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The Łojasiewicz gradient inequality in a neighbourhood of the fibre

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Abstract. Some estimates of the Łojasiewicz gradient exponent at infinity near any fibre of a polynomial in two variables are given. An important point in the proofs is a new Charzyński–Kozłowski–Smale estimate of critical values of a polynomial in one variable.

1. Introduction. In this paper, effective estimates relating to the Łojasiewicz gradient inequality at infinity for polynomials in two variables are given. To achieve them, we prove an estimate for critical values of a polynomial in one variable (Theorem 2.1), which is a version of the Charzyński–Kozłowski [2] and Smale [24] theorems. Namely, if $P:\mathbb{C}\to\mathbb{C}$ is a polynomial of degree $d>1$, $\varphi_1,\dots,\varphi_d\in\mathbb{C}$ and $\xi_1,\dots,\xi_{d-1}\in\mathbb{C}$ are all roots of P and of its derivative P', respectively, and a is the leading coefficient of P, then

$$\min_{1\le k\le d-1}|P(\xi_k)|\le 4|a|^{-1/(d-1)}[\min_{1\le i\le d}|P'(\varphi_i)|]^{d/(d-1)}.$$

The first result (Theorem 3.1) on the Łojasiewicz inequality is the following generalisation of the Bochnak–Łojasiewicz inequality ([1, Lemma 2], [25, Theorem 1]); for a polynomial $f:\mathbb{C}^n\to\mathbb{C}$, there exist $C,\varepsilon>0$ such that

$$|f(z)|\le\varepsilon\ \Rightarrow\ |z|\,|\operatorname{grad}f(z)|\ge C|f(z)|,$$

where $|\cdot|$ is a norm in $\mathbb{C}^n$.

From the above two inequalities, we obtain an estimate of the Łojasiewicz gradient exponent in a neighbourhood of the bifurcation fibre of a polynomial. By definition, the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(F|X)$ of a polynomial mapping $F:\mathbb{C}^n\to\mathbb{C}^m$ on an unbounded set $X\subset\mathbb{C}^n$ is the

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best exponent ν in the inequality

$$|F(z)| \geq C|z|^{\nu} \quad \text{as } z \in X,\, z \to \infty,$$

for some constant $C > 0$. In the case $F = \operatorname{grad} f$, where f is a polynomial in two variables, and $X = \{z \in \mathbb{C}^2 : |f(z)| < \varepsilon\}$ is a neighbourhood of $f^{-1}(0)$, $\mathcal{L}_\infty(\operatorname{grad} f|X)$ is equal to the exponent $\mathcal{L}_{\infty,0}(f)$ considered by Ha [8] and by Chądzyński and Krasiński [5], provided $\varepsilon > 0$ is sufficiently close to 0. It is shown in Theorem 4.2 and Corollary 6.1 (cf. [8, Theorem 1.3.2], [12, Theorem 3.1]) that if $0 \in \mathbb{C}$ is a bifurcation point of f at infinity (see Section 6), and $d = \deg f > 2$, then for any neighbourhood X of $f^{-1}(0)$,

$$(*) \qquad \mathcal{L}_\infty(\operatorname{grad} f|X) \leq -1 - \frac{1}{d-2}$$

or equivalently

$$\mathcal{L}_\infty(f, \operatorname{grad} f) < 0.$$

Estimate $(*)$ is sharp as regards the degree of f (see Remark 6.3). In particular, we obtain the following result of Gwoździewicz and Płoski (see [7, Theorem 5.2]): if the bifurcation set of f at infinity is nonempty, then $\mathcal{L}_\infty(\operatorname{grad} f) \leq -1 - 1/(d-2)$ (Corollary 6.2).

Theorem 4.2 gives a sharper version of the Malgrange condition (condition (m) in [15]), namely the inequality

$$|z|^{\alpha}|\operatorname{grad} f(z)| \geq \eta \quad \text{as } z \to \infty \text{ and } f(z) \to 0$$

does not depend on the choice of α such that $0 \leq \alpha < 1 + 1/(d-2)$ (Proposition 5.1).

In Theorem 7.5 we prove the following separation condition of $\operatorname{grad} f$ and f (introduced by Płoski and Tworzewski in [20], see also [25]):

$$\text{(PT)} \qquad |f(z)| \leq \varepsilon \;\Rightarrow\; |\operatorname{grad} f(z)| \geq C|f(z)|^{q}$$

for some $C, \varepsilon, q > 0$. Moreover one can take $q = (d-1)^2$. In the general case, i.e. $f : \mathbb{C}^n \to \mathbb{C}$, $n > 2$, condition (PT) may not be satisfied (see [25, Remark 2] and [21, Remark 9.1]). The description of polynomials for which (PT) holds is given in Remark 7.6.

2. The Charzyński–Kozłowski–Smale Theorem. In this section we give a version of the Charzyński–Kozłowski (see [2, Theorem 3]) and Smale (see [24, p. 33]) Theorem.

THEOREM 2.1. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$, and let $\varphi_1, \ldots, \varphi_d \in \mathbb{C}$ and $\xi_1, \ldots, \xi_{d-1} \in \mathbb{C}$ be all roots of P and P', respectively. Then*

$$(1) \qquad \min_{1 \leq k \leq d-1} |P(\xi_k)| \leq 4 \min_{i \neq j} |\varphi_i - \varphi_j|\, |P'(\varphi_i)|.$$

In particular,

$$\min_{1\le k\le d-1}|P(\xi_k)| \le 4|a|^{-1/(d-1)}[\min_{1\le i\le d}|P'(\varphi_i)|]^{d/(d-1)}, \tag{2}$$

where $a\in\mathbb{C}$ *is the leading coefficient of* P.

Proof. If $A\subset\mathbb{C}$ and $b\in\mathbb{C}$, we put $A-b=\{a-b : a\in A\}$ and $bA=\{ab : a\in A\}$. For $a\in\mathbb{C}$ and $r>0$, we denote by $D(a,r)$ the disc with centre at a and radius r.

Let

$$R=\min_{1\le k\le d-1}|P(\xi_k)|.$$

It suffices to consider the case $R>0$. Then obviously $\varphi_i\ne\varphi_j$ for $i\ne j$. Let $G=P^{-1}(D(0,R))$. Since $P|_G : G\to D(0,R)$ has no critical values, it is a d-sheeted covering. As $D(0,R)$ is a simply connected domain, $G=G_1\cup\dots\cup G_d$, where $G_1,\dots,G_d$ are domains such that $P|_{G_i}:G_i\to D(0,R)$ is a biholomorphism, $i=1,\dots,d$. Write $f_i=(P|_{G_i})^{-1}:D(0,R)\to G_i$. We may renumber f_i so that $\varphi_i=f_i(0)$, $i=1,\dots,d$. Take any $i\in\{1,\dots,d\}$ and put

$$g_i(w)=\frac{1}{Rf_i'(0)}\,[f_i(wR)-\varphi_i],\qquad w\in D(0,1).$$

Each g_i is an injective holomorphic function such that $g_i(0)=0$ and $g_i'(0)=1$. Therefore, by the Koebe Theorem (see [10]), $D(0,1/4)\subset g_i(D(0,1))$. In consequence,

$$D(0,R|f_i'(0)|/4)\subset Rf_i'(0)g_i(D(0,1))=f_i(D(0,R))-\varphi_i.$$

Hence,

$$D(\varphi_i,R|f_i'(0)|/4)\subset f_i(D(0,R))=G_i.$$

Since $\varphi_j\notin G_i$ for $i\ne j$, by the above we have

$$\frac{R}{4|P'(\varphi_i)|}=\frac{R|f_i'(0)|}{4}\le|\varphi_i-\varphi_j|,\qquad i\ne j.$$

Hence (1) follows. From (1) we see that

$$\min_{1\le k\le d-1}|P(\xi_k)|\le 4|\varphi_i-\varphi_j|\,|P'(\varphi_i)|\quad\text{ for any } j\ne i.$$

Thus, from $P'(\varphi_i)=a\prod_{j\ne i}(\varphi_i-\varphi_j)$, we deduce (2). ∎

The inequality (1) cannot be improved, except for the constant 4. Namely we have

PROPOSITION 2.2. *Let* $P:\mathbb{C}\to\mathbb{C}$ *be a polynomial of degree* $d>1$, *and let* $\varphi_1,\dots,\varphi_d\in\mathbb{C}$ *and* $\xi_1,\dots,\xi_{d-1}\in\mathbb{C}$ *be all roots of* P *and* P', *respectively. Then*

$$\frac{2^{d-2}}{3^d}\min_{i\ne j}|\varphi_i-\varphi_j|\,|P'(\varphi_i)|\le\min_{1\le k\le d-1}|P(\xi_k)|. \tag{3}$$

Proof. Let $r = \min_{i\neq j} |\varphi_i - \varphi_j|\,|P'(\varphi_i)|$. Under the notation of the proof of Theorem 2.1, it suffices to consider the case $R > 0$. For any $i \in \{1,\dots,d\}$, let $j_i \in \{1,\dots,d\}$ be such that $|\varphi_i - \varphi_{j_i}| = \min_{i\neq j} |\varphi_i - \varphi_j|$, and let $r_i = (1/2)|\varphi_i - \varphi_{j_i}|$, $M_i = (3/2)^{d-1} r_i |P'(\varphi_i)|$. Set $D_i = D(\varphi_i, r_i)$, $i = 1,\dots,d$. Then for any $z \in \overline{D}_i$ we easily obtain

$$|P(z)| = |z-\varphi_i|\,|a| \prod_{j\neq i} |z-\varphi_j| \le \frac{1}{2}\,|\varphi_i - \varphi_{j_i}|\,|a| \prod_{j\neq i} \left(\frac{3}{2}\,|\varphi_i - \varphi_j|\right) = M_i.$$

Hence, by [23, Ch. VII, Theorem 12.7],

$$D\biggl(0, \frac{r_i^2|P'(\varphi_i)|^2}{6M_i}\biggr) \subset P(D_i), \qquad i = 1,\dots,d.$$

For any $i \in \{1,\dots,d\}$, from the definition of r, r_i and M_i, we have

$$\frac{2^{d-2}}{3^d}\, r \le \frac{r_i^2|P'(\varphi_i)|^2}{6M_i}.$$

In consequence for $\widetilde{r} = 2^{d-2} r/3^d$, we obtain $D(0,\widetilde{r}) \subset P(D_i)$, $i = 1,\dots,d$. Since $D_i \cap D_j = \emptyset$ for $i \neq j$, for any $w \in D(0,\widetilde{r})$ we have $\#P^{-1}(w) = d$. Summing up, P has no critical values in $D(0,\widetilde{r})$, so $R \ge \widetilde{r}$ and we get (3). ■

REMARK 2.3. From Theorem 2.1 and Proposition 2.2, it is easy to prove the Kuo and Lu formula for the Łojasiewicz gradient exponent of a holomorphic function at zero [11] and the Ha formula for the Łojasiewicz gradient exponent at infinity of a polynomial [8] as is shown in [26].

3. The Łojasiewicz gradient inequality. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial in $z_1,\dots,z_n$ and let $\operatorname{grad} f = (\partial f/\partial z_1,\dots,\partial f/\partial z_n) : \mathbb{C}^n \to \mathbb{C}^n$. We will prove the following version of the Bochnak–Łojasiewicz inequality [1] (cf. the main result of [25]).

THEOREM 3.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then there exist $C, \varepsilon > 0$ such that*

$$|f(z)| \le \varepsilon \;\Rightarrow\; |z|\,|\operatorname{grad} f(z)| \ge C|f(z)|.$$

We begin with a definition. A curve $\varphi : (r,\infty) \to \mathbb{R}^k$, $r \in \mathbb{R}$, is called *meromorphic at* ∞ if φ is the sum of a Laurent series of the form

$$\varphi(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \cdots, \qquad \alpha_i \in \mathbb{R}^k,\ p \in \mathbb{Z}.$$

If $\varphi \neq 0$, then we may assume that $\alpha_p \neq 0$. Then the number p is called the *degree* of φ and denoted by $\deg\varphi$. Additionally we put $\deg 0 = -\infty$.

Proof of Theorem 3.1. As in [25], we use Hörmander's method. The Łojasiewicz inequality does not depend on a particular norm in $\mathbb{C}^n$, so we shall use the Euclidean norm $\|\cdot\|$. Assume to the contrary that for any $\varepsilon > 0$ there exists $z \in \mathbb{C}^n$ such that $|f(z)| \le \varepsilon$ and $\|\operatorname{grad} f(z)\|\|z\| < \varepsilon|f(z)|$. Then there exists $z^0 \in \mathbb{C} \cup \{\infty\}$ such that $(z^0, 0)$ is an accumulation point of the semi-algebraic set

$$X = \{(z, \varepsilon) \in \mathbb{C}^n \times \mathbb{R} : \varepsilon > 0 \wedge |f(z)| \le \varepsilon \wedge \|z\| \, \|\mathrm{grad}\, f(z)\| < \varepsilon |f(z)|\}.$$

Thus, by the Curve Selection Lemma ([13, Lemma 3.1]), there exists a curve $\psi = (\varphi, \varphi_{n+1}) : (r, \infty) \to X$ meromorphic at ∞, such that $\lim_{t\to\infty} \psi(t) = (z^0, 0)$. Hence, $\deg \varphi_{n+1} < 0$. By the definition of X we have

$$\deg((\mathrm{grad}\, f) \circ \varphi) + \deg \varphi \le \deg \varphi_{n+1} + \deg f \circ \varphi < \deg f \circ \varphi$$

and $\deg f \circ \varphi \ne 0$. This is impossible, because, for $\varphi = (\varphi_1, \ldots, \varphi_n)$, we get

$$\begin{aligned} \deg f \circ \varphi = 1 + \deg(f \circ \varphi)' &= 1 + \deg \left(\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(\varphi)\varphi_i' \right) \\ &\le \deg((\mathrm{grad}\, f) \circ \varphi) + \deg \varphi. \ \blacksquare \end{aligned}$$

REMARK 3.2. From Theorem 3.1 we easily obtain the analogous Łojasiewicz inequality for a real polynomial $f : \mathbb{R}^n \to \mathbb{R}$.

4. The Łojasiewicz exponent of the gradient. Let us start from the precise definition of the Łojasiewicz exponent. Let $F : \mathbb{C}^2 \to \mathbb{C}^m$ be a polynomial mapping, and let $X \subset \mathbb{C}^2$ be an unbounded set. Put

$$N(F|X) := \{\nu \in \mathbb{R} : \exists_{A,B>0} \forall_{z\in X} \, (|z| \ge B \Rightarrow A|z|^\nu \le |F(z)|)\}.$$

By the *Łojasiewicz exponent at infinity of F on X* we mean $\mathcal{L}_\infty(F|X) = \sup N(F|X)$ if $N(F|S) \ne \emptyset$, and $\mathcal{L}_\infty(F|X) = -\infty$ if $N(F|X) = \emptyset$. If $X = \mathbb{C}^2$, we write $\mathcal{L}_\infty(F)$ and call it the *Łojasiewicz exponent at infinity* of F.

Let $U \subset \mathbb{C}$ be a neighbourhood of infinity, i.e. the complement of a compact set. Analogously to the real case, a mapping $h : U \to \mathbb{C}^m$ is called *meromorphic at infinity* if h is the sum of a Laurent series of the form

$$h(t) = \alpha_p t^p + \alpha_{p-1} t^{p-1} + \cdots, \quad t \in U, \ \alpha_i \in \mathbb{C}^m, \ p \in \mathbb{Z}.$$

If $m = 1$, then h is called a *function meromorphic at infinity.*

Throughout the remainder of this section, let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial in $z = (x, y)$, and let $d = \deg f \ge 2$. Let $V = \{z \in \mathbb{C}^2 : f(z) = 0\}$ and

$$V_x = \left\{ z \in \mathbb{C}^2 : \frac{\partial f}{\partial x}(z) = 0 \right\}, \quad V_y = \left\{ z \in \mathbb{C}^2 : \frac{\partial f}{\partial y}(z) = 0 \right\}.$$

If $\deg f = \deg_y f$ then, by the Puiseux Theorem at infinity (see [3, Lemmas 4.1 and 4.2]), there exist $N \in \mathbb{Z}$, $N > 0$, $a \in \mathbb{C}$, and functions $\varphi_1, \ldots, \varphi_d, \xi_1, \ldots, \xi_{d-1} : U \to \mathbb{C}$ meromorphic at infinity such that

$$(4) \qquad f(t^N, y) = a \prod_{i=1}^{d} (y - \varphi_i(t)), \quad \frac{\partial f}{\partial y}(t^N, y) = ad \prod_{k=1}^{d-1} (y - \xi_k(t)).$$

From (2) in Theorem 2.1 we immediately obtain (cf. [5, Theorem 3.3]).

PROPOSITION 4.1. *Under the above assumptions*,

$$\mathcal{L}_\infty(f|V_y) \le \frac{d}{d-1}\,\mathcal{L}_\infty(\operatorname{grad} f|V).$$

Proof. It is easy to see that

$$\mathcal{L}_\infty(f|V_y) = \frac{1}{N}\min_{1\le k\le d-1} \deg f(t^N, \xi_k(t))$$

and

$$\frac{1}{N}\min_{1\le i\le d} \deg \frac{\partial f}{\partial y}(t^N, \varphi_i(t)) \le \mathcal{L}_\infty(\operatorname{grad} f|V).$$

Hence and from (2) in Theorem 2.1 the assertion follows. ∎

Let us state a generalisation of the Ha Theorem (cf. [8, Theorem 1.3.2], [12, Theorem 3.1], [5, Corollary 3.5]).

THEOREM 4.2. *For every polynomial* $f : \mathbb{C}^2 \to \mathbb{C}$ *with* $d = \deg f > 2$, *the following conditions are equivalent*:

(i) $\mathcal{L}_\infty(f, \operatorname{grad} f) < 0$,
(ii) $\mathcal{L}_\infty(f, \operatorname{grad} f) \le -1/(d-2)$,
(iii) $\mathcal{L}_\infty(\operatorname{grad} f|X) \le -1-1/(d-2)$ *for any* $X = \{z \in \mathbb{C}^2 : |f(z)| < \varepsilon\}$, *where* $\varepsilon > 0$.

Proof. The implication (iii)⇒(i) follows from Theorem 3.1, and (ii)⇒(i) is trivial.

Assume (i). We prove (ii) and (iii). Since $\mathcal{L}_\infty(\operatorname{grad} f)$ and $\mathcal{L}_\infty(f, \operatorname{grad} f)$ do not depend on the choice of the coordinate system, after a linear change of coordinates one can assume that $\deg f = \deg_x f = \deg_y f$. Moreover one can assume that f has no multiple factors and (4) holds. Take any $X = \{z \in \mathbb{C}^2 : |f(z)| < \varepsilon\}$, where $\varepsilon > 0$. By [4, Theorem 1] and (i),

$$\min\Big\{\mathcal{L}_\infty(\operatorname{grad} f|V),\ \mathcal{L}_\infty\Big(\Big(f, \frac{\partial f}{\partial x}\Big)\Big| V_y\Big),\ \mathcal{L}_\infty\Big(\Big(f, \frac{\partial f}{\partial y}\Big)\Big| V_x\Big)\Big\} = \mathcal{L}_\infty(f, \operatorname{grad} f) < 0.$$

If $\mathcal{L}_\infty(\operatorname{grad} f|V) = \mathcal{L}_\infty(f, \operatorname{grad} f)$, then, by Proposition 4.1, $\mathcal{L}_\infty(f|V_y) < 0$. So, by [7, Theorem 2.9],

$$\mathcal{L}_\infty(f|V_y) \le -\frac{1}{d-2}. \tag{5}$$

Moreover, there exists $k_0 \in \{1, \ldots, d-1\}$ such that $(1/N)\deg f(t^N, \xi_{k_0}(t)) = \mathcal{L}_\infty(f|V_y)$, and, for some $R \ge r$, $Y = \{(t^N, \xi_{k_0}(t)) \in \mathbb{C}^2 :\ |t| \ge R^{1/N}\}$ is an unbounded subset of $V_y \cap X$. Therefore,

$$(6)\quad \mathcal{L}_\infty(\operatorname{grad} f|Y) = \frac{1}{N}\deg\frac{\partial f}{\partial x}(t^N, \xi_{k_0}(t)) \le -1 + \frac{1}{N}\deg f(t^N, \xi_{k_0}(t)) = -1 + \mathcal{L}_\infty(f|V_y),$$

and (5) gives (ii) in this case. Moreover, by (6),

$$\mathcal{L}_\infty(\operatorname{grad} f|X) \le \mathcal{L}_\infty(\operatorname{grad} f|Y) \le -1 - \frac{1}{d-2}.$$

This gives (iii) in this case.

If $\mathcal{L}_\infty(f, \frac{\partial f}{\partial x}|V_y) = \mathcal{L}_\infty(f, \operatorname{grad} f)$, then $\mathcal{L}_\infty(f|V_y) < 0$ and as above we deduce (ii) and (iii).

If $\mathcal{L}_\infty(f, \frac{\partial f}{\partial y}|V_x) = \mathcal{L}_\infty(f, \operatorname{grad} f)$, then, by exchanging the roles of x and y, from the second case we obtain (ii) and (iii) in this case. This ends the proof. ∎

REMARK 4.3. The omitted case $\deg f \le 2$ in Theorem 4.2 is not essential. Indeed, for $\deg f = 1$, the gradient of f is a constant mapping. For $\deg f = 2$, either f is a square function of a linear polynomial or $\mathcal{L}_\infty(\operatorname{grad} f) > 0$.

From the proof of Theorem 4.2 we obtain the following version of Theorem 3.4 in [5].

COROLLARY 4.4. *If* $\mathcal{L}_\infty(f, \operatorname{grad} f) < 0$, *then*

$$\mathcal{L}_\infty(\operatorname{grad} f|X) = \mathcal{L}_\infty(f, \operatorname{grad} f) - 1,$$

where $X = \{z \in \mathbb{C}^2 : |f(z)| < \varepsilon\}$ *and* $\varepsilon > 0$ *is sufficiently close to* 0.

Proof. By Theorem 3.1 we have $\mathcal{L}_\infty(\operatorname{grad} f|X) \ge \mathcal{L}_\infty(f, \operatorname{grad} f) - 1$. As in the proof of Theorem 4.2 we deduce $\mathcal{L}_\infty(\operatorname{grad} f|X) \le \mathcal{L}_\infty(f, \operatorname{grad} f) - 1$. ∎

REMARK 4.5. We define

$$l(f) = \frac{1}{N}\min_{i\ne j}\biggl(\deg(\varphi_i(t) - \varphi_j(t)) + \deg\frac{\partial f}{\partial y}(t^N, \varphi_i(t))\biggr).$$

Then, by Theorem 2.1, we obtain $\mathcal{L}_\infty(f|V_y) = l(f)$ and $\mathcal{L}_\infty(\operatorname{grad} f|V_y) \le l(f) - 1$, so

$$\mathcal{L}_\infty((f, \operatorname{grad} f)|V_y) = l(f)$$

(cf. [7, B.2], [5, Proposition 2.3], [22, Proposition 2]). If $\mathcal{L}_\infty(f, \operatorname{grad} f) < 0$, then by [5, Theorem 3.3] and Corollary 4.4,

$$\mathcal{L}_\infty(f, \operatorname{grad} f) = \mathcal{L}_\infty(f|V_y) = l(f)$$

(cf. [8, Theorem 1.4.1]).

5. The Fedoryuk and Malgrange conditions. From Theorem 4.2 we easily obtain the following proposition (cf. [8, Theorem 1.3.2], [12, Theorem 3.1]).

Proposition 5.1. *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial with* $d = \deg f > 2$, *and let* $\lambda \in \mathbb{C}$ *and* $0 < \alpha < 1 + 1/(d-2)$. *Then the following conditions are equivalent*:

(i) *there exist* $\eta, R, \varepsilon > 0$ *such that*

$$|z| \geq R \wedge |f(z) - \lambda| \leq \varepsilon \;\Rightarrow\; |\operatorname{grad} f(z)| \geq \eta, \tag{F}$$

(ii) *there exist* $\eta, R, \varepsilon > 0$ *such that*

$$|z| \geq R \wedge |f(z) - \lambda| \leq \varepsilon \;\Rightarrow\; |z|^{\alpha} |\operatorname{grad} f(z)| \geq \eta. \tag{M}$$

Conditions (F) and (M) are called the *Fedoryuk condition* (see [6]) and the *Malgrange condition* (cf. [17]), respectively. Denote by $K_\infty(f)$ the set of $\lambda \in \mathbb{C}$ for which condition (F) fails, and by $\widetilde{K}_\infty(f)$ the set of λ for which (M) with $\alpha = 1$ fails.

Using Proposition 5.1 and the known fact that $\widetilde{K}_\infty(f)$ is finite (cf. [9, Theorem 1.1], see also [25, Corollary 4]) we deduce the following known fact.

Corollary 5.2. *If* $f : \mathbb{C}^2 \to \mathbb{C}$ *is a polynomial*, *then* $K_\infty(f)$ *is finite.*

6. The Łojasiewicz exponent and bifurcation points. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. The smallest set $B(f) \subset \mathbb{C}$ such that f is a fibration outside $B(f)$ is called the *bifurcation set* of f. The smallest set $B_\infty(f) \subset \mathbb{C}$ such that f is a fibration at infinity outside $B_\infty(f)$ is called the *bifurcation set* of f *at infinity*. More precisely, $\lambda \notin B_\infty(f)$ if there exists a compact $H \subset \mathbb{C}^n$ such that $f|_{\mathbb{C}^n \setminus H} : \mathbb{C}^n \setminus H \to \mathbb{C}$ is a trivial fibration over a neighbourhood $U \subset \mathbb{C}$ of λ. It is known that $B_\infty(f) \subset \widetilde{K}_\infty(f)$, and $B(f) \subset K_0(f) \cup \widetilde{K}_\infty(f)$, where $K_0(f)$ is the set of critical values of f ([15, Lemma 1.2 and Remark 1.3]). In the case $n = 2$ we have $B_\infty(f) = \widetilde{K}_\infty(f)$ ([15, Theorem 1.4]).

From Theorem 4.2 we immediately obtain (cf. [25, Corollary 4])

Corollary 6.1. *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial with* $d = \deg f > 2$. *The following conditions are equivalent*:

(i) $\lambda \in B_\infty(f)$,
(ii) $\mathcal{L}_\infty(f - \lambda, \operatorname{grad} f) < 0$,
(iii) $\mathcal{L}_\infty(f - \lambda, \operatorname{grad} f) \leq -1/(d-2)$,
(iv) $\mathcal{L}_\infty(\operatorname{grad} f|X) \leq -1 - 1/(d-2)$ *for any* $X = \{z \in \mathbb{C}^2 : |f(z) - \lambda| \leq \varepsilon\}$, $\varepsilon > 0$.

Proof. Since $B_\infty(f) = \widetilde{K}_\infty(f)$, it follows that $\lambda \in B_\infty(f)$ if and only if there exists a sequence $\{z^m\} \subset \mathbb{C}^2$ with $z^m \to \infty$ such that $(f, \operatorname{grad} f)(z^m) \to (\lambda, 0, 0)$. Thus $\lambda \in B_\infty(f)$ if and only if $\mathcal{L}_\infty(f - \lambda, \operatorname{grad} f) < 0$. This gives the equivalence (i)$\Leftrightarrow$(ii). The remaining equivalences immediately follow from Theorem 4.2. ∎

Corollary 6.1 implies the following result Gwoździewicz and Płoski (see [7, Theorem 5.2]):

COROLLARY 6.2. *Let* $f : \mathbb{C}^2 \to \mathbb{C}$ *be a polynomial with* $d = \deg f > 2$. *If* $B_\infty(f) \neq \emptyset$, *then* $\mathcal{L}_\infty(\operatorname{grad} f) \leq -1 - 1/(d-2)$.

Proof. Since $B_\infty(f) = \widetilde{K}_\infty(f)$, it follows that there exist $\lambda \in \mathbb{C}$ and a sequence $\{z^m\} \subset \mathbb{C}^2$ with $z^m \to \infty$ such that $(f, \operatorname{grad} f)(z^m) \to (\lambda, 0, 0)$. Thus $\mathcal{L}_\infty(f - \lambda, \operatorname{grad} f) < 0$, and by Corollary 6.1 we get the assertion. ∎

REMARK 6.3. The estimate in Corollary 6.1(iv) cannot be improved as regards the degree. Indeed, for the polynomial $f(x, y) = y^d + xy^{d-1} + y$, $d > 2$ and $\lambda = 0$ equality holds (see [5, Example 4.11(b)]).

REMARK 6.4. In Theorem 4.2 and Corollary 6.1, we require no special form of the polynomial f. Under an additional assumption on the form of f, i.e. $f(x, y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x)$, where $d = \deg f > 2$, one can obtain Corollary 6.1 and Theorem 4.2 from [7, Theorem 2.9] and [5, Theorem 3.3 and Corollary 3.5]. Indeed, if 0 is a bifurcation point of f at infinity then one can prove that $\mathcal{L}_\infty(f, \operatorname{grad} f) = \mathcal{L}_\infty(f, \partial f/\partial y)$ and $\mathcal{L}_\infty(f, \partial f/\partial y) \leq -1/(d-2)$. Hence, we easily obtain the assertions of the above-mentioned theorems.

7. Separation of the gradient. In this section we show that in the two-dimensional case the gradient of a polynomial and the polynomial are separated. We begin with definitions and general properties.

Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. We say that F and f are *separated at infinity* (see [20]) if there exist $C, R > 0$ and $q \in \mathbb{R}$ such that

$$|f(z)| \geq R \ \Rightarrow\ |F(z)| \geq C|f(z)|^q.$$

The basic characterisation of separation at infinity is given in [20].

PROPOSITION 7.1 ([20, Proposition 1.1]). *Let* $F : \mathbb{C}^n \to \mathbb{C}^m$ *be a polynomial mapping and let* $f : \mathbb{C}^n \to \mathbb{C}$ *be a polynomial. Then the following conditions are equivalent*:

(i) F *and* f *are separated at infinity*,
(ii) $\{0\} \times \mathbb{C} \not\subset \overline{(F, f)(\mathbb{C}^n)}$,
(iii) *there is a polynomial* $P : \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}$ *such that* $P(F, f) = 0$ *and* $P|_{\{0\}\times\mathbb{C}} \neq 0$.

We shall say that F and f are *separated in a neighbourhood of the fibre* $f^{-1}(\lambda)$, where $\lambda \in \mathbb{C}$, if there exist $C, \varepsilon > 0$ and $q \in \mathbb{R}$ such that

$$|f(z) - \lambda| \leq \varepsilon \ \Rightarrow\ |F(z)| \geq C|f(z)|^q.$$

From Proposition 7.1, it is easy to see that the above two definitions of separation are equivalent. Namely we have

PROPOSITION 7.2. *Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then the following conditions are equivalent*:

(i) *F and f are separated at infinity*,
(ii) *F and f are separated in a neighbourhood of the fibre $f^{-1}(0)$*,
(iii) *F and f are separated in a neighbourhood of any fibre $f^{-1}(\lambda)$, $\lambda \in \mathbb{C}$.*

According to Proposition 7.2, we shall call F and f *separated* if F and f are separated at infinity or in a neighbourhood of the fibre of f.

Let us pass to a separation condition for the gradient.

PROPOSITION 7.3. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial and let $d = \deg f > 0$. Then the following conditions are equivalent*:

(i) $\operatorname{grad} f$ *and f are separated*,
(ii) *there exist $R, C > 0$ such that*
$$|f(z)| \geq R \ \Rightarrow\ |\operatorname{grad} f(z)| \geq C|f(z)|^{-(d-1)^n},$$
(iii) *for any $\lambda \in \mathbb{C}$, there exist $C, \varepsilon > 0$ such that*
$$|f(z) - \lambda| \leq \varepsilon \ \Rightarrow\ |\operatorname{grad} f(z)| \geq C|f(z) - \lambda|^{(d-1)^n},$$

The above proposition is a generalisation of Theorem 2 in [25]. The proof will be preceded by a lemma.

LEMMA 7.4. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $d = \deg f > 1$. If* $\operatorname{grad} f$ *and f are separated, then there exists a polynomial $P \in \mathbb{C}[y_1, \ldots, y_n, t]$ such that $P(\operatorname{grad} f, f) = 0$, $P|_{\{0\}\times\mathbb{C}} \neq 0$ and $\deg_t P \leq (d-1)^n$.*

Proof. We shall use the method developed in [18] (see proof of the main result). Let $V = \overline{(\operatorname{grad} f, f)(\mathbb{C}^n)} \subset \mathbb{C}^n \times \mathbb{C}$ and $k = \dim V$. Obviously $k \leq n$. Then, by Proposition 6.1, $\{0\} \times \mathbb{C} \not\subset V$. Without loss of generality, we may assume that $(0,0) \notin V$. Then there exists a linear mapping $L : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{k+1}$ such that $L|_V$ is a proper mapping and $L(0,0) = 0 \notin L(V)$. After composition of L with some linear automorphism $\mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ we may assume that for $G = L \circ (\operatorname{grad} f, f)$ we have $\deg g_i \leq d-1$, $i = 1, \ldots, k$, and $\deg g_{k+1} \leq d$, where $G = (g_1, \ldots, g_{k+1})$. Thus there exists a polynomial $\widetilde{P} : \mathbb{C}^{k+1} \to \mathbb{C}$ such that $L(V) = \{y \in \mathbb{C}^{k+1} : \widetilde{P}(y) = 0\}$ and $\widetilde{P}(0) \neq 0$.

It is easy to see that there exists an affine subspace $M \subset \mathbb{C}^n$ with $\dim M = k$ such that $V = \overline{(\operatorname{grad} f, f)(M)}$, so $L(V) = \overline{G(M)}$. In consequence, by the Perron Theorem ([16, Satz 57]) there exists a nonzero polynomial $Q \in \mathbb{C}[y_1, \ldots, y_{k+1}]$ of the form
$$Q = \sum_{(\nu_1+\cdots+\nu_k)(d-1)+\nu_{k+1}d \leq (d-1)^k d} c_{\nu_1,\ldots,\nu_{k+1}} y_1^{\nu_1} \cdots y_k^{\nu_k} y_{k+1}^{\nu_{k+1}}$$
such that $Q(g_1, \ldots, g_{k+1}) = 0$. Since $\dim L(V) = k$ and $Q|_{L(V)} = \widetilde{P}|_{L(V)} = 0$, the polynomials Q and $\widetilde{P}$ have a common divisor R such that $R(G) = 0$.

Moreover, by the definition of Q, $\deg_{y_{k+1}} R \le \deg_{y_{k+1}} Q \le (d-1)^n$. Putting $P = R \circ L : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ we easily get the assertion. ∎

Proof of Proposition 7.3. If $d = 1$, then the assertion is trivial. Assume that $d > 1$.

Assume that (i) holds. By Lemma 7.4 there exists a polynomial $P \in \mathbb{C}[y_1, \dots, y_n, t]$ such that $P(\operatorname{grad} f, f) = 0$, $P_{\{0\}\times\mathbb{C}} \neq 0$ and $\deg_t P \le (d-1)^n$. Hence, by Lemma 3.1 in [19], we get (ii). By using an analogous method, we shall prove (iii). Take any $\lambda \in \mathbb{C}$ and put $\widetilde{P}(y_1, \dots, y_n, t) = P(y_1, \dots, y_n, t+\lambda)$. Then $\widetilde{P}(\operatorname{grad} f, f-\lambda) = 0$ and $\widetilde{P}|_{\{0\}\times\mathbb{C}} \neq 0$. If $\widetilde{P}(0) \neq 0$, then (iii) is obvious. Assume that $\widetilde{P}(0) = 0$. Then P is t-regular. Thus, by the Weierstrass Preparation Theorem, there exist neighbourhoods $\Omega = \{y \in \mathbb{C}^m : |y| \le \eta\}$, $\Delta = \{t \in \mathbb{C} : |t| \le \varepsilon\}$, $\eta, \varepsilon > 0$, of the origins and a distinguished pseudopolynomial g in t, $0 < N = \deg_t g \le (d-1)^n$ of the form $g = t^N + g_1 t^{N-1} + \dots + g_N$, where $g_i : \Omega \to \mathbb{C}$ are holomorphic, $i = 1, \dots, N$, such that

$$\{(y,t) \in \Omega \times \Delta : P(y,t) = 0\} = \{(y,t) \in \Omega \times \Delta : g(y,t) = 0\}.$$

Diminishing η and ε if necessary, we may assume that $\varepsilon^{(d-1)^n} \le \eta < 1$. Then for any $|f(z)| \le \varepsilon$, we have

$$|f(z)| \le 2 \max_{i=1,\dots,N} |g_i(\operatorname{grad} f(z))|^{1/i} \le C_1 |\operatorname{grad} f(z)|^{1/(d-1)^n}$$

for some $C_1 > 0$. This gives (iii). The implications (iii)⇒(i), (ii)⇒(i) are obvious. ∎

Let us give the main result of this section.

THEOREM 7.5. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial with $d = \deg f > 0$. Then*

(i) *there exist C, $R > 0$ such that*

$$|f(z)| \ge R \;\Rightarrow\; |\operatorname{grad} f(z)| \ge C|f(z)|^{-(d-1)^2},$$

(ii) *for any $\lambda \in \mathbb{C}$ there exist $C, \varepsilon > 0$ such that*

$$|f(z) - \lambda| \le \varepsilon \;\Rightarrow\; |\operatorname{grad} f(z)| \ge C|f(z) - \lambda|^{(d-1)^2}.$$

Proof. By [25, Theorem 2] and Proposition 7.2 we know that the set $K_\infty(f)$ is finite if and only if $\operatorname{grad} f$ and f are separated. Then by Corollary 5.2 and Proposition 7.3 we obtain the assertion. ∎

REMARK 7.6. As in Proposition 5.1 we may define the Fedoryuk condition (F) for a polynomial f in several variables. In this case the set of values for which (F) fails is also denoted by $K_\infty(f)$. By [25, Theorem 2] and Proposition 7.2 we conclude that $\operatorname{grad} f$ and f are separated if and only if the set $K_\infty(f)$ is finite.

REMARK 7.7. From Theorem 7.5 we deduce that for any real polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, $\operatorname{grad} f$ and f are separated.

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