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Thom polynomials and Schur functions: the singularities $III_{2,3}(-)$

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Abstract. We give a closed formula for the Thom polynomials of the singularities $III_{2,3}(-)$ in terms of Schur functions. Our computations combine the characterization of the Thom polynomials via the "method of restriction equations" of Rimányi et al. with the techniques of Schur functions.

1. Introduction. This paper is a part of the project of investigating the structures of Schur function expansions of Thom polynomials of the singularities associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter k. The project was started by Pragacz in [12], where techniques of Schur functions were combined with the "method of restriction equations" of Rimányi et al. (cf. [17]).

The first results of the project appeared in [12], [13], [14] and [8] (see also [4]). These include

- more transparent proofs of formulas of Thom, Porteous and Ronga (cf. [19], [11], [18]);
- new formulas for the Thom polynomials of the singularities $I_{2,2}(-)$ and $A_3(-)$, for all $k \geq 0$ (cf. [13], [8]);
- a structure result: Theorem 11 in [13], asserting that the appearing partitions contain sufficiently large rectangles (this leads to decomposition of Thom polynomials into "h-parts");
- structure results bounding the number of rows of appearing partitions (see [13], [10]);
- formula for the "1-part" of the Thom polynomials for $A_i(-)$ (the "1-part" of the Thom polynomial of $A_i(r)$ is the sum of all Schur functions appearing (multiplied with their coefficients) with partitions containing the row (r) but not $(r+1, r+1)$, cf. [14]).

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Then Pragacz and Weber proved that the coefficients of Schur function expansions of the Thom polynomials of stable singularities are nonnegative (see [15]), and generalized this result to Thom polynomials of nonstable singularities and invariant cones in [16].

In this paper we give a closed formula (Theorem 4.1) for the Thom polynomials of the singularities $III_{2,3}(-)$ associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 1$. Our computations combine the characterization of Thom polynomials via the "method of restriction equations" of Rimányi et al. with the techniques of Schur functions. In Lemma 2.1 we compute the Euler class of the singularity $III_{2,3}(-)$, needed to apply Theorem 2.4 from [17]. The key algebraic calculations are performed in Lemmas 4.2–4.4. The proof of the main result relies heavily on the factorization property for Schur functions from [2].

2. Thom polynomials. Our main reference for this section is [17]. Let $k \geq 0$ be a fixed integer and $\bullet \in \mathbf{N}$. A *singularity* η is an equivalence class of the relation on stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ generated by contact equivalence and suspension. According to Mather's classification, singularities are in one-to-one correspondence with finite-dimensional $\mathbf{C}$-algebras (cf. Chapter 8 in [3], see also [6]). Our notation for singularities is as follows:

- A_i will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;
- $I_{a,b}$ (of Thom–Boardman type $\Sigma^{2,0}$) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a + y^b)$, $b \geq a \geq 2$;
- $III_{a,b}$ (of Thom–Boardman type $\Sigma^{2,0}$) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a, y^b)$, $b \geq a \geq 2$ (here $k \geq 1$).

Let η be a singularity and $f : X \to Y$ be a general map between complex analytic manifolds. Let $V^\eta(f)$ denote the closure of the set of η-points of f. The *Thom polynomial* $\mathcal{T}^\eta$ of η is a polynomial such that $\mathcal{T}^\eta(c_1, c_2, \ldots)$ gives the Poincaré dual of $V^\eta(f)$, where c_i are the Chern classes of the virtual bundle $f^*TY - TX$ (cf. [1] or [17]). By $\operatorname{codim}(\eta)$, we mean the codimension of $V^\eta(f)$ in X.

Let $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$ be a prototype of a singularity η. It is possible to choose a maximal compact subgroup G_η of the *right-left symmetry group* $\operatorname{Aut}(\kappa)$ such that the images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear (cf. [17]). That is, projecting on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$ we obtain representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$. Let E'_η and E_η denote the vector bundles associated with the universal principal G_η-bundles $EG_\eta \to BG_\eta$ that correspond to $\lambda_1(\eta)$ and $\lambda_2(\eta)$, respectively. The *total Chern class*, $c(\eta) \in H^\bullet(BG_\eta; \mathbf{Z})$, and the *Euler class*,

$e(\eta) \in H^{2\,\mathrm{codim}(\eta)}(BG_\eta;\mathbf{Z})$, of η are defined as

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)} \quad \text{and} \quad e(\eta) := e(E'_\eta). \tag{2.1}$$

The method of computing these classes is described in [17]. The Chern classes that we shall use were computed in [17], [12], and [13]. For $A_i : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, a suitable maximal compact subgroup can be chosen as $G_{A_i} = U(1) \times U(k)$. Then the Chern class becomes

$$c(A_i) = \frac{1+(i+1)x}{1+x}\prod_{j=1}^{k}(1+y_j), \tag{2.2}$$

where x and $y_1, \dots, y_k$ are the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$.

In the case of $\eta = I_{2,2}$, we consider the extension of $U(1) \times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$. Denoting this group by H, a maximal compact subgroup is $G_\eta = H \times U(k)$ (for $k \geq 0$). But to make computations easier, we use the subgroup $U(1) \times U(1) \times U(k)$ as G_η. (In concrete computations it is possible to use the action of a subgroup of G_η, instead of G_η itself; see [17, p. 502].) We get

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)}\prod_{j=1}^{k}(1+y_j). \tag{2.3}$$

Here x_1, x_2 and $y_1, \dots, y_k$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k)$,

Next we set $\eta = III_{2,2}$. This time we use the group $G_\eta = U(2) \times U(k-1)$ for $k \geq 1$. We have

$$c(III_{2,2}) = \frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)}\prod_{j=1}^{k-1}(1+y_j), \tag{2.4}$$

where x_1, x_2 and $y_1, \dots, y_{k-1}$ denote the Chern roots of the universal bundles on $BU(2)$ and $BU(k-1)$.

For the singularity $III_{2,3}$ we can use the action of the subgroup $U(1) \times U(1) \times U(k-1)$ to obtain

$$c(III_{2,3}) = \frac{(1+2x_1)(1+3x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)}\prod_{j=1}^{k-1}(1+y_j). \tag{2.5}$$

This time x_1, x_2 and $y_1, \dots, y_{k-1}$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k-1)$,

We shall also need the Euler class $e(III_{2,3})$ which we now compute. First assume that $k = 1$ and consider the germ $g(x, y) = (x^2, y^3, xy)$. A prototype of $III_{2,3}$ can be written as the unfolding $g + \sum_{i=1}^{8} u_i h_i$ where the h_i form a basis of the space

$$\frac{\mathfrak{m}^3_{x,y}}{\mathfrak{m}_{x,y}\cdot\left\{\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right\}+\mathbf{C}^3\cdot I(g)},$$

and where $I(g)$ is the subspace generated by the component functions of g. We shall work with the basis consisting of the germs:

$$\begin{aligned} h_1(x,y)&=(x,0,0), & h_5(x,y)&=(0,y,0),\\ h_2(x,y)&=(y,0,0), & h_6(x,y)&=(0,y^2,0),\\ h_3(x,y)&=(y^2,0,0), & h_7(x,y)&=(0,0,x),\\ h_4(x,y)&=(0,x,0), & h_8(x,y)&=(0,0,y).\end{aligned}$$

Let ρ_{h_i} denote the representation of the action of the group $U(1)\times U(1)$ on the space generated by h_i. Then denoting the 1-dimensional representations of the first and the second copies of $U(1)$ by λ and μ we have

$$\begin{aligned} \rho_{h_1}&=\lambda, & \rho_{h_5}&=\mu^2,\\ \rho_{h_2}&=\lambda^2\otimes\mu^{-1}, & \rho_{h_6}&=\mu,\\ \rho_{h_3}&=\lambda^2\otimes\mu^{-2}, & \rho_{h_7}&=\mu,\\ \rho_{h_4}&=\lambda^3\otimes\mu^{-1}, & \rho_{h_8}&=\lambda.\end{aligned}$$

Therefore for $k=1$, using the representation $\bigoplus\rho_{h_i}$, we can write the Euler class as

$$e(III_{2,3})=4x_1^2x_2^3(x_1-x_2)(x_1-3x_2)(x_2-2x_1), \tag{2.6}$$

where x_1 and x_2 denote the Chern roots of the universal bundles on the two copies of $BU(1)$.

For $k=2$, in addition to h_i above, we need to consider the representations of the action of the group $U(k-1)=U(1)$ on the spaces generated by $(x,y)\mapsto(0,0,0,x)$, $(x,y)\mapsto(0,0,0,y)$ and $(x,y)\mapsto(0,0,0,y^2)$. These can be written as $\nu\otimes\lambda^{-1}$, $\nu\otimes\mu^{-1}$, and $\nu\otimes\mu^{-2}$, where ν denotes the 1-dimensional representation of this copy of $U(1)$. Hence, in this case, the Euler class can be written as

$$\begin{aligned} &e(III_{2,3})\\ &\quad=4x_1^2x_2^3(x_1-x_2)(x_1-3x_2)(x_2-2x_1)(x_1-y_1)(x_2-y_1)(2x_2-y_1),\end{aligned} \tag{2.7}$$

where x_i are as above and y_1 denotes the Chern root of the universal bundle on $BU(1)$.

In the general case ($k\geq 1$), we need to consider $U(k-1)$ instead of $U(1)$. Let $y_1,\dots,y_{k-1}$ denote the Chern roots of the universal bundle on $BU(k-1)$. The argument above proves the following lemma:

Lemma 2.1. *Let $k \geq 1$. The Euler class of $III_{2,3}$ singularity is*

$$e(III_{2,3}) = 4x_1^2x_2^3(x_1-x_2)(x_1-3x_2)(x_2-2x_1)\prod_{j=1}^{k-1}(x_1-y_j)(x_2-y_j)(2x_2-y_j).$$

We end this section by recalling a theorem of Rimányi that explains the name "method of restriction equations".

Theorem 2.2 ([17, Theorem 2.4]). *Suppose, for a singularity η, that the Euler class of no singularity of codimension at most* $\operatorname{codim}(\eta)$ *is a zero-divisor* ([1]). *Then we have:*

(i) *if $\xi \neq \eta$ and* $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$, *then* $\mathcal{T}^\eta(c(\xi)) = 0$;
(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$.

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

3. Schur functions. Our main reference for this section is [7] (see also [9]). By a *partition* $I = (i_1, \ldots, i_h)$ we mean a weakly increasing sequence $0 \leq i_1 \leq \cdots \leq i_h$ of natural numbers. To simplify the notation we shall also write $i_1 \ldots i_h$. When we refer to an arbitrary partition I, the length of I (the number of nonzero parts of I) will be denoted by $\ell(I)$.

For positive integers m and n, we shall say that a partition I *is not contained in the* (m,n)*-hook* if $\ell(I) > m$ and $i_{\ell(I)-m} > n$.

An *alphabet* $\mathbb{A}$ means a finite multi-set of elements from a commutative ring. The symbol $\mathbb{A}_m$ denotes the alphabet $(a_1, \ldots, a_m)$ with m elements and we identify it with the sum $a_1 + \cdots + a_m$. Similarly, $\mathbb{B}_n$ denotes the alphabet $(b_1, \ldots, b_n)$ which is identified with $b_1 + \cdots + b_n$, etc.

Definition 3.1. Given a partition $I = (i_1, \ldots, i_h)$, and alphabets $\mathbb{A}$ and $\mathbb{B}$:

(i) The *ith complete symmetric function* $S_i(\mathbb{A} - \mathbb{B})$ is defined as the coefficient of z^i in the generating series

$$\sum S_i(\mathbb{A} - \mathbb{B})z^i = \frac{\prod_{b\in\mathbb{B}}(1-bz)}{\prod_{a\in\mathbb{A}}(1-az)}. \tag{3.1}$$

(ii) The *Schur function* $S_I(\mathbb{A} - \mathbb{B})$ is defined as the following determinant:

$$S_I(\mathbb{A} - \mathbb{B}) := |S_{i_q+q-p}(\mathbb{A} - \mathbb{B})|_{1\leq p,q\leq h}. \tag{3.2}$$

(iii) The *resultant* $R(\mathbb{A}, \mathbb{B})$ of the alphabets $\mathbb{A}$ and $\mathbb{B}$ is the product

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b). \tag{3.3}$$

([1]) This condition holds true for the singularities $III_{2,3}(-)$.

In the rest of this section we recall three properties of Schur functions that we use frequently:

Let q be a positive integer. Then

$$S_j(-\mathbb{E}-\mathbb{B}_q) = S_j(-\mathbb{E}-\mathbb{B}_{q-1}) - b_q S_{j-1}(-\mathbb{E}-\mathbb{B}_{q-1}). \tag{3.4}$$

If a partition I is not contained in the (m,n)-hook then we have the following *vanishing property*:

$$S_I(\mathbb{A}_m - \mathbb{B}_n) = 0. \tag{3.5}$$

For instance, $I = (2,5,6,8)$ is not contained in the $(2,4)$-hook, therefore $S_{2568}(\mathbb{A}_2 - \mathbb{B}_4) = 0$.

If a partition is contained in the (m,n)-hook and contains the rectangular partition (n^m) then we have the following *factorization property* (cf. [2]): For partitions $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_h)$,

$$S_{(j_1,\dots,j_h,i_1+n,\dots,i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_J(\mathbb{A}_m) R(\mathbb{A}_m, \mathbb{B}_n) S_I(-\mathbb{B}_n). \tag{3.6}$$

For example,

$$S_{1367}(\mathbb{A}_2 - \mathbb{B}_4) = S_{23}(\mathbb{A}_2) R(\mathbb{A}_2, \mathbb{B}_4) S_{13}(-\mathbb{B}_4).$$

Remark 3.2. Expressions appearing inside a box, such as $\boxed{x_1 + x_2}$, should be understood as a single variable. For example, $S_2(\boxed{x_1 + x_2}) = x_1^2 + 2x_1x_2 + x_2^2$ where $S_2(x_1 + x_2) = x_1^2 + x_1x_2 + x_2^2$ (cf. Convention 10 in [13]).

4. Thom polynomials for $III_{2,3}(-)$. In this section we shall use *Segre classes* S_i of the virtual bundle $TX^* - f^*(TY^*)$, replacing their equals, the Chern classes $c_i(f^*TY - TX)$. That is, we write complete symmetric functions $S_i(\mathbb{A} - \mathbb{B})$ for the alphabets of the *Chern roots* $\mathbb{A}, \mathbb{B}$ of TX^* and TY^*. Also, instead of k we will use a "shifted" parameter r:

$$r := k + 1. \tag{4.1}$$

We shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet + r - 1}, 0)$, and $\mathcal{T}_r^{III_{2,3}}$ will denote the Thom polynomial for $III_{2,3}(r)$ singularity.

Theorem 4.1. *Let $r \geq 2$. The Thom polynomial for $III_{2,3}(r)$ singularity is given by the following formula* (2)*:*

$$\mathcal{T}_r^{III_{2,3}} = \sum_{i=1}^{r+1} 2^i S_{r+1-i, r+1, r+i}.$$

(2) The case $r = 2$ was computed in the monomial basis by Rimányi in [17].

To prove the theorem we shall need some preparation. Let

$$\mathcal{Q}_r = \sum_{i=1}^{r+1} 2^i S_{r+1-i,r+1,r+i}.$$

We will show that $\mathcal{Q}_r = \mathcal{T}_r^{III_{2,3}}$ by using Theorem 2.2. Set $\mathbb{E} = \boxed{2x_1} + \boxed{2x_2}$ and $\mathbb{F} = \boxed{2x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}$. Let us rewrite the equations coming from Theorem 2.2 with the notation of Schur functions: The equations imposed by $A_0(r)$, $A_1(r)$, $A_2(r)$ and $A_3(r)$ are

$$\begin{aligned}\mathcal{Q}_r(-\mathbb{B}_{r-1}) &= \mathcal{Q}_r(x - \mathbb{B}_{r-1} - \boxed{2x}) = \mathcal{Q}_r(x - \mathbb{B}_{r-1} - \boxed{3x}) \\ &= \mathcal{Q}_r(x - \mathbb{B}_{r-1} - \boxed{4x}) = 0.\end{aligned} \tag{4.2}$$

The equation

$$\mathcal{Q}_r(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2} - \boxed{x_1 + x_2} - \mathbb{B}_{r-2}) = 0 \tag{4.3}$$

is imposed by $III_{2,2}(r)$. Next, we have the equation imposed by $I_{2,2}(r)$:

$$\mathcal{Q}_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = 0, \tag{4.4}$$

and the normalizing equation

$$\mathcal{Q}_r(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}) = 2x_2(x_1 - x_2)R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2}) \prod_{j=1}^{r-2}(2x_2 - b_j). \tag{4.5}$$

Since the partitions appearing in $\mathcal{Q}_r$ are not contained in the corresponding hooks we see that equations (4.2) are valid for any r. Also (4.3) can be obtained from (4.4). Therefore it is enough to show that (4.4) and (4.5) are satisfied. To do this we will proceed with a number of lemmas.

Lemma 4.2. *Let $p \geq 2$ be an integer. Then*

(i) $$\sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{E}) S_{i-1}(\mathbb{X}_2) = 0,$$

(ii) $$\sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{F}) S_{i-1}(\mathbb{X}_2) = 2^{p-2} x_2^{p-2}(x_1 - x_2).$$

Proof. Observing that $S_{p-i}(-\mathbb{E})$ vanishes for $i < p-2$ and $S_{p-i}(-\mathbb{F})$ vanishes for $i < p-3$, we get the desired equalities. ∎

Lemma 4.3. *For any $r \geq 2$ we have*

$$\sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{E} - \mathbb{B}_{r-1}) S_{i-1}(\mathbb{X}_2) = 0.$$

Proof. First notice that for $p \geq 3$ we have

$$\sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{E} - b_1) S_{i-1}(\mathbb{X}_2) = 0.$$

Therefore assuming the equality for $s < r$ we get

$$\begin{aligned}
&\sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{E} - \mathbb{B}_{r-1}) S_{i-1}(\mathbb{X}_2) \\
&= \sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{E} - \mathbb{B}_{r-2}) S_{i-1}(\mathbb{X}_2) - b_{r-1} \sum_{i=1}^{r+1} 2^i S_{r-i}(-\mathbb{E} - \mathbb{B}_{r-2}) S_{i-1}(\mathbb{X}_2) \\
&= \sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{E} - \mathbb{B}_{r-2}) S_{i-1}(\mathbb{X}_2).
\end{aligned}$$

But after finitely many applications of (3.4), this last expression reduces to

$$\sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{E}) S_{i-1}(\mathbb{X}_2).$$

Then using Lemma 4.2(i) with $p = r + 1$ we complete the proof. ∎

LEMMA 4.4. *Let $p \geq 3$ and $p - 3 \geq q \geq 0$. Then*

$$\sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{F} - \mathbb{B}_q) S_{i-1}(\mathbb{X}_2) = (2x_2)^{p-q-2}(x_1 - x_2) \prod_{j=1}^{q} (2x_2 - b_j).$$

Proof. If $p = 3$ then $q = 0$. Hence Lemma 4.2(ii) forms the base step of induction where $q = 0$. Assume that the equality holds true if $s < p$ (and $t \leq q$) or $t < q$ (and $s \leq p$). Then

$$\begin{aligned}
&\sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{F} - \mathbb{B}_q) S_{i-1}(\mathbb{X}_2) \\
&= \sum_{i=1}^{p} 2^i S_{p-i}(-\mathbb{F} - \mathbb{B}_{q-1}) S_{i-1}(\mathbb{X}_2) - b_q \sum_{i=1}^{p-1} 2^i S_{p-1-i}(-\mathbb{F} - \mathbb{B}_{q-1}) S_{i-1}(\mathbb{X}_2) \\
&= (2x_2)^{p-q-1}(x_1 - x_2) \prod_{j=1}^{q-1} (2x_2 - b_j) - b_q (2x_2)^{p-q-2}(x_1 - x_2) \prod_{j=1}^{q-1} (2x_2 - b_j) \\
&= (2x_2)^{p-q-2}(x_1 - x_2) \prod_{j=1}^{q} (2x_2 - b_j). \ \blacksquare
\end{aligned}$$

Proof of Theorem 4.1. By the factorization property and Lemma 4.3 we obtain

$$\mathcal{Q}_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = 0.$$

This proves (4.4) for any $r \geq 2$. Setting $p = r + 1$ and $q = r - 2$ in Lemma 4.4 we get

$$\sum_{i=1}^{r+1} 2^i S_{r+1-i}(-\mathbb{F} - \mathbb{B}_{r-2}) S_{i-1}(\mathbb{X}_2) = 2x_2(x_1 - x_2) \prod_{j=1}^{r-2} (2x_2 - b_j).$$

Then using the factorization property once more, we have

$$\mathcal{Q}_r(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}) = 2x_2(x_1 - x_2) R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2}) \prod_{j=1}^{r-2} (2x_2 - b_j)$$

for any $r \geq 2$. Therefore (4.5) is satisfied for any $r \geq 2$ and the theorem is proved. ∎

Remark 4.5. Let Φ denote the linear endomorphism on the $\mathbf{Z}$-module spanned by the Schur functions indexed by partitions of length ≤ 3, that sends a Schur function S_{i_1,i_2,i_3} to S_{i_1+1,i_2+1,i_3+1}. Let $\overline{\mathcal{T}_r}$ denote the sum of those terms in the Schur function expansion of $\mathcal{T}_r$ which are indexed by partitions of length 2. We have

$$\mathcal{T}_r^{III_{2,3}} = 2^{r+1} S_{r+1,2r+1} + \Phi(\mathcal{T}_{r-1}^{III_{2,3}}) \tag{4.6}$$

$$= \overline{\mathcal{T}_r^{III_{2,3}}} + \Phi(\mathcal{T}_{r-1}^{III_{2,3}}). \tag{4.7}$$

Note that a recursion of the same form also appeared in Thom polynomials for other singularities (cf. [8], [13] and [10]).

Remark 4.6. As the referee points out, Fehér and Rimányi [5] developed another method for computing Thom polynomials of contact singularities. In particular, they computed the same Thom polynomials for singularities $III_{2,3}(-)$ with different techniques.

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