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On families of trajectories of an analytic gradient vector field

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To the memory of Professor Stanisław Łojasiewicz

Abstract. For an analytic function $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ having a critical point at the origin, we describe the topological properties of the partition of the family of trajectories of the gradient equation $\dot{x} = \nabla f(x)$ attracted by the origin, given by characteristic exponents and asymptotic critical values.

1. Introduction. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function defined in a neighbourhood of the origin, having a critical point at 0. We consider the trajectories of the gradient vector field $\dot{x} = \nabla f(x)$. Take $y > 0$ such that $-y$ is a regular value of f. One can show that there exists a closed set $\Gamma \subset f^{-1}(-y)$ such that a non-trivial trajectory of the gradient field is attracted by the origin if and only if it intersects $f^{-1}(-y)$ transversally at a point belonging to Γ. Thus one may equip the set of non-trivial trajectories attracted by 0 with the topology induced from Γ.

By [18], the Čech–Alexander cohomology groups $\check{H}^*(\Gamma)$ are isomorphic to the cohomology groups $H^*(F_y)$ of the real Milnor fibre $F_y = \{x \in f^{-1}(-y) \mid |x| \leq d\}$, where $0 < y \ll d \ll 1$. A more general version concerning analytic functions on manifold is presented in [19].

By [8], if $n = 3$ and f is harmonic then Γ may be stratified.

Kurdyka *et al.* [11], in the course of proving Thom's conjecture, showed in particular that to each trajectory attracted by 0 (and so to each point in Γ) one may associate an element of a finite subset $L' \subset \mathbb{Q}^+ \times \mathbb{R}_-$. This way we obtain a natural partition

$$\Gamma = \bigcup_{(l,a) \in L'} \Gamma(l, a).$$

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In $\mathbb{Q}^+ \times \mathbb{R}_-$ we may introduce the lexicographic order, so we may enumerate the elements of L' according to this order: $L' = \{(l_1, a_1), \ldots, (l_j, a_j), \ldots, (l_s, a_s)\}$.

We will show that

$$\Gamma(l_1, a_1) \subset \cdots \subset \bigcup_{j=1}^{i} \Gamma(l_j, a_j) \subset \cdots \subset \bigcup_{j=1}^{s} \Gamma(l_j, a_j) = \Gamma$$

is a filtration of Γ by closed sets, and that there are regular values $0 < z_1 < \cdots < z_i < \cdots < z_s$ of the distance function $|x|$ restricted to the Milnor fibre F_y such that each inclusion

$$\bigcup_{j=1}^{i} \Gamma(l_j, a_j) \hookrightarrow \{x \in F_y \mid |x| \leq z_i\}$$

induces isomorphism of Čech–Alexander cohomology groups. Hence one may apply techniques of differential topology to investigate the topology of the partition $\{\Gamma(l_i, a_i)\}$ of the set of trajectories attracted by the origin.

Among the references we list several papers [2–7, 9, 10, 12, 13, 15, 17, 20–22] devoted to geometric and topological properties of solutions of the gradient equation.

2. Preliminaries. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function defined in a neighbourhood of the origin, having a critical point at 0. We consider the gradient ∇f of f. We will denote by $x(t)$ a trajectory of this vector field, that is, a curve satisfying

$$\dot{x}(t) = \nabla f(x(t)).$$

It is easy to see that $\frac{d}{dt} f(x(t)) > 0$ unless $x(t)$ is constant, that is, f is increasing along the trajectory $x(t)$. For x with $f(x) \leq 0$ and sufficiently close to the origin, we denote by τ_x the set of points on the trajectory passing through x belonging to $\{y \mid f(y) \geq f(x)\}$. Denote by $\omega(x) \in f^{-1}(0)$ either the intersection point of τ_x and $f^{-1}(0)$ or the limit point of the trajectory if it tends to $f^{-1}(0)$. It is well known that ω is a strong deformation retraction.

There is a neighbourhood U_0 of the origin, $0 < \varrho < 1$ and $c_\varrho, c_f > 0$ such that

$$|\nabla f(x)| \geq c_\varrho |f(x)|^\varrho, \tag{2.1}$$

$$|x|\,|\nabla f(x)| \geq c_f |f(x)|, \tag{2.2}$$

for $x \in U_0$. Inequality (2.1) is due to Łojasiewicz (see [14]), and (2.2) is known as the Bochnak–Łojasiewicz inequality (see [1]). In particular as a consequence of (2.1) we have $\nabla f^{-1}(0) \subseteq f^{-1}(0)$.

The gradient $\nabla f(x)$ splits into its radial component $\frac{\partial f}{\partial r}(x)\frac{x}{|x|}$ and the spherical one $\nabla' f(x) = \nabla f(x) - \frac{\partial f}{\partial r}(x)\frac{x}{|x|}$. We shall denote $x/|x|$ by $\partial/\partial r$ and $\partial f/\partial r$ by $\partial_r f$. We will also often write r instead of $|x|$. Then

$$\nabla f = \nabla' f + \partial_r f \, \frac{\partial}{\partial r}$$

and

$$|\nabla f|^2 = |\nabla' f|^2 + |\partial_r f|^2.$$

Now let y, d be such that $0 < y \ll d \ll 1$, and $-y \in \mathbb{R}$ is a regular value of f. We call the set $F_y = \{x \mid |x| \le d,\ f(x) = -y\}$ the *real Milnor fibre* of f. It is either an $(n-1)$-dimensional compact manifold with boundary or an empty set (see [16]). If $f(x) \le -y$ and $0 \in \overline{\tau}_x$ then $\tau_x \cap f^{-1}(-y) \neq \emptyset$, because the function is increasing along the trajectory. The intersection is transversal and consists exactly of one point. This justifies

Definition. $\Gamma = \{x \in F_y \mid 0 \in \overline{\tau}_x\} = \{x \in F_y \mid \omega(x) = 0\}$.

Nowel and the second-named author showed that each trajectory attracted by the origin intersects F_y at a point in Γ and the topology of the set Γ is related to the topology of the Milnor fibre. We have (see [18])

Theorem 1. *The inclusion $\Gamma \hookrightarrow F_y$ induces an isomorphism*

$$\check{H}^*(\Gamma) \simeq H^*(F_y),$$

where $\check{H}^*$ denotes the Čech–Alexander cohomology groups.

3. Invariants associated with trajectories. In order to say more about the topology of the set Γ, we need some notions introduced in [11]. For $\varepsilon > 0$ define

$$W^\varepsilon = \{x \mid f(x) \neq 0,\ \varepsilon|\nabla' f| \le |\partial_r f|\}.$$

Kurdyka *et al.* have defined the characteristic exponents, which are characterised by the following proposition ([11, Proposition 4.2]).

Proposition 2. *There exists a finite subset of positive rationals $L \subset \mathbb{Q}^+$ such that for any sequence $W^\varepsilon \ni x \to 0$ there is a subsequence $W^\varepsilon \ni x' \to 0$ and $l \in L$ such that*

$$\frac{|x'|\partial_r f(x')}{f(x')} \to l.$$

In particular, as a germ at the origin, each W^ε is the disjoint union

$$W^\varepsilon = \bigcup_{l \in L} W^\varepsilon_l,$$

where

$$W^\varepsilon_l = \left\{x \in W^\varepsilon \;\middle|\; \left|\frac{|x|\partial_r f}{f} - l\right| \le |x|^\delta\right\},$$

for $\delta > 0$ sufficiently small. Moreover, there exist constants $0 < c_\varepsilon < C_\varepsilon$, which depend on ε, such that

$$c_\varepsilon \le \frac{|f|}{|x|^l} \le C_\varepsilon \quad \text{on } W_l^\varepsilon.$$

Fix $l > 0$, not necessarily in L, and consider $F = f/|x|^l$ defined in the complement of the origin. We say that $a \in \mathbb{R}$ is an *asymptotic critical value* of F at the origin if there exists a sequence $x \to 0$, $x \neq 0$, such that

(a) $$|x|\,|\nabla F(x)| \to 0,$$

(b) $$F(x) \to a.$$

By [11, Propositions 5.1 and 5.4] we have

PROPOSITION 3. *The set of asymptotic critical values of $F = f/|x|^l$ is finite. The real number $a \neq 0$ is an asymptotic critical value if and only if there exists a sequence $x \to 0$, $x \neq 0$, such that*

(a′) $$\frac{|\nabla' f(x)|}{|\partial_r f(x)|} \to 0,$$

(b) $$F(x) \to a.$$

By the above proposition, the set

$$L' = \{(l, a) \mid l \in L, a < 0 \text{ is an asymptotic critical value of } f/|x|^l\}$$

is a finite subset of $\mathbb{Q}^+ \times \mathbb{R}_-$. For a given characteristic exponent $l \in L$ there can be more than one asymptotic critical value a. By Section 6 of [11] we have

THEOREM 4. *For every trajectory $x(t) \to 0$ of the gradient vector field there exists a unique pair $(l, a) \in L'$ such that $\frac{f}{r^l}(x(t)) \to a$.*

4. Partition of the set of trajectories

DEFINITION. There is a natural partition of Γ associated with L'. Namely for $(l, a) \in L'$,

$$\Gamma(l, a) = \{x \in \Gamma \mid f(x(t))/|x(t)|^l \to a \text{ on the trajectory } \tau_x\}.$$

DEFINITION. In the set $\mathbb{Q}^+ \times \mathbb{R}_-$ we may introduce the lexicographic order

$$(l, a) \le (l', a') \quad \text{if } l < l', \text{ or } l = l' \text{ and } a \le a'.$$

It is obvious that $(l, a) \le (l', a')$ if and only if $a|x|^l \le a'|x|^{l'}$ near the origin. We enumerate the elements of L' according to this order.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^n$. We have the following

LEMMA 5. *If $(l, a) \in (\mathbb{Q}^+ \times \mathbb{R}_-) \setminus L'$ then*

$$\langle \nabla (f - a|x|^l)(x), \nabla f(x) \rangle > 0$$

for $x \in (f - a|x|^l)^{-1}(0) \setminus \{0\}$ near 0.

Proof. Suppose, contrary to our claim, that there is a sequence $x \to 0$, $x \neq 0$, such that $f(x) - a|x|^l = 0$ and

$$\begin{aligned}(4.3)\qquad 0 &\geq \langle \nabla(f - a|x|^l), \nabla f\rangle \\ &= |\nabla f|^2 - \left\langle la|x|^{l-1}\frac{\partial}{\partial r}, \nabla' f + \partial_r f\frac{\partial}{\partial r}\right\rangle \\ &= |\nabla f|^2 - lar^{l-1}\partial_r f = |\nabla f|^2 - \frac{lf}{r}\partial_r f.\end{aligned}$$

Using (2.2) we have

$$l|f|\,|\partial_r f| \geq r|\nabla f|^2 \geq c_f|f|\,|\nabla f|.$$

Hence

$$\frac{c_f}{l}|\nabla f| \leq |\partial_r f|, \tag{4.4}$$

which means that $x \in W^{c_f/l}$. By Proposition 2, there are $l' \in L$ and a subsequence x' such that

$$\frac{|x'|\partial_r f}{f} \to l'.$$

All x' lie in $W_{l'}^{c_f/l}$, hence

$$c \leq \frac{f}{|x'|^{l'}} \leq C,$$

where $c = c_{c_f/l}$ and $C = C_{c_f/l}$. Since $f(x') = a|x'|^l$, $l = l'$ is a characteristic exponent.

We shall now prove that a is an asymptotic critical value. Let us transform the inequality (4.3):

$$0 \geq |\nabla' f|^2 + |\partial_r f|^2 - \frac{lf}{r}\frac{|\partial_r f|^2}{\partial_r f} = |\nabla' f|^2 + |\partial_r f|^2\left(1 - \frac{lf}{r\partial_r f}\right).$$

Hence

$$\frac{|\nabla' f|^2}{|\partial_r f|^2} \leq \left|1 - \frac{lf}{r\partial_r f}\right|. \tag{4.5}$$

Since

$$\frac{r\partial_r f}{f} = \frac{|x'|\partial_r f(x')}{f(x')} \to l' = l,$$

the right-hand side of the inequality (4.5) tends to 0. So does the left-hand side and we have

$$\frac{|\nabla' f|}{|\partial_r f|}(x') \to 0 \quad \text{and} \quad \frac{f(x')}{|x'|^l} = a.$$

By Proposition 3, a is an asymptotic critical value of f/r^l. ∎

Take $(l,a)\in\mathbb{Q}^+\times\mathbb{R}_-\setminus L'$ and $y>0$ close to 0 such that $-y$ is a regular value of f. Define

$$\Theta(l,a)=F_y\cap\{f-a|x|^l\le 0\}=F_y\cap\{|x|\le(y/(-a))^{1/l}\}.$$

We will show a relation between the cohomologies of $\Theta(l,a)$ and

$$\widetilde{\Gamma}(l,a)=\bigcup_{(l_i,a_i)<(l,a)}\Gamma(l_i,a_i),\qquad \text{where } (l_i,a_i)\in L'.$$

Theorem 6. *For every $(l,a)\in\mathbb{Q}^+\times\mathbb{R}_-\setminus L'$ and every $y>0$ small enough, $\widetilde{\Gamma}(l,a)$ is closed, and there is an inclusion*

$$\widetilde{\Gamma}(l,a)\hookrightarrow\Theta(l,a),$$

which induces an isomorphism

$$\check{H}^*(\widetilde{\Gamma}(l,a))\cong H^*\left(\Theta(l,a)\right).$$

Lemma 7. *For every $\varepsilon>0$ there exists $\eta=\eta(\varepsilon)>0$ such that if $|x|<\eta$ then for every point y on τ_x between x and $\omega(x)$ we have $|y|<\varepsilon$.*

Proof. For $a\in\tau_x$ denote by $\ell(x,a)$ the length of the trajectory between x and a. From the Łojasiewicz inequality (2.1) it follows (see [11]) that for x close to the origin

$$\ell(x,a)\le c_\varrho(1-\varrho)^{-1}[|f(x)|^{1-\varrho}-|f(a)|^{1-\varrho}].$$

As $a\to\omega(x)$ we get

$$\ell(x,\omega(x))\le c_\varrho(1-\varrho)^{-1}|f(x)|^{1-\varrho}=c_1|f(x)|^{1-\varrho}.$$

By continuity of f there exists η, $0<\eta<\varepsilon/2$, such that for $|x|<\eta$,

$$\ell(x,\omega(x))\le c_1|f(x)|^{1-\varrho}<\varepsilon/2.$$

That is, for x' between x and $\omega(x)$,

$$|x'|\le|x|+\ell(x,x')<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \blacksquare$$

Define $A_\le=\{x\mid -y\le f(x)\le a|x|^l\}$ and $A_==\{x\mid -y\le f(x)=a|x|^l\}$. If y is small enough then $A_\le$ is bounded by $A_=$ and $\Theta(l,a)$. By Corollary 6, $A_=$ and $\Theta(l,a)$ intersect transversally.

If $x\in\Theta(l,a)$ then $\nabla f(x)$ is normal to $\Theta(l,a)$ and points into $A_\le$. If $x\in A_=\setminus\{0\}$ then $\nabla(f-a|x|^l)$ is normal to $A_=$ and points away from $A_\le$.

We consider a mapping $\gamma:\Theta(l,a)\to A_=$ such that $\gamma(x)$ is the point of intersection of the trajectory τ_x with the set $A_=$ or $\gamma(x)=\omega(x)=0$ if τ_x does not intersect $A_=$.

Lemma 8. *γ is well defined, and $\gamma^{-1}(0)=\widetilde{\Gamma}(l,a)$.*

Proof. Consider trajectories starting from $\Theta(l,a)$. Some of them will stay in the set $A_\le$ and others will leave it forever. (A trajectory cannot get back to $A_\le$, because for a point $x\in A_=\setminus\{0\}$ we have $\langle\nabla(f-a|x|^l)(x),\nabla f(x)\rangle>0$.

The angle between the gradients $\nabla(f-a|x|^l)(x)$ and $\nabla f(x)$ is less than $\pi/2$, so the trajectory passing through x leaves $A_{\le}$.)

Consider a trajectory τ_x which stays in $A_{\le}$. By the Łojasiewicz inequality (2.1), ∇f does not vanish on $A_{\le}\setminus\{0\}$. Hence $x(t)\to 0$, i.e. $\gamma(x)=\omega(x)=0$ and $x\in\Gamma$. That is, we proved γ is well defined. By Theorem 4 there is $(l_i,a_i)\in L'$ such that $f(x(t))/|x(t)|^{l_i}\to a_i$.

The trajectory stays inside $A_{\le}$, so

$$f(x(t))-a|x(t)|^l\le 0.$$

For every $\varepsilon>0$, if $x(t)$ is sufficiently close to the origin we have

$$(a_i-\varepsilon)|x(t)|^{l_i}<f(x(t))\le a|x(t)|^l.$$

Therefore $l_i<l$ or $l_i=l$ and $a_i-\varepsilon<a$ for every $\varepsilon>0$. Hence

$$(l_i,a_i)\le(l,a).$$

Since $(l,a)\notin L'$, $(l_i,a_i)<(l,a)$.

Now consider a trajectory τ_x which leaves $A_{\le}$, i.e. $\gamma(x)\ne 0$. Then for t large enough we have $f(x(t))>a|x(t)|^l$. If τ_x starts from Γ, then $x(t)\to 0$ and there is $(l_i,a_i)\in L'$ such that $f(x(t))/|x(t)|^{l_i}\to a_i$. For every $\varepsilon>0$,

$$(a_i+\varepsilon)|x(t)|^{l_i}>f(x(t))>a|x(t)|^l$$

if $x(t)$ is sufficiently close to the origin. Applying similar arguments to the above we have $(l_i,a_i)>(l,a)$. Similarly for a trajectory which starts from Γ outside $\Theta(l,a)$: it cannot enter the set $A_{\le}$ and hence (l_i,a_i) corresponding to that trajectory is greater than (l,a). ∎

Lemma 9. *γ is continuous, and γ restricted to $\Theta(l,a)\setminus\widetilde{\Gamma}(l,a)$ is a homeomorphism onto* $\operatorname{Im}\gamma\setminus\{0\}=A_{=}\setminus\{0\}$. *In particular, $\widetilde{\Gamma}(l,a)$ is compact.*

Proof. Consider $x\in\Theta(l,a)$ such that $\gamma(x)\ne 0$. Then τ_x is transversal to $\Theta(l,a)$ at x and to $A_{=}$ at $\gamma(x)$, therefore γ is a Poincaré mapping in some neighbourhood of x. Hence γ is a local homeomorphism at x.

Now take x such that $\gamma(x)=0$. Then $\tau_x\subset A_{\le}$ and $0\in\overline{\tau}_x$. Fix an $\varepsilon>0$. There is $x'\in\tau_x$ such that $|x'|<\eta/2$, where $\eta=\eta(\varepsilon)$ comes from Lemma 7. Now consider a neighbourhood V of x' of diameter $\eta/2$ contained in $A_{\le}$. Reversing trajectories we get an open neighbourhood $W\subset\Theta(l,a)$ of x such that $|\gamma(y)|<\varepsilon$ for $y\in W$. ∎

Lemma 10. *For every open neighbourhood U of $\widetilde{\Gamma}(l,a)$ in $\Theta(l,a)$, $\gamma(U)$ is an open neighbourhood of* 0 *in* $\operatorname{Im}\gamma=A_{=}$.

Proof. Rewrite the proof of Lemma 9 in [18] substituting $\Theta(l,a)$ for F_r and $\operatorname{Im}\gamma$ for Z_r. ∎

Proof of Theorem 6. The inclusion $\widetilde{\Gamma}(l,a)\subseteq\Theta(l,a)$ follows from the fact that $\widetilde{\Gamma}(l,a)=\gamma^{-1}(0)$ as stated in Lemma 8.

In order to prove that the inclusion induces an isomorphism of Čech–Alexander cohomology groups, we will construct a descending family $\Theta(l,a) = U_1 \supset U_2 \supset \cdots$ of open neighbourhoods of $\widetilde{\Gamma}(l,a)$ in $\Theta(l,a)$, which satisfies

(u1) every inclusion $U_{n+1} \subset U_n$ is a homotopy equivalence,

(u2) for every neighbourhood U of $\widetilde{\Gamma}(l,a)$ in $\Theta(l,a)$ there is n such that $U_n \subset U$.

The set $\operatorname{Im}\gamma = A_= = \{x \mid f = a|x|^l,\ |x| \le (y/(-a))^{1/l}\}$, for y small enough, is homeomorphic to a cone with vertex at 0, so there is a descending family $A_= = V_1 \supset V_2 \supset \cdots$ of open neighbourhoods of 0 in $A_=$ such that every inclusion is a homotopy equivalence and for every open neighbourhood V of 0 in $A_=$ there is n such that $V_n \subset V$. We put $U_n = \gamma^{-1}(V_n)$. Clearly $\{U_n\}$ is a family of open neighbourhoods of $\widetilde{\Gamma}(l,a)$ in $\Theta(l,a)$. The mapping γ restricted to $\Theta(l,a)\setminus\widetilde{\Gamma}(l,a)$ is a homeomorphism onto $A_=\setminus\{0\}$, hence (u1) holds. If U is an open neighbourhood of $\widetilde{\Gamma}(l,a)$ then by Lemma 10, $\gamma(U)$ is an open neighbourhood of 0. There is n such that $V_n \subset \gamma(U)$; then $U_n \subset U$, so (u2) holds.

As the family $\{U_n\}$ is cofinal in the family of all open neighbourhoods of $\widetilde{\Gamma}(l,a)$ in $\Theta(l,a)$ ordered by $\supseteq$, we have an isomorphism of direct limits

$$\varinjlim_{U} H^*(U) \cong \varinjlim_{U_n} H^*(U_n) = \check{H}^*(\widetilde{\Gamma}(l,a)).$$

Since $H^*(U_n) \cong H^*(\Theta(l,a))$ by (u1), the theorem holds. ∎

For given $l \in \mathbb{Q}^+$ and y, $(y/(-a))^{1/l}$ is a regular value of $|x|_{|F_y}$, for almost all $a \in \mathbb{R}_-$. In that case $\Theta(l,a)$ is either void or a compact $(n-1)$-manifold with boundary.

Proposition 11. *For each $(l,a) \in (\mathbb{Q}^+ \times \mathbb{R}_-)\setminus L'$ and each $y > 0$ small enough, $z = (y/(-a))^{1/l}$ is a regular value for $|x|_{|F_y}$ and the inclusion*

$$\widetilde{\Gamma}(l,a) = \bigcup_{(l_i,a_i)<(l,a)} \Gamma(l_i,a_i) \hookrightarrow F_y \cap \{|x| \le z\}$$

induces an isomorphism of Čech–Alexander cohomology groups.

Proof. Consider the set of critical values of $|x|_{|F_y}$. For a given y we have finitely many critical values $w_1(y),\ldots,w_p(y)$. We can treat $w_j(y)$ as a real function. The graph of w_j is a subanalytic set. Since it lies in the plane, it is semianalytic. Hence we can write the Puiseux expansion for each w_j (see [14]):

$$w_j(y) = by^m + \cdots \qquad (b>0,\ m \in \mathbb{Q}_+).$$

We will show that $(1/m, -b^{-1/m}) \in L'$.

By the curve selection lemma we can choose a curve $\xi(r)$ of critical points corresponding to w_j. We parametrize the curve by the distance to the origin.

Put $y(r) = -f(\xi(r))$. That is, $\xi(r) \in F_{y(r)}$ is a critical point of $|x|_{|F(y(r))}$ such that

$$r = |\xi(r)| = w_j(y(r)) = b(y(r))^m + \cdots . \tag{4.6}$$

We can also write a Puiseux expansion of f along this curve,

$$f(\xi(r)) = -\alpha r^q + \cdots \quad (\alpha > 0,\ q \in \mathbb{Q}_+).$$

Thus

$$y(r) = \alpha r^q + \cdots \tag{4.7}$$

By (4.7) and (4.6) we get

$$r = b(\alpha r^q)^m + \cdots = b\alpha^m r^{qm} + \cdots \tag{4.8}$$

along the curve $\xi(r)$. Hence $qm = 1$ and $b\alpha^m = 1$. That is,

$$f(\xi(r)) = -b^{-1/m} r^{1/m} + \cdots . \tag{4.9}$$

The curve $\xi(r)$ consists of critical points of $|x|_{|F_{y(r)}}$ and therefore on $\xi(r)$ we have $|\nabla' f| \equiv 0$, $|\nabla f| = |\partial_r f|$. For every $\varepsilon > 0$ we have $\varepsilon|\nabla' f| < |\partial_r f|$, and that means the curve ξ lies in every W^ε, so there exists a characteristic exponent l' such that ξ lies in $W^\varepsilon_{l'}$.

Since $f(\xi(r))/|\xi(r)|^{1/m} \to -b^{-1/m}$, it follows that $l' = 1/m \in L$ by the last statement of Proposition 2. By Proposition 3, $-b^{-1/m}$ is the corresponding asymptotic critical value for $f/r^{1/m}$. In particular, $(1/m, -b^{-1/m}) \in L'$. Assume that $(l, a) \notin L'$. If y is small enough, then $(y/(-a))^{1/l} = (-a)^{-1/l} y^{1/l}$ is different from any $w_j(y)$. Hence it is a regular value for $|x|_{|F_y}$.

Now it is enough to apply Theorem 7. ■

The proof above gives us even more:

THEOREM 12. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function defined in a neighbourhood of the origin, having a critical point at* 0. *For each y small enough there is a finite sequence $0 < z_1 < \cdots < z_i < \cdots < z_s$ of regular values of $|x|_{|F_y}$ such that*

$$\Gamma(l_1, a_1) \subset \cdots \subset \bigcup_{j=1}^{i} \Gamma(l_j, a_j) \subset \cdots \subset \bigcup_{j=1}^{s} \Gamma(l_j, a_j) = \Gamma$$

is a filtration of Γ by closed sets, and the inclusions

$$\bigcup_{j=1}^{i} \Gamma(l_j, a_j) \hookrightarrow \{x \in F_y \mid |x| \le z_i\}$$

induce isomorphisms of Čech–Alexander cohomology groups. One can take $z_i = (y/(-a))^{1/l}$, where $(l_i, a_i) < (l, a) < (l_{i+1}, a_{i+1})$.

Proof. Let s be the cardinality of L'. As we have seen in the proof of Corollary 11, if $(l, a) \notin L'$ then $(y/(-a))^{1/l}$ is a regular value of $|x|_{|F_y}$. Since

L' is totally ordered by the lexicographic ordering, for every i we can choose a pair (l,a) such that

$$(l_i,a_i) < (l,a) < (l_{i+1},a_{i+1}),$$

where (l_{s+1},a_{s+1}) is greater than any pair in L'. Set $z_i(y) = (y/(-a))^{1/l}$. One can easily see that $z_i < z_{i+1}$ and $z_i(y) \neq w_j(y)$ for sufficiently small y.

By Proposition 11, the vertical inclusions induce isomorphisms of the Čech–Alexander cohomology groups. ∎

The above theorem shows that applying well known methods of differential topology and Morse theory to the distance function $|x|$ on the Milnor fibre may provide important information about the topology of families of trajectories of an analytic gradient vector field with given characteristic exponent and asymptotic critical value.

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