mathrm{Re}(\lambda z))(1 + (\overline{\lambda} a - \lambda) z - a z^2)} \\
&= r(z,\lambda) \frac{e^{-i\gamma} a z^2}{|z|^2} \, \frac{|1 + (\overline{\lambda} a - \lambda) z - a z^2|^2}{(1 + (\overline{\lambda} a - \lambda) z - a z^2)^2}.
\end{aligned}
$$

Now by substituting $a = e^{i\theta}$ we easily see that

$$
H_{e^{i\theta},\lambda}(z) - c(z,\lambda) = r(z,\lambda) \frac{e^{-i\gamma} e^{i\theta} z^2}{|z|^2} \, \frac{|1 + (\overline{\lambda} e^{i\theta} - \lambda) z - e^{i\theta} z^2|^2}{(1 + (\overline{\lambda} e^{i\theta} - \lambda) z - e^{i\theta} z^2)^2}.
$$

For $G(z)$ as in Lemma 4.4, we get

$$
H_{e^{i\theta},\lambda}(z) - c(z,\lambda) = r(z,\lambda) e^{-i\gamma} \, \frac{G'(z)}{|G'(z)|} \tag{4.7}
$$

and there exists a starlike univalent function G_0 in $\mathbb{D}$ such that $G = \frac{1}{3} e^{i\theta} G_0^3$ and $G_0(0) = G_0'(0) - 1 = 0$. As G_0 is starlike, for any $z_0 \in \mathbb{D} \setminus \{0\}$ the linear segment joining 0 and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$.

Now, we define γ_0 by

$$
\gamma_0 : z(t) = G_0^{-1}(t G_0(z_0)), \qquad 0 \le t \le 1. \tag{4.8}
$$

Since $G(z(t)) = \frac{1}{3} e^{i\theta} (G_0(z(t)))^3 = \frac{1}{3} e^{i\theta} (t G_0(z_0))^3 = t^3 G(z_0)$, we have

$$
G'(z(t)) z'(t) = 3t^2 G(z_0), \qquad t \in [0,1]. \tag{4.9}
$$

Using (4.9) and (4.7) we have

$$
\begin{aligned}
\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta - C(\lambda,\gamma_0) &= \int_0^1 \{ H_{e^{i\theta},\lambda}(z(t)) - c(z(t),\lambda) \} z'(t)\, dt \\
&= e^{-i\gamma} \int_0^1 r(z(t),\lambda) \frac{G'(z(t)) z'(t)}{|G'(z(t)) z'(t)|} |z'(t)|\, dt \\
&= e^{-i\gamma} \frac{G(z_0)}{|G(z_0)|} \int_0^1 r(z(t),\lambda) |z'(t)|\, dt \\
&= e^{-i\gamma} \frac{G(z_0)}{|G(z_0)|} R(\lambda,\gamma_0),
\end{aligned} \tag{4.10}
$$

where $C(\lambda,\gamma_0)$ and $R(\lambda,\gamma_0)$ are defined as in Corollary 4.3. Thus

$$\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta \in \partial\overline{\mathbb{D}}(C(\lambda,\gamma_0),R(\lambda,\gamma_0)).$$

Also, from Corollary 4.3, we have

$$\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta \in V_{\mathcal{P}}(z_0,\lambda) \subset \overline{\mathbb{D}}(C(\lambda,\gamma_0),R(\lambda,\gamma_0)).$$

Hence, we conclude that $\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta \in \partial V_{\mathcal{P}}(z_0,\lambda)$.

Finally, we prove the uniqueness of the curve. Suppose that

$$\int_0^{z_0} P(\zeta)\,d\zeta = \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta$$

for some $P \in \mathcal{P}(\lambda)$ and $\theta \in (-\pi,\pi]$. We introduce

$$h(t) = e^{i\gamma}\frac{\overline{G(z_0)}}{|G(z_0)|}\{P(z(t)) - c(z(t),\lambda)\}z'(t),$$

where $\gamma_0 : z(t)$, $0 \le t \le 1$, is given by (4.8). Then $h(t)$ is a continuous function in $[0,1]$ and satisfies

$$|h(t)| \le r(z(t),\lambda)|z'(t)|.$$

Furthermore, from (4.10) we have

$$\begin{aligned}\int_0^1 \operatorname{Re} h(t)\,dt &= \int_0^1 \operatorname{Re}\left\{e^{i\gamma}\frac{\overline{G(z_0)}}{|G(z_0)|}\{P(z(t)) - c(z(t),\lambda)\}z'(t)\right\}dt\\ &= \operatorname{Re}\left\{e^{i\gamma}\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta - C(\lambda,\gamma_0)\right\}\right\}\\ &= \int_0^1 r(z(t),\lambda)|z'(t)|\,dt.\end{aligned}$$

Thus

$$h(t) = r(z(t),\lambda)|z'(t)| \quad \text{for all } t \in [0,1].$$

From (4.7) and (4.9), it follows that

$$\int_0^{z_0} P(\zeta)\,d\zeta = \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\,d\zeta \quad \text{on } \gamma_0.$$

In view of the identity theorem for analytic functions, we see that it holds for all $z_0 \in \mathbb{D}$, and hence, by the normalization, $P = H_{e^{i\theta},\lambda}$ in $\mathbb{D}$. ∎

5. Main theorem

THEOREM 5.1. *For $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V_{\mathcal{P}}(z_0, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta$$

$$= \int_0^{z_0} \frac{1 + [2(1-\beta)(\cos\gamma)e^{-i\gamma} - 1]\delta(e^{i\theta}\zeta, \lambda)\zeta}{1 - \delta(e^{i\theta}\zeta, \lambda)\zeta}\, d\zeta.$$

If $\int_0^{z_0} P(\zeta)\, d\zeta = \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta$ for some $P \in \mathcal{P}(\lambda)$ and $\theta \in (-\pi, \pi]$, then $P(z) = H_{e^{i\theta},\lambda}(z)$, where $\delta(z, \lambda)$ is defined by (3.3).

Proof. We need to prove that the closed curve

$$(-\pi, \pi] \ni \theta \mapsto \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta \tag{5.1}$$

is simple. Suppose that

$$\int_0^{z_0} H_{e^{i\theta_1},\lambda}(\zeta)\, d\zeta = \int_0^{z_0} H_{e^{i\theta_2},\lambda}(\zeta)\, d\zeta$$

for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$. Then, from Proposition 4.5, we have

$$H_{e^{i\theta_1},\lambda} = H_{e^{i\theta_2},\lambda}. \tag{5.2}$$

From (3.5) and (4.4) we obtain the identity

$$\tau\left(\frac{\omega_{H_{e^{i\theta},\lambda}}}{z}, \lambda\right) = \frac{e^{i\theta} z(1 - \overline{\lambda}^2) + \lambda - \overline{\lambda}}{e^{i\theta} z(\overline{\lambda} - \lambda) + 1 - \lambda^2}. \tag{5.3}$$

From (5.2) and (5.3) we have

$$\frac{e^{i\theta_1} z(1 - \overline{\lambda}^2) + \lambda - \overline{\lambda}}{e^{i\theta_1} z(\overline{\lambda} - \lambda) + 1 - \lambda^2} = \frac{e^{i\theta_2} z(1 - \overline{\lambda}^2) + \lambda - \overline{\lambda}}{e^{i\theta_2} z(\overline{\lambda} - \lambda) + 1 - \lambda^2}. \tag{5.4}$$

A simplification of (5.4) gives

$$e^{i\theta_1} z = e^{i\theta_2} z,$$

which is a contradiction to the choice of θ_1 and θ_2. Thus, the curve must be simple.

Since $V_{\mathcal{P}}(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V_{\mathcal{P}}(z_0, \lambda)$ is a simple closed curve. From Proposition 4.1, the curve $\partial V_{\mathcal{P}}(z_0, \lambda)$ contains the curve (5.1). Recall the fact that a simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial V_{\mathcal{P}}(z_0, \lambda)$ is given by (5.1). ∎

Remark 5.2. The integral in (5.1) can be simplified as follows: Set $b = \mathrm{Im}(\overline{\lambda} e^{i\theta/2}) \in \mathbb{R}$. Then a computation shows that

$$1 + (\overline{\lambda} e^{i\theta} - \lambda)z - e^{i\theta} z^2 = (1 - z/z_1)(1 - z/z_2),$$

where

$$z_1 = e^{-i\theta/2}(ib + \sqrt{1-b^2}) \quad \text{and} \quad z_2 = e^{-i\theta/2}(ib - \sqrt{1-b^2}).$$

From (3.4) and (3.3) we have

$$H_{e^{i\theta},\lambda}(z) = 1 + 2(1-\beta)e^{-i\gamma}(\cos\gamma)\frac{(e^{i\theta}z + \lambda)z}{1 + (\overline{\lambda} e^{i\theta} - \lambda)z - e^{i\theta} z^2}. \tag{5.5}$$

Since

$$\frac{(e^{i\theta}z + \lambda)z}{1 + (\overline{\lambda} e^{i\theta} - \lambda)z - e^{i\theta} z^2} = -1 - \frac{e^{-i\theta}}{z_1 - z_2}\left(\frac{1 + \overline{\lambda} e^{i\theta} z_1}{z - z_1} - \frac{1 + \overline{\lambda} e^{i\theta} z_2}{z - z_2}\right),$$

the equation (5.5) becomes

$$\begin{aligned} H_{e^{i\theta},\lambda}(z) &= 1 - 2(1-\beta)e^{-i\gamma}\cos\gamma \\ &\quad - \frac{2e^{-i\gamma}(1-\beta)e^{-i\theta}\cos\gamma}{z_1 - z_2}\left(\frac{1 + \overline{\lambda} e^{i\theta} z_1}{z - z_1} - \frac{1 + \overline{\lambda} e^{i\theta} z_2}{z - z_2}\right). \end{aligned}$$

By integrating on both sides from 0 to z_0, we can easily obtain the following representation:

$$\begin{aligned} \int_0^{z_0} H_{e^{i\theta},\lambda}(\zeta)\, d\zeta &= (1 - 2(1-\beta)e^{-i\gamma}\cos\gamma)z_0 \\ &\quad + K(\gamma,\beta,\theta,b)\left[(1 + \overline{\lambda} e^{i\theta/2}(-\sqrt{1-b^2} + ib))\log\left(1 + \frac{e^{i\theta/2}z_0}{\sqrt{1-b^2} - ib}\right)\right. \\ &\quad \left. - (1 + \overline{\lambda} e^{i\theta/2}(\sqrt{1-b^2} + ib))\log\left(1 - \frac{e^{i\theta/2}z_0}{\sqrt{1-b^2} + ib}\right)\right], \end{aligned}$$

where

$$K(\gamma,\beta,\theta,b) = \frac{e^{-i\gamma}(1-\beta)e^{-i\theta/2}\cos\gamma}{\sqrt{1-b^2}}.$$

For $\lambda = 0$, Theorem 5.1 takes the following simple form.

Corollary 5.3. *For $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda = 0$ the boundary $\partial V_{\mathcal{P}}(z_0, 0)$ is the Jordan curve given by*

$$\begin{aligned} (-\pi, \pi] \ni \theta \mapsto &\int_0^{z_0} H_{e^{i\theta},0}(\zeta)\, d\zeta \\ &= (1 - 2(1-\beta)e^{-i\gamma}\cos\gamma)z_0 + e^{-i\gamma}(1-\beta)e^{-i\theta/2}\cos\gamma \log\frac{1 + e^{i\theta/2}z_0}{1 - e^{i\theta/2}z_0}. \end{aligned}$$

If $\int_0^{z_0} P(\zeta)\,d\zeta = \int_0^{z_0} H_{e^{i\theta},0}(\zeta)\,d\zeta$ *for some* $P \in \mathcal{P}(0)$ *and* $\theta \in (-\pi, \pi]$, *then* $P(z) = H_{e^{i\theta},0}(z)$.

6. Some special cases

6.1. The class $\mathcal{R}_\beta$. In order to discuss a special situation, we consider $P = f'$ and $\gamma = 0$ in the class $\mathcal{P}_{\gamma,\beta}$. Thus, $\mathcal{P}_{\gamma,\beta}$ reduces to $\mathcal{R}_\beta$, where

$$\mathcal{R}_\beta = \{f \in \mathcal{A}\colon\ \mathrm{Re}\, f'(z) > \beta \text{ in } \mathbb{D}\}.$$

Then $\mathcal{R}_\beta \subset \mathcal{S}$ for $0 \le \beta < 1$. As with $\mathcal{P}(\lambda)$, for $\lambda \in \overline{\mathbb{D}}$ and $z_0 \in \mathbb{D}$ being fixed, we define

$$\mathcal{R}(\lambda) = \{f \in \mathcal{R}_\beta : f''(0) = 2(1-\beta)\lambda\}, \qquad V_{\mathcal{R}}(z_0, \lambda) = \{f(z_0)\colon f \in \mathcal{R}(\lambda)\}.$$

We remark that if $f \in \mathcal{R}(\lambda)$, then necessarily

$$f'''(0) = 4(1-\beta)[(1-|\lambda|^2)a + \lambda^2]$$

for some $a \in \overline{\mathbb{D}}$.

For $P = f'$, a computation shows that the extremal function $H_{e^{i\theta},\lambda}(z)$ for the class $\mathcal{R}(\lambda)$ takes the form

$$H_{e^{i\theta},\lambda}(z) = z_0 + 2(1-\beta)\int_0^{z_0} \frac{(e^{i\theta}\zeta + \lambda)\zeta}{1 + \overline{\lambda}e^{i\theta}\zeta - (e^{i\theta}\zeta + \lambda)\zeta}\,d\zeta.$$

It is not difficult to obtain the following result, which is the analog of Theorem 5.1 for the class $\mathcal{R}(\lambda)$.

Corollary 6.1. *For* $\lambda \in \mathbb{D}$ *and* $z_0 \in \mathbb{D} \setminus \{0\}$, *the boundary* $\partial V_{\mathcal{R}}(z_0, \lambda)$ *is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto H_{e^{i\theta},\lambda}(z_0) = z_0 + 2(1-\beta)\int_0^{z_0} \frac{(e^{i\theta}\zeta + \lambda)\zeta}{1 + \overline{\lambda}e^{i\theta}\zeta - (e^{i\theta}\zeta + \lambda)\zeta}\,d\zeta.$$

If $f(z_0) = H_{e^{i\theta},\lambda}(z_0)$ *for some* $f \in \mathcal{R}(\lambda)$ *and* $\theta \in (-\pi, \pi]$, *then* $f(z) = H_{e^{i\theta},\lambda}(z)$.

For $0 \le \beta < 1$ and $\lambda = 0$, set

$$\mathcal{R}(0) = \{f \in \mathcal{A}\colon f''(0) = 0 \text{ and } \mathrm{Re}\, f'(z) > \beta \text{ in } \mathbb{D}\} \subset \mathcal{R}_\beta.$$

In particular, the choices $\gamma = 0$ and $P(z) = f'(z)$ in Corollary 4.2 give the following: if $f \in \mathcal{R}(0) \subset \mathcal{R}_\beta$ for some $0 \le \beta < 1/2$, then by (4.6), one has

$$|f'(z)| \le \frac{1 + (1-2\beta)|z|^4 + 2(1-\beta)|z|^2}{1-|z|^4} = \frac{1 + (1-2\beta)|z|^2}{1-|z|^2}, \qquad z \in \mathbb{D},$$

so that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| \le 2(1-\beta).$$

Equality holds for

$$f(z) = \beta z + \frac{1-\beta}{2}\log\frac{1+z}{1-z}, \qquad z \in \mathbb{D}.$$

6.2. The class $\mathcal{F}(\alpha,\beta)$. For $\alpha \in \mathbb{C}$ satisfying $\operatorname{Re}\alpha > 0$ and $\beta \in \mathbb{R}$ with $\beta < 1$, let $\mathcal{F}(\alpha,\beta)$ denote the class of functions $f \in \mathcal{A}$ satisfying

$$f'(z) + \alpha z f''(z) \prec \frac{1+(1-2\beta)z}{1-z}, \qquad z \in \mathbb{D}, \tag{6.1}$$

where $\prec$ denotes the usual subordination [MM]. In [Po1] conditions on α and β for which

$$\mathcal{F}(\alpha,\beta) \subset \mathcal{S}^*$$

have been established (also for certain complex values of α), and in [FR] it has been shown that $\mathcal{F}(\alpha,\beta) \subset \mathcal{S}^*$ if $\alpha \geq 1/3$ and $\beta \geq \beta_0(\alpha)$, where

$$\beta_0(\alpha) = \frac{-\frac{1}{\alpha}\int_0^1 t^{1/\alpha-1}\,\frac{1+t}{1-t}\,dt}{1-\frac{1}{\alpha}\int_0^1 t^{1/\alpha-1}\,\frac{1+t}{1-t}\,dt}.$$

This is indeed a reformulated version of a theorem from [FR] and the inclusion is sharp in the following sense: for $\beta < \beta_0(\alpha)$ the functions in $\mathcal{F}(\alpha,\beta)$ are not even univalent in $\mathbb{D}$. For an extension of this inclusion result, we refer to [PR1, PR2].

Now, we present an alternative representation for functions in $\mathcal{F}(\alpha,\beta)$. If $f \in \mathcal{F}(\alpha,\beta)$, then (6.1) is equivalent to

$$\frac{f(z)}{z} * \Big(1 + \sum_{n=2}^{\infty} n(1+(n-1)\alpha)z^{n-1}\Big) \prec 1 + 2(1-\beta)\frac{z}{1-z},$$

where $*$ denotes the Hadamard product (or convolution) of two analytic functions in $\mathbb{D}$ represented by power series about the origin. By a well-known convolution theorem (cf. [RS]) this gives

$$\frac{f(z)}{z} \prec \beta + (1-\beta)\left[1 + \frac{2}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{(n+1)(n+1/\alpha)}\right]$$

and a computation shows that

$$\frac{f(z)}{z} \prec \begin{cases} \beta + (1-\beta)\left[1 - \dfrac{2}{1-\alpha}\left(\dfrac{\log(1-z)}{z} + 1 + \displaystyle\int_0^1 t^{1/\alpha}\frac{z}{1-tz}\,dt\right)\right] & \text{if } \alpha \neq 1, \\ \beta + (1-\beta)\left[1 + 2z\displaystyle\int_0^1 \frac{t\log(1/t)}{1-tz}\,dt\right] & \text{if } \alpha = 1. \end{cases}$$

The definition of subordination gives the following representation of functions in $\mathcal{F}(\alpha,\beta)$:

$$f(z)=\begin{cases} z-\dfrac{2(1-\beta)z}{1-\alpha}\left\{1+\dfrac{1}{\omega(z)}\log(1-\omega(z))+\omega(z)\displaystyle\int_0^1\frac{t^{1/\alpha}}{1-t\omega(z)}\,dt\right\} \\ \hfill \text{if } \alpha\neq 1, \\ z+2(1-\beta)z\omega(z)\displaystyle\int_0^1\frac{t\log(1/t)}{1-t\omega(z)}\,dt \hfill \text{if } \alpha=1, \end{cases}$$

for $z\in\mathbb{D}$, and for some $\omega\in\mathcal{B}_0$.

If $f\in\mathcal{F}(\alpha,\beta)$, then according to the Herglotz representation there exists a unique positive unit measure μ on $(-\pi,\pi]$ such that

$$f'(z)+\alpha zf''(z)=\int_{-\pi}^{\pi}\frac{1+(1-2\beta)ze^{-it}}{1-ze^{-it}}\,d\mu(t),$$

or equivalently

$$\frac{f(z)}{z}=\left[1+\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{z^n}{(n+1)(n+1/\alpha)}\right]*\int_{-\pi}^{\pi}\frac{1+(1-2\beta)ze^{-it}}{1-ze^{-it}}\,d\mu(t).$$

A simplification of the last equality gives the following representation of functions in the class $\mathcal{F}(\alpha,\beta)$:

$$f(z)=\begin{cases} \dfrac{z}{1-\alpha}\displaystyle\int_0^1\int_{-\pi}^{\pi}(1-s^{1/\alpha-1})\left(\frac{1+(1-2\beta)sze^{-it}}{1-sze^{-it}}\right)d\mu(t)\,ds \\ \hfill \text{if } \alpha\neq 1, \\ z+2(1-\beta)z\displaystyle\int_0^1\int_{-\pi}^{\pi}\left(\log\frac{1}{s}\right)\left(\frac{sze^{-it}}{1-sze^{-it}}\right)d\mu(t)\,ds \\ \hfill \text{if } \alpha=1. \end{cases}$$

To state our special case in precise form, for convenience we let $\gamma=0$, and define

$$P(z)=f'(z)+\alpha zf''(z),\qquad f\in\mathcal{F}(\alpha,\beta),$$

so that

$$P'(0)=(1+\alpha)f''(0)\quad\text{and}\quad P''(0)=(1+2\alpha)f''(0).$$

In view of these observations, the analogs of the sets $\mathcal{P}(\lambda)$ and $V_{\mathcal{P}}(z_0,\lambda)$ will be

$$\mathcal{G}(\lambda)=\left\{f\in\mathcal{F}(\alpha,\beta): f''(0)=2\,\frac{1-\beta}{1+\alpha}\,\lambda\right\},$$
$$V_{\mathcal{G}}(z_0,\lambda)=\{(1-\alpha)f(z_0)+\alpha z_0f'(z_0): f\in\mathcal{G}(\lambda)\},$$

where $0 \le \beta < 1$. We observe that for functions f in $\mathcal{G}(\lambda)$,

$$f'''(0) = 4((1-|\lambda|^2)a + \lambda^2)\frac{1-\beta}{1+2\alpha}$$

for some $a \in \overline{\mathbb{D}}$.

With $P(z) = f'(z) + \alpha z f''(z)$, the corresponding extremal function $F_{e^{i\theta},\lambda}(z)$ for $\mathcal{G}(\lambda)$ can be computed to be

$$(1-\alpha)F_{e^{i\theta},\lambda}(z) + \alpha z F'_{e^{i\theta},\lambda}(z) = \int_0^z \frac{1+(1-2\beta)\delta(a\zeta,\lambda)\zeta}{1-\delta(a\zeta,\lambda)\zeta}\, d\zeta,$$

where $\delta(z,\lambda)$ is defined by (3.3). In this setting, Proposition 4.1 (for $\gamma = 0$) takes the following form:

Proposition 6.2. *For $f \in \mathcal{G}(\lambda)$ and $\lambda \in \mathbb{D}$, we have*

$$|f'(z) + \alpha z f''(z) - c(z,\lambda)| \le r(z,\lambda), \qquad z \in \mathbb{D},$$

where

$$c(z,\lambda) = \frac{(1+(1-2\beta)\lambda z)(1-\overline{\lambda}\overline{z}) + |z|^2(\overline{z}-\lambda)(\overline{\lambda}+(1-2\beta)z)}{(1-|z|^2)(1+|z|^2-2\operatorname{Re}(\lambda z))},$$

$$r(z,\lambda) = \frac{2(1-\beta)(1-|\lambda|^2)|z|^2}{(1-|z|^2)(1+|z|^2-2\operatorname{Re}(\lambda z))}.$$

For each $z \in \mathbb{D}\setminus\{0\}$, equality holds if and only if $f = F_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$.

Using Theorem 5.1, we get the following result.

Corollary 6.3. *For $\lambda \in \mathbb{D}$, $z_0 \in \mathbb{D}\setminus\{0\}$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$, the boundary $\partial V_{\mathcal{G}}(z_0,\lambda)$ is the Jordan curve given by*

$$\begin{aligned}(-\pi,\pi] \ni \theta \mapsto (1-\alpha)F_{e^{i\theta},\lambda}(z_0) + \alpha z_0 F'_{e^{i\theta},\lambda}(z_0) &= (2\beta-1)z_0 \\ + \frac{(1-\beta)e^{-i\theta/2}}{\sqrt{1-b^2}}\Bigg[(1+\overline{\lambda}e^{i\theta/2}(-\sqrt{1-b^2}+ib))\log\left(1+\frac{e^{i\theta/2}z_0}{\sqrt{1-b^2}-ib}\right) \\ - (1+\overline{\lambda}e^{i\theta/2}(\sqrt{1-b^2}+ib))\log\left(1-\frac{e^{i\theta/2}z_0}{\sqrt{1-b^2}+ib}\right)\Bigg],\end{aligned}$$

where $b = \operatorname{Im}(\overline{\lambda}e^{i\theta/2})$. If $(1-\alpha)f(z_0) + \alpha z_0 f'(z_0) = (1-\alpha)F_{a,\lambda}(z_0) + \alpha z_0 F'_{a,\lambda}(z_0)$ for some $f \in \mathcal{G}(\lambda)$ and $\theta \in (-\pi,\pi]$, then $f(z) = F_{e^{i\theta},\lambda}(z)$.

The proof of this corollary follows by taking $\gamma = 0$ in Remark 5.2 and so we omit the details.

In the case of $\lambda = 0$ in Corollary 6.3, the corresponding extremal function $F_{a,0}(z)$ can be obtained easily by solving

$$F'_{a,0}(z) + \alpha z F''_{a,0}(z) = \frac{1+(1-2\beta)\delta(az,0)z}{1-\delta(az,0)z}.$$

This may be rewritten as

$$\frac{F_{a,0}(z)}{z} * \left[1+\sum_{n=0}^{\infty}(n+1)(1+n\alpha)z^n\right] = 2\beta-1+2(1-\beta)\frac{1}{1-az^2}$$

or equivalently as

$$\frac{F_{a,0}(z)}{z} = \left[1+\sum_{n=1}^{\infty}2(1-\beta)a^n z^{2n}\right] * \left[1+\sum_{n=0}^{\infty}\frac{z^n}{(n+1)(1+n\alpha)}\right].$$

A simple calculation gives

$$F_{a,0}(z) = \begin{cases} z+\dfrac{(1-\beta)az^3}{1-\alpha}\displaystyle\int_0^1 \frac{t^{1/2}-t^{1/2\alpha}}{1-taz^2}\,dt & \text{if } \alpha\neq 1, \\ z+\dfrac{(1-\beta)az^3}{2}\displaystyle\int_0^1 \frac{t^{1/2}\log(1/t)}{1-taz^2}\,dt & \text{if } \alpha = 1. \end{cases}$$

7. Geometric view of Theorem 5.1. Using Mathematica (see [R]), we describe the boundary of the sets $V_{\mathcal{P}}(z_0,\lambda)$ and $V_{\mathcal{G}}(z_0,\lambda)$. In the program below, "`z0`" stands for z_0, "`lam`" for λ, "`g`" for γ and "`b`" for β.

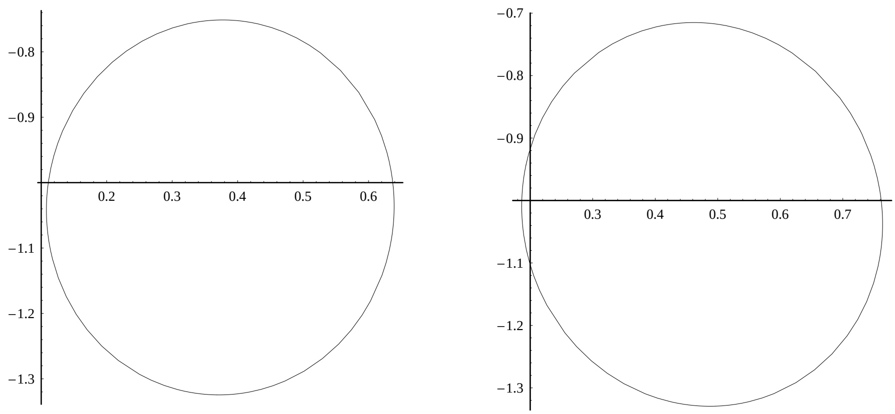
```
(* Geometric view of the main Theorem 5.1  and Corollary 6.3 *)

Remove["Global`*"];

z0 = Random[]Exp[I*Random[Real, {-Pi, Pi}]]
lam = Random[]Exp[I*Random[Real, {-Pi, Pi}]]
g = Random[Real, {-Pi/2, Pi/2}]
b = Random[Real, {0, 1}]

Print["z0=", z0]
Print["lam=", lam]
Print["g=", g]
Print["b=", b]

Q1[b_, g_, lam_,the_] := ((1 +Conjugate[lam]Exp[I*the]*z) +
(lam*z +Exp[I*the]*z^2)(Exp[-I*g] - 2b*Cos[g])Exp[-I*g])/
((1 + ( Conjugate[lam]*Exp[I*the] - lam)*z )-Exp[I*the]*z*z);

Q2[b_, lam_,the_] := ((1 +Conjugate[lam]Exp[I*the]*z) +
(1 - 2b)(lam*z +Exp[I*the]*z^2))/
((1 + ( Conjugate[lam]*Exp[I*the] - lam)*z )-Exp[I*the]*z*z);

myf1[b_, g_, lam_, the_, z0_] :=
NIntegrate[Q1[b, g, lam, the], {z, 0, z0}];

myf2[b_, lam_, the_, z0_] := NIntegrate[Q2[b, lam, the],
                             {z, 0, z0}];
```

```
image1 = ParametricPlot[{Re[myf1[b, g,  lam, the, z0]],
Im[myf1[b, g, lam, the, z0]]}, {the, -Pi, Pi},AspectRatio ->
Automatic,TextStyle -> {FontFamily -> "Times",FontSize ->14},
AxesStyle -> {Thickness[0.0035]} ];

image2 = ParametricPlot[{Re[myf2[b,  lam, the, z0]],
Im[myf2[b, lam, the, z0]]},{the, -Pi, Pi}, AspectRatio ->
Automatic,TextStyle -> {FontFamily -> "Times",FontSize ->14},
AxesStyle -> {Thickness[0.0035]} ];

Clear[b, g, lam, z0, myf1, myf2];
```

The following figures show the boundary of $V_{\mathcal{P}}(z_0,\lambda)$ and $V_{\mathcal{G}}(z_0,\lambda)$ for certain values of $z_0 \in \mathbb{D}\setminus\{0\}$, $\lambda \in \mathbb{D}$, $0 \le \beta < 1$ and $|\gamma| < \pi/2$. Table 1 gives the list of these parameter values corresponding to Figs. 1–5. We recall that according to Proposition 3.1 the region bounded by the curve $\partial V_{\mathcal{P}}(z_0,\lambda)$ is compact and convex.

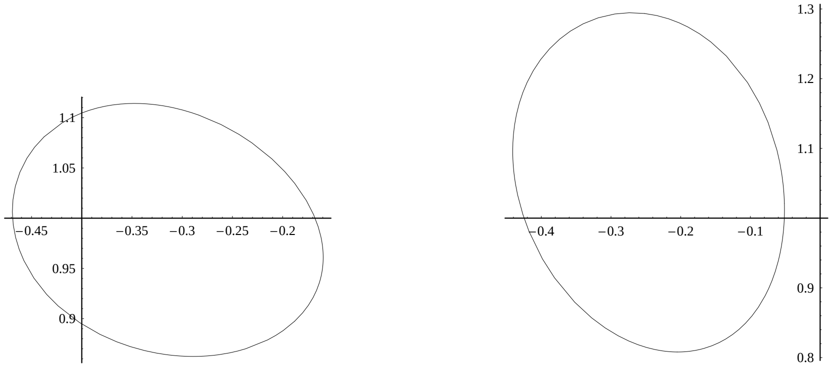
Table 1

Fig.	z_0	λ	β	γ
1	$0.335192 - 0.787333i$	$0.0737292 + 0.466706i$	0.591244	0.383292
2	$-0.261209 + 0.926935i$	$-0.28588 + 0.307498i$	0.700318	-0.87825
3	$-0.41227 - 0.521734i$	$-0.0875648 + 0.0714166i$	0.602203	0.910581
4	$0.771264 + 0.151204i$	$-0.391149 - 0.294747i$	0.928608	1.55854
5	$0.335626 + 0.929093i$	$0.00010443 + 0.0255256i$	0.76622	1.5449



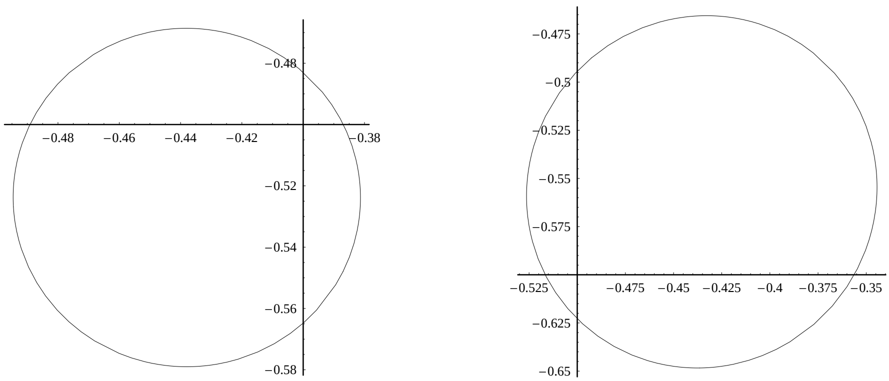
$\partial V_{\mathcal{P}}(z_0,\lambda)$ $\partial V_{\mathcal{G}}(z_0,\lambda)$

Fig. 1. $z_0 = 0.335192 - 0.787333i$ and $\beta = 0.591244$



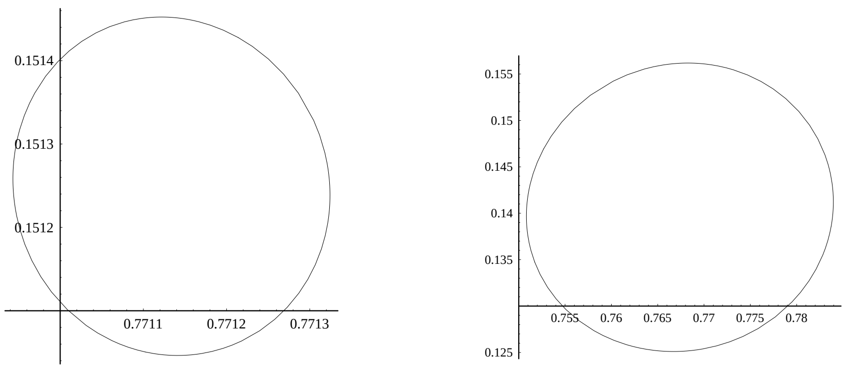
$\partial V_{\mathcal{P}}(z_0, \lambda)$ $\partial V_{\mathcal{G}}(z_0, \lambda)$

Fig. 2. $z_0 = -0.261209 + 0.926935i$ and $\beta = 0.700$



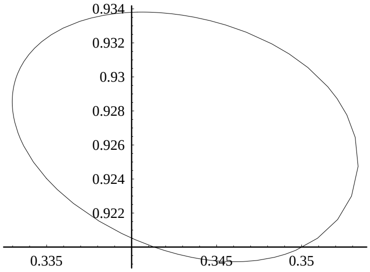
$\partial V_{\mathcal{P}}(z_0, \lambda)$ $\partial V_{\mathcal{G}}(z_0, \lambda)$

Fig. 3. $z_0 = -0.41227 - 0.521734i$ and $\beta = 0.602203$

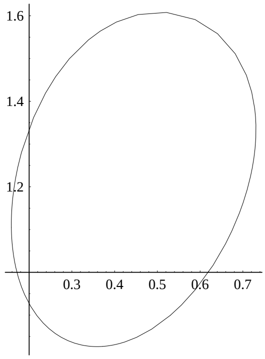


$\partial V_{\mathcal{P}}(z_0, \lambda)$ $\partial V_{\mathcal{G}}(z_0, \lambda)$

Fig. 4. $z_0 = 0.771264 + 0.151204i$ and $\beta = 0.928608$



$\partial V_{\mathcal{P}}(z_0, \lambda)$



$\partial V_{\mathcal{G}}(z_0, \lambda)$

Fig. 5. $z_0 = 0.335626 + 0.929093i$ and $\beta = 0.76622$

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