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On convergence of integrals in o-minimal structures on archimedean real closed fields

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Abstract. We define a notion of volume for sets definable in an o-minimal structure on an archimedean real closed field. We show that given a parametric family of continuous functions on the positive cone of an archimedean real closed field definable in an o-minimal structure, the set of parameters where the integral of the function converges is definable in the same structure.

Introduction. So far a good notion of volume for sets definable in an o-minimal structure on an arbitrary real closed field is missing. In [4] A. Berarducci and M. Otero defined a real-valued volume for the finite parts of definable sets. In Section 1 we recall these definitions which lead to a good notion in the archimedean case having the desired properties like additivity and monotonicity. We also obtain the standard connection with antiderivatives for integrals.

In Section 2 we show the following

Theorem. *Let R be an archimedean real closed field and let $f\colon R^n \times R_{\geq 0} \to R_{\geq 0}$ be a continuous function which is definable in an o-minimal structure on R. Then the set $\{a \in R^n \mid \int_0^\infty f(a,t)\,dt < \infty\}$ of parameters where the integrals converge is definable in the same structure.*

G. Comte, J.-M. Lion and J.-P. Rolin established a multivariable version of this theorem for semialgebraic and globally subanalytic functions on the reals (cf. [5], [7]). In the case $R = \mathbb{R}$ the above theorem is proven for arbitrary o-minimal structures on $\mathbb{R}$ by O. Le Gal in [6]. He uses a fact about o-minimal structures on $\mathbb{R}$ (cf. [8]) which heavily relies on the completeness of $\mathbb{R}$ and which is not true for arbitrary archimedean real closed fields. In this paper we use ideas of the theory of Hardy fields as in the work of C. Miller and P. Speissegger [9]. But instead of model-theoretic we give

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geometric arguments to prove a stronger result with the above theorem as a consequence, as follows: By the properties of the integral we can translate the problem into the value group of the Hardy field of the given o-minimal structure, i.e. the Hardy field of germs at $+\infty$ of definable functions. The convergence or divergence of the integrals of the given family gives us a Dedekind cut $(\mathcal{I},\mathcal{F})$ in the value group. This Dedekind cut $(\mathcal{I},\mathcal{F})$ has a property we call "width 0". With this we can show that the values of the given family have an upper bound in $\mathcal{I}$ and a lower bound in $\mathcal{F}$. The theorem follows.

In Section 3 we give examples of such Dedekind cuts $(\mathcal{I},\mathcal{F})$ and study them in the case $R=\mathbb{R}$. We also give some consequences for certain classes of ordinary differential equations. And in Section 4 we show how to recognize the freeness of this Dedekind cut. From this we deduce a definable asymptotic integration for families.

1. Volume of definable sets in o-minimal structures on an archimedean real closed field. Let R be an archimedean real closed field and let $\mathcal{M}$ be an o-minimal structure on R.

We want to speak about volume of sets and functions which are definable in $\mathcal{M}$. We use the definition of an additive measure in o-minimal expansions of fields due to A. Berarducci and M. Otero [4]. They define it for the finite parts of definable sets also in nonarchimedean real closed fields. Since we are in the archimedean case we start with definitions more adapted to this situation; the result is the same.

1.1. Definition.

(a) For $a=(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in R^n$ we define $a\le b$ if $a_i\le b_i$ for all $1\le i\le n$, and $a<b$ if $a_i<b_i$ for all $1\le i\le n$.
(b) A *rectangle* is a set of the form

$$[a,b[\,:=\prod_{i=1}^{n}[a_i,b_i[$$

with $a\le b\in R^n$.
(c) A *polytope* is a finite union of rectangles.

1.2. Definition.

(a) Let $a,b\in R^n$ with $a\le b$. For a rectangle $Q:=[a,b[$ we set

$$\mathrm{Vol}(Q):=\prod_{i=1}^{n}(b_i-a_i).$$

(b) Let P be a polytope and $P = \dot{\bigcup}_{1\le j\le r} Q_j$ be a finite partition into rectangles. We set

$$\mathrm{Vol}(P) := \sum_{i=1}^{r} \mathrm{Vol}(Q_j).$$

1.3. Remark.

(a) For $a \le b \in R^n$ we have

$$\mathrm{Vol}([a,b[) > 0 \;\Leftrightarrow\; a < b.$$

(b) Each polytope can be written as a disjoint union of rectangles.
(c) Definition 1.2(b) is correct. If there are two finite partitions $P = \dot{\bigcup}_{1\le j\le r} Q_j = \dot{\bigcup}_{1\le s\le t} Q'_s$ of a polytope into rectangles, then

$$\sum_{j=1}^{r} \mathrm{Vol}(Q_j) = \sum_{s=1}^{t} \mathrm{Vol}(Q'_s).$$

This can be seen by applying a common subpartition.

1.4. Definition. For an arbitrary set $A \subset R^n$ we define

$$I(A) := \{\mathrm{Vol}(P) \mid P \subset A \text{ is a polytope}\}.$$

If A is bounded we define

$$O(A) := \{\mathrm{Vol}(P) \mid P \supset A \text{ is a polytope}\}.$$

(I stands for inner, O for outer measure.)

1.5. Remark.

(a) If A is bounded, then $I(A) < O(A)$.
(b) $I(A) \subset R_{\ge 0}$ is convex, i.e. if $0 \le r < s < t \in R$ with $r, t \in \mathrm{Vol}(A)$, then $s \in I(A)$. For A bounded, $O(A)$ is convex.

In [4, Theorem 2.4], the following is shown:

1.6. Theorem. *If $A \subset R^n$ is definable and bounded then $(I(A), O(A))$ is a Dedekind cut of $R_{\ge 0}$, i.e. for each $\varepsilon > 0$ (resp. $n \in \mathbb{N}$) there are $v_1 \in I(A)$ and $v_2 \in O(A)$ with $v_2 - v_1 < \varepsilon$ (resp. $< 1/n$).*

Let $R \hookrightarrow \mathbb{R}$ be the embedding as real closed fields. Then the Dedekind cut $(I(A), O(A))$ of R is realized by a unique real number denoted by $\mathrm{Vol}(A)$ ($\mu(A)$ in [4]).

1.7. Remark. For a polytope P the two notions of $\mathrm{Vol}(P)$ coincide. A. Berarducci and M. Otero define for an unbounded definable set A the volume of A by

$$\mathrm{Vol}(A) := \lim_{\substack{r\to\infty \\ r\in R}} \mathrm{Vol}(A \cap B_r(0)) \in \mathbb{R}_{\ge 0} \cup \{\infty\}.$$

This is also equivalent to the definition by the inner measure. Let $A \subset R^n$ be unbounded and definable. Then $\mathrm{Vol}(A) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ realizes the Dedekind cut $(I(A), R_{\geq 0} \setminus I(A))$.

We deduce from [4] the following properties of the volume:

1.8. Proposition.

(a) Vol *is finitely additive and translation-invariant.*
(b) *For a definable set* $A \subset R^n$ *we have*

$$\mathrm{Vol}(A) > 0 \ \Leftrightarrow\ \mathring{A} \neq \emptyset.$$

From the proposition we get $\mathrm{Vol}(A) = \mathrm{Vol}(\bar{A})$ for a definable set $A \subset R^n$, since $(\bar{A} \setminus A)^\circ = \emptyset$.

1.9. Observation. The volume introduced above can also be understood in the following way. Let $\mathcal{M}$ be the given o-minimal structure on the archimedean real closed field R and let $R \subset \mathbb{R}$ be the unique embedding of R into its Dedekind completion $\mathbb{R}$. Then there is a unique o-minimal structure $\mathcal{M}'$ on $\mathbb{R}$ such that $\mathcal{M} \prec \mathcal{M}'$, i.e. $\mathcal{M}$ is an elementary substructure of $\mathcal{M}'$ (cf. [12]).

Let $A \subset R^n$ be definable in $\mathcal{M}$ and let $A_{\mathbb{R}}$ be its realization in $\mathbb{R}^n$. Then $A_{\mathbb{R}}$ is definable in $\mathcal{M}'$. By stratification theorems for o-minimal structures $A_{\mathbb{R}}$ is obviously Lebesgue-measurable. By Definition 1.4 and the density of R in $\mathbb{R}$ one can deduce Theorem 1.6 and we see that $\mathrm{Vol}(A)$ is the Lebesgue volume of $A_{\mathbb{R}}$. This immediately gives Proposition 1.8.

1.10. Definition. Let $f\colon A \to R_{\geq 0}$ be definable. Then

$$\int_A f\,dx = \mathrm{Vol}(\{(x,t) \in A \times R_{\geq 0} \mid 0 \leq t \leq f(x)\}).$$

This can be extended in the usual way to arbitrary definable functions whenever it makes sense, for example to functions with bounded graphs.

We get the usual linearity of the integral:

1.11. Proposition. *Let* $A \subset R^n$ *be bounded and let* $f, g\colon A \to R$ *be definable and bounded. Let* $\lambda, \mu \in R$. *Then*

$$\int_A (\lambda f + \mu g)\,dx = \lambda \int_A f\,dx + \mu \int_A g\,dx.$$

We also get the well known connection with the antiderivative:

1.12. Theorem. *Let* $f\colon R \to R$ *be definable and continuous and let* $F\colon R \to R$ *be definable and differentiable with* $F' = f$. *Then for* $a < b$ *we get*

$$\int_{[a,b]} f(t)\,dt = F(b) - F(a).$$

Proof 1. By o-minimality we find a finite partition of $[a,b]$ into intervals such that f has constant sign on each of them. So we may assume that $f \geq 0$ on $[a,b]$. Again by o-minimality we find a finite partition of $[a,b]$ into intervals such that f is monotonic on each of the intervals. So we may assume that f is increasing on $[a,b]$.

Let $n \in \mathbb{N}$. Define $a_0^{(n)} := a$ and $a_j^{(n)} := a + j(b-a)/n$ for $1 \leq j \leq n$. We set

$$P_1^{(n)} := \bigcup_{0\leq j<n} [a_j^{(n)}, a_{j+1}^{(n)}[\times [0, f(a_j^{(n)})[,$$

$$P_2^{(n)} := \bigcup_{0\leq j<n} [a_j^{(n)}, a_{j+1}^{(n)}[\times [0, f(a_{j+1}^{(n)})[.$$

Then $\mathrm{Vol}(P_1^{(n)}) \in I(A)$ and $\mathrm{Vol}(P_2^{(n)}) \in O(A)$. We will show that for each $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that

$$F(b) - F(a) - \varepsilon < \mathrm{Vol}(P_1^{(n)}) < \mathrm{Vol}(P_2^{(n)}) < F(b) - F(a) + \varepsilon;$$

this completes the proof. So let $\varepsilon > 0$. Since f is absolutely continuous on $[a,b]$ there is some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(b-a)$ for $x, y \in [a,b]$ with $|x-y| < \delta$. We choose $n \in \mathbb{N}$ such that $1/n \leq \delta$. By the mean-value property there are points $\xi_j^{(n)}$, $0 \leq j < n$, with $a_j^{(n)} \leq \xi_j^{(n)} \leq a_{j+1}^{(n)}$ and $F(a_{j+1}^{(n)}) - F(a_j^{(n)}) = f(\xi_j^{(n)})(a_{j+1}^{(n)} - a_j^{(n)})$. Hence we get

$$\begin{aligned}
\mathrm{Vol}(P_1^{(n)}) &= \sum_{j=0}^{n-1} f(a_j^{(n)})(a_{j+1}^{(n)} - a_j^{(n)}) \\
&= \sum_{j=0}^{n-1} f(\xi_j^{(n)})(a_{j+1}^{(n)} - a_j^{(n)}) \\
&\qquad - \sum_{j=0}^{n-1} (f(\xi_j^{(n)}) - f(a_{j+1}^{(n)}))(a_{j+1}^{(n)} - a_j^{(n)}) \\
&> \sum_{j=0}^{n-1} f(\xi_j^{(n)})(a_{j+1}^{(n)} - a_j^{(n)}) - \frac{\varepsilon}{b-a}\Big(\sum_{j=0}^{n-1} a_{j+1}^{(n)} - a_j^{(n)}\Big) \\
&= \sum_{j=0}^{n-1} (F(a_{j+1}^{(n)}) - F(a_j^{(n)})) - \varepsilon = F(b) - F(a) - \varepsilon.
\end{aligned}$$

In the same way we get $\mathrm{Vol}(P_2^{(n)})) < F(b) - F(a) + \varepsilon$.

Proof 2. We may assume that $f \geq 0$ on $[a,b]$. We set $A := \{(t,y) \in R^2 \mid a \leq t \leq b,\ 0 \leq y \leq f(t)\}$. Let $f_{\mathbb{R}}$, $F_{\mathbb{R}}$ and $A_{\mathbb{R}}$ be the realizations of f, F and A in $\mathbb{R}$. Then $f_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is continuous, $F_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is differentiable with

$F'_{\mathbb{R}} = f_{\mathbb{R}}$ and $A_{\mathbb{R}} = \{(t,y) \in R^2 \mid a \le t \le b,\ 0 \le y \le f_{\mathbb{R}}(t)\}$. So in $\mathbb{R}$ we have

$$\mathrm{Vol}(A_{\mathbb{R}}) = \int_a^b f_{\mathbb{R}}(t)\,dt = F_{\mathbb{R}}(a) - F_{\mathbb{R}}(b).$$

The claim now follows from the fact that $\mathrm{Vol}(A) = \mathrm{Vol}(A_{\mathbb{R}})$ and $F_{\mathbb{R}}(t) = F(t)$ for all $t \in R$. ■

2. The main theorem. We fix an archimedean real closed field R. We prove the following two equivalent theorems:

2.1. THEOREM. *Let $\mathcal{M}$ be an o-minimal structure on R. Let $f\colon R^n \times R_{\ge 0} \to R_{\ge 0}$ be a continuous function, definable in $\mathcal{M}$. Then the set*

$$\Big\{a \in R^n \,\Big|\, \int_0^\infty f(a,t)\,dt < \infty\Big\}$$

is definable in $\mathcal{M}$.

2.2. THEOREM. *Let $\mathcal{M}$ be an o-minimal structure on the field R. Let $f\colon R^n \times R_{\ge 0} \to R_{\ge 0}$ be a continuous function, definable in $\mathcal{M}$. Then the set*

$$\Big\{a \in R^n \,\Big|\, \sum_{k=0}^\infty f(a,k) < \infty\Big\}$$

is definable in $\mathcal{M}$. (For $a_k \in R_{\ge 0} \subset \mathbb{R}_{\ge 0}$ we define $\sum_{k=0}^\infty a_k \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ as the limit of the real-valued partial sums.)

These two theorems are clearly equivalent since a function definable in an o-minimal structure is ultimately monotone.

Proof of Theorem 2.1. The o-minimal structure on the field R will be fixed and denoted by $\mathcal{M}$. Definable means definable with parameters in $\mathcal{M}$.

Let $g\colon R_{\ge 0} \to R_{\ge 0}$ be definable and continuous. Then the question if $\int_0^\infty g(t)\,dt < \infty$ depends only on the germ of g at $+\infty$. Let $\mathcal{H}$ be the set of germs at $+\infty$ of definable functions. It is well known that $\mathcal{H}$ is a Hardy field; it is a field with canonical ordering, derivation and the additively written valuation $\nu\colon \mathcal{H}^* \to \nu(\mathcal{H}^*) =: \Gamma$ given by

$$\nu(h) \begin{cases} > 0 & \text{if } \lim\limits_{t\to\infty} h(t) = 0, \\ = 0 & \text{if } \lim\limits_{t\to\infty} h(t) \in R^*, \\ < 0 & \text{if } \lim\limits_{t\to\infty} |h(t)| = \infty, \end{cases}$$

for $h \in \mathcal{H}^*$. For $h \in \mathcal{H}$ we define

$$\int^\infty |h| < \infty \quad \text{if} \quad \int_{t_0}^\infty |h(t)|\,dt < \infty \quad \text{for all } t_0 \text{ large},$$

and similarly we define $\int^\infty |h| = \infty$.

The question if $\int^\infty |h| < \infty$ depends, by the monotonicity and the linearity of the integral, only on the valuation $\nu(h)$. Again by the monotonicity of the integral we get a Dedekind cut $(\mathcal{I}, \mathcal{F})$ of Γ (i.e. $\mathcal{I} < \mathcal{F}$ and $\mathcal{I} \dot{\cup} \mathcal{F} = \Gamma$) with

$$\mathcal{I} = \Big\{\nu(h) \in \Gamma \Bigm| \int^\infty |h| = \infty\Big\}, \qquad \mathcal{F} = \Big\{\nu(h) \in \Gamma \Bigm| \int^\infty |h| < \infty\Big\}.$$

We show that there are $\nu(g) \in \mathcal{I}$ and $\nu(h) \in \mathcal{F}$ such that for each $a \in R^n$,

$$\nu(f(a,\cdot)) \le \nu(g) \quad \text{or} \quad \nu(f(a,\cdot)) \ge \nu(h).$$

Hence

$$\Big\{a \in R^n \Bigm| \int_0^\infty f(a,t)\,dt < \infty\Big\} = \Big\{a \in R^n \Bigm| \lim_{t\to\infty} \frac{f(a,t)}{|h(t)|} < \infty\Big\},$$

which is obviously definable in $\mathcal{M}$, so Theorem 2.1 is an easy consequence of this. ∎

Miller and Speisegger showed a similar fact in a slightly different setting with model-theoretic means (see [9, Theorem 1]). We show it in a more general setting.

2.3. Definition. A Dedekind cut $(\mathcal{A}, \mathcal{B})$ of Γ is *of width* 0 if for each $\gamma \in \Gamma_{>0}$ there are $\gamma_1 \in \mathcal{A}$ and $\gamma_2 \in \mathcal{B}$ such that $\gamma_2 - \gamma_1 < \gamma$.

2.4. Examples.

(a) Every Dedekind cut which is not free is of width 0.
(b) $(\mathcal{I}, \mathcal{F})$ is of width 0: Let $\nu(h) \in \Gamma_{<0}$. Then $\nu((1/h)') \in \mathcal{F}$ and $\nu(h') \in \mathcal{I}$. We have

$$\nu\Big(\Big(\frac{1}{h}\Big)'\Big) - \nu(h') = \nu\Big(\frac{h'}{h^2}\Big) - \nu(h') = \nu\Big(\frac{1}{h^2}\Big) = -2\nu(h).$$

(c) Changing the archimedean class is not of width 0:

$$\mathcal{M} = \mathbb{R}_{\exp}, \qquad \mathcal{B} = \{\nu(h) \mid |h(t)| \le t^n \text{ for some } n \in \mathbb{N}\}.$$

We now prove the following

2.5. Theorem. *Let $(\mathcal{A}, \mathcal{B})$ be a cut of Γ of width* 0, *and let $f\colon R^n \times R_{\ge 0} \to R_{\ge 0}$ be definable. Then there are $\gamma \in \mathcal{A}$ and $\delta \in \mathcal{B}$ such that for each $a \in R^n$,*

$$\nu(f(a,\cdot)) \le \gamma \quad \text{or} \quad \nu(f(a,\cdot)) \ge \delta.$$

Proof. Without restriction, $f(a,t) > 0$ for all $(a,t) \in R^n \times R_{\ge 0}$. Assume the statement does not hold. We define

$$g\colon R^n \times R^n \times R_{\ge 0} \to R_{>0}, \qquad (a,b,t) \mapsto f(a,t)/f(b,t).$$

Let $W := \{(a,b) \in R^n \times R^n \mid \lim_{t\to\infty} g(a,b,t) = \infty\}$. For $(a,b) \in W$ let $t_{(a,b)} := \min\{s \geq 0 \mid g(a,b,\cdot)|_{[s,\infty[}$ is strictly increasing$\}$. We define

$$h\colon \{(a,b,\tau) \in V \times R_{\geq 0} \mid \tau \geq g(t_{(a,b)})\} \to R_{>0}, \qquad h(a,b,\tau) := g^{-1}_{(a,b)}(\tau).$$

Then h is definable in $\mathcal{M}$. Since $(\mathcal{A},\mathcal{B})$ is of width 0, h satisfies the following condition: For each $\widetilde{\gamma} \in \Gamma_{<0}$ there is some $(a,b) \in V$ such that $\nu(h(a,b,\cdot)) \leq \widetilde{\gamma}$. But by universal boundedness of growth of definable functions (cf. [13]), this is impossible. ∎

3. Examples and consequences. We look closer at the Dedekind cut $(\mathcal{I},\mathcal{F})$. It is related to an important invariant of a Hardy field (cf. [10], [11]) which also has consequences in applications (cf. [1]–[3]). We concentrate on the case $R = \mathbb{R}$, where the antiderivative of a continuous function exists. If the function is definable in an *o*-minimal structure then the antiderivative lives in the Pfaffian closure, i.e. in an *o*-minimal expansion of the given structure.

We start with some facts from the theory of Hardy fields.

3.1. Definition.

(a) Let $\nu(h) \in \Gamma$. Then $\nu(g)$ is called an *asymptotic antiderivative* of $\nu(h)$ if $\nu(g') = \nu(h)$.
(b) $\mathcal{M}$ is *closed under asymptotic integration* if each $\nu(h) \in \Gamma$ has an asymptotic antiderivative.

3.2. Definition ([10]). Let

$$\Psi := \{\nu(h'/h) \mid \nu(h) \in \Gamma^*\} = \{\nu(h'/h) \mid \nu(h) \in \Gamma_{<0}\}.$$

3.3. Rosenlicht's Theorem ([10]). *$\nu(h) \in \Gamma$ has asymptotic antiderivative if and only if $\nu(h) \neq \sup\Psi$. In particular, $\mathcal{M}$ is closed under asymptotic integration if and only if $\sup\Psi$ does not exist.*

Proof. This was shown by Rosenlicht in [10]. For $\nu(h) \in \Gamma^*$, $\nu(h) \neq \sup\Psi$, an asymptotic antiderivative is given by

$$\nu\left(\frac{h \cdot \left(\frac{h\cdot u}{u'}\right)}{\left(\frac{h\cdot u}{u'}\right)'}\right)',$$

with $|\nu(u)| \leq |\nu(u_0)|$, $\nu(u_0) \in \Gamma^*$ depending on $\nu(h)$. ∎

3.4. Remark. The theorem of Rosenlicht is stated for the more general case of Hardy fields.

For *o*-minimal structures we get a connection between the Dedekind cut $(\mathcal{I},\mathcal{F})$ and Ψ:

3.5. LEMMA. *Let $\mathcal{M}$ be an o-minimal structure. Then* $\sup \mathcal{I}$ *exists, if and only if* $\sup \Psi$ *exists and then*

$$\sup \mathcal{I} = \sup \Psi.$$

Proof. CASE 1: *$\mathcal{M}$ is polynomially bounded.* This case is clear (cf. also Example 3.7(a)).

CASE 2: *$\mathcal{M}$ is not polynomially bounded.* Then the exponential function is definable in $\mathcal{M}$. Now, it was shown in [9] that

$$\mathcal{I} \setminus \{\sup \mathcal{I}\} = \Psi \setminus \{\sup \Psi\}. \ \blacksquare$$

3.6. OBSERVATION. Let $\mathcal{M}$ be an o-minimal structure. Assume that $\sup \mathcal{I}$ exists. Let $f\colon [t_0, \infty[\to \mathbb{R}_{\geq 0}$ be definable and continuous with $\nu(f) = \sup \mathcal{I}$. Let $F\colon [t_0, \infty[\to \mathbb{R}_{\geq 0}$, $t \mapsto \int_{t_0}^{t} f(s)\, ds$, be the antiderivative of f. Then F is definable in an o-minimal expansion $\mathcal{M}^*$ of $\mathcal{M}$.

(a) If $\nu(f) = \max \mathcal{I}$, then by Rosenlicht's theorem and the last lemma,

$$0 > \nu(F) > \nu(h) \quad \text{for all } \nu(h) \in \Gamma_{<0}.$$

(b) If $\nu(f) = \min \mathcal{F}$, then

$$0 > \nu\left(\frac{1}{a - F}\right) > \nu(h) \quad \text{for all } \nu(h) \in \Gamma_{<0}$$

with $a := \int_{t_0}^{\infty} f(t)\, dt$.

So in the o-minimal expansion $\mathcal{M}^*$ we have a function going to infinity "more slowly" than every function definable in $\mathcal{M}$ and tending to infinity.

3.7. EXAMPLE.

(a) Let $\mathcal{M}$ be polynomially bounded, e.g. $\mathcal{M} = \mathbb{R}$. Then

$$\mathcal{I} = \{\nu(t^\alpha) \mid \alpha \geq -1,\ t^\alpha \text{ definable in } \mathcal{M}\},$$
$$\mathcal{F} = \{\nu(t^\alpha) \mid \alpha < -1,\ t^\alpha \text{ definable in } \mathcal{M}\},$$
$$\Psi = \{\nu(1/t)\}.$$

So $\max \mathcal{I} = \nu(1/t)$ exists and the Dedekind cut $(\mathcal{I}, \mathcal{F})$ is not free. The antiderivative of $1/t$, $\log t$, goes to ∞ "more slowly" than every t^α, $\alpha > 0$.

(b) Let $\mathcal{M}$ be the o-minimal structure $\mathbb{R}_{\exp}$. Since the Pfaffian closure of $\mathbb{R}_{\exp}$ is exponentially bounded, by the previous observation the Dedekind cut $(\mathcal{I}, \mathcal{F})$ is free and $\mathbb{R}_{\exp}$ is closed under asymptotic integration. Each definable function is bounded by an iterated exp-function $\exp_n(t) := \exp \circ \cdots \circ \exp(t)$ (n times). So $(\nu(\log_n(t)'))_{n \in \mathbb{N}}$ is cofinal in $\mathcal{I}$ and $(\nu(1/\log_n(t))')_{n \in \mathbb{N}}$ is coinitial in $\mathcal{F}$. Since

$$(\log_n(t))' = \frac{1}{\prod_{k=0}^{n-1} \log_k(t)} \quad \text{and} \quad \left(\frac{1}{\log_n(t)}\right)' = \frac{1}{\prod_{k=0}^{n-1} \log_k(t) \cdot (\log_n(t))^2}$$

we see that $(-\sum_{k=0}^{n}\gamma_k)_{n\in\mathbb{N}}$ is cofinal in $\mathcal{I}$ and $(-\sum_{k=0}^{n}\gamma_k-\gamma_n)_{n\in\mathbb{N}}$ is coinitial in $\mathcal{F}$, where $\gamma_n:=\nu(\log_n(t))$.

3.8. Remark. So far every o-minimal structure on $\mathbb{R}$ known fits in the above examples. If the o-minimal structure is polynomially bounded, then we are in case (a); if the exponential function is definable, the o-minimal structure is the reduct of the Pfaffian closure of a polynomially bounded o-minimal structure, so that the statements of case (b) hold.

To end this section we give some consequences of Theorem 2.1 on boundedness and lifetime of ODE's on o-minimal structures. The first two deal with ODE's with separated variables.

3.9. Corollary. *Let $f\colon \mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ and $g\colon \mathbb{R}^n\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{>0}$ be definable and differentiable. For $a\in\mathbb{R}^n$ consider the first order ordinary differential equation*

$$D_a\colon\quad y'=g(a,y)\cdot f(a,t),\qquad y(0)=0.$$

Then the set

$$\{a\in\mathbb{R}^n \mid \textit{the solution } y_a(t) \textit{ of } D_a \textit{ is bounded at } +\infty\}$$

is definable in $\mathcal{M}$.

Proof. Let

$$G_a\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{>0},\qquad s\mapsto\int_0^s\frac{dt}{g(a,t)},$$

and

$$F_a\colon \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0},\qquad t\mapsto\int_0^t f(a,\tau)\,d\tau.$$

Then the maximal solution of D_a is given by

$$y_a\colon I_a\to\mathbb{R}_{\geq 0},\qquad t\mapsto y_a(t)=G_a^{-1}(F_a(t)),$$

where $I_a\subset\mathbb{R}_{\geq 0}$ is given by

$$I_a=[0,\sup_{t\in\mathbb{R}_{\geq 0}}\{F_a(t)<\lim_{s\to\infty}G_a(s)\}[.$$

Hence

$$\begin{aligned}&\{a\in\mathbb{R}^n\mid \text{the solution } y_a(t) \text{ of } D_a \text{ is not bounded at } +\infty\}\\ &\quad=\Big\{a\in\mathbb{R}^n\;\Big|\;\int_0^\infty f(a,t)\,dt=\infty\Big\}\cap\Big\{a\in\mathbb{R}^n\;\Big|\;\int_0^\infty\frac{ds}{g(a,s)}\,ds=\infty\Big\}\end{aligned}$$

and is therefore definable by Theorem 2.1. ∎

3.10. Corollary. *Let f, g and D_a for $a \in \mathbb{R}^n$ be as in Corollary 3.9. Further assume that $\int_0^\infty f(a,t)\,dt = \infty$ for all $a \in \mathbb{R}^n$. Then the set*

$$\{a \in \mathbb{R}^n \mid \text{the solution } y_a(t) \text{ of } D_a \text{ lives forever}\}$$

is definable in $\mathcal{M}$.

Proof. By the proof of Corollary 3.9, $y_a(t)$ lives forever if and only if $\lim_{s\to\infty} G_a(s) = \infty$. So

$$\{a \in \mathbb{R}^n \mid \text{the solution } y_a(t) \text{ of } D_a \text{ lives forever}\} = \left\{a \in \mathbb{R}^n \;\middle|\; \int_0^\infty \frac{ds}{g(a,s)} = \infty\right\},$$

hence is definable in $\mathcal{M}$ by Theorem 2.1. ■

The last corollary deals with a special case of a linear differential equation:

3.11. Corollary. *Let $f, g\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be definable and continuous with $\int_0^\infty |f(a,t)|\,dt < \infty$ for all $a \in \mathbb{R}^n$. For $a \in \mathbb{R}^n$ consider the linear differential equation*

$$L_a: \quad y' = f(a,t)\cdot y + g(a,t).$$

Then given $a \in \mathbb{R}^n$ all the solutions of L_a are simultaneously bounded or unbounded and the set

$$\{a \in \mathbb{R}^n \mid \text{the solutions of } L_a \text{ are bounded}\}$$

is definable in $\mathcal{M}$.

Proof. Let $F_a\colon \mathbb{R}_{\geq 0} \to \mathbb{R},\ t \mapsto \int_0^t f(a,\tau)\,d\tau$. The set of solutions of the homogeneous equation

$$L_a^h\colon \quad y' = f(a,t)\cdot y$$

is given by $\mathbb{R}\cdot e^{F_a}$. By variation of constants a special solution of L_a is given by

$$\varphi_a\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad t \mapsto \left(\int_0^t \frac{g(a,\tau)}{e^{F_a(\tau)}}\,d\tau\right)\cdot e^{F_a(t)},$$

and the set of all solutions of L_a by $\varphi_a + \mathbb{R}\cdot e^{F_a}$. Since e^{F_a} is bounded by assumption, we get

$$\{a \in \mathbb{R}^n \mid \text{the solutions of } L_a \text{ are bounded}\} = \left\{a \in \mathbb{R}^n \;\middle|\; \int_0^\infty |g(a,t)|\,dt < \infty\right\},$$

which is definable by Theorem 2.1. ■

4. The Dedekind cut $(\mathcal{I}, \mathcal{F})$**.** In the discussion of Theorem 2.1 in Section 3 we saw that it is important whether the Dedekind cut $(\mathcal{I}, \mathcal{F})$ defined by integration is free or not. In this section we show how to recognize this by operations on the value group and we discuss several consequences. We start with

4.1. Proposition. *Let $f \in \mathcal{H}$ with $\nu(f)$ in the archimedean class of $\nu(t)$. Then the map*

$$_f\circ\colon \Gamma_{<0} \to \Gamma, \qquad \nu(\varphi) \mapsto \nu(f \circ |\varphi|),$$

is well defined.

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{H}$ with

$$\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} \in \mathbb{R}^*.$$

We have to show that

$$\lim_{t\to\infty} \frac{(f\circ\varphi_1)(t)}{(f\circ\varphi_2)(t)} \in \mathbb{R}^*.$$

By changing f to $1/f$ we may assume that f is increasing. Further we may assume that $f > 0$. There is a $C > 1$ such that

$$\frac{1}{C} < \frac{\varphi_1(t)}{\varphi_2(t)} < C \qquad \text{for all } t > 0 \text{ large.}$$

Since $\nu(f)$ is in the archimedean class of $\nu(t)$ there is some $n \in \mathbb{N}$ with

$$f(t) \le t^n \quad \text{for all } t > 0 \text{ large.}$$

Hence $\lim_{t\to\infty} t^{2n}/f(t) = \infty$. We get

$$\begin{aligned}\frac{f\circ\varphi_1(t)}{f\circ\varphi_2(t)} &\le \frac{f\circ(C\varphi_2)(t)}{f\circ\varphi_2(t)} = \frac{f(C\varphi_2(t))}{f(\varphi_2(t))}\\ &= C^{2n}\,\frac{f(C\varphi_2(t))}{(C\varphi_2(t))^{2n}}\cdot\frac{(\varphi_2(t))^{2n}}{f(\varphi_2(t))} \le C^{2n}\end{aligned}$$

for all $t > 0$ large since $t^{2n}/f(t)$ is eventually increasing, and

$$\frac{f\circ\varphi_1(t)}{f\circ\varphi_2(t)} \ge \frac{f\circ\varphi_1(t)}{f\circ(C\varphi_1)(t)} = \frac{f(\varphi_1(t))}{f(C\varphi_1(t))} \ge \frac{1}{C^{2n}}$$

for all $t > 0$ large. ■

The converse of Proposition 4.1 is also true:

4.2. Proposition. *Let $f \in \mathcal{H}^*$ with*

$$\lim_{t\to\infty} \frac{f(ct)}{f(t)} \in \mathbb{R}^* \qquad \textit{for all } c > 0.$$

Then $\nu(f)$ is in the archimedean class of $\nu(t)$.

Proof. We may assume that f is increasing and that $f > 0$. We have to show the existence of some $n \in \mathbb{N}$ with

$$f(t) \le t^n \qquad \text{for all } t > 0 \text{ large.}$$

We define

$$g\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \qquad c \mapsto \lim_{t\to\infty} \frac{f(ct)}{f(t)}.$$

Then g is definable and satisfies the functional equation

$$g(c \cdot d) = g(c) \cdot g(d), \qquad c, d > 0.$$

Indeed,

$$\begin{aligned} g(c \cdot d) &= \lim_{t\to\infty} \frac{f(c \cdot d \cdot t)}{f(t)} = \lim_{t\to\infty} \frac{f(c \cdot d \cdot t)}{f(d \cdot t)} \cdot \frac{f(d \cdot t)}{f(t)} \\ &= \lim_{dt\to\infty} \frac{f(c \cdot dt)}{f(dt)} \cdot \lim_{t\to\infty} \frac{f(d \cdot t)}{f(t)} = g(c) \cdot g(d). \end{aligned}$$

Hence there is some $\alpha > 0$ with

$$g(c) = c^{\alpha} \qquad \text{for all } c > 0 \text{ large.}$$

Choose $n > \alpha$. Let $t_0 > 0$ be such that f is defined on $[t_0, \infty[$. Fix $c > 1$ and $t_1 \geq t_0$ such that

$$\frac{f(ct)}{f(t)} < c^n \qquad \text{for all } t \geq t_0.$$

For $k \in \mathbb{N}$ we get

$$\frac{f(c^k t_0)}{f(t_0)} = \prod_{l=1}^{k} \frac{f(c^l t_0)}{f(c^{l-1} t_0)} = \prod_{l=1}^{k} \frac{f(c(c^{l-1} t_0))}{f(c^{l-1} t_0)} < (c^n)^k = (c^k)^n.$$

So $f(c^k t_0) < (c^k)^n \cdot f(t_0)$ for all $k \in \mathbb{N}$ and hence by o-minimality

$$f(x \cdot t_0) < x^n \cdot f(t_0) \qquad \text{for all } x > 0 \text{ large.}$$

This gives the claim. ■

We denote the archimedean class of $\nu(t)$ by p (p stands for polynomial). Then $\mathcal{I} \cap p$ is cofinal in $\mathcal{I}$ and $\mathcal{F} \cap p$ is coinitial in $\mathcal{F}$.

Now we characterize the Dedekind cut $(\mathcal{I}, \mathcal{F})$. This cut is not free if and only if $\sup \mathcal{I} = \inf \mathcal{F}$ exists. We frequently use the fact from Section 3 that if $\nu(f) \neq \sup \mathcal{I}$, then there exists some $\nu(g) \in \Gamma_{<0}$ with $\nu(g') = \nu(f)$.

4.3. Theorem. *For $\nu(f) \in p$ we define the mapping s_f on $\Gamma_{<0}$ by*

$$s_f\colon \Gamma_{<0} \to \Gamma, \qquad \nu(\varphi) \mapsto \nu((f \circ |\varphi|) \cdot \varphi')$$

(s_f stands for substitution). Then the Dedekind cut $(\mathcal{I}, \mathcal{F})$ is not free if and only if there is some $\nu(f) \in p$ such that the mapping s_f is constant, and then $\nu(f) = \sup \mathcal{I}$.

Proof. Taking $\varphi(t) = t$ we see that $\nu(f) \in s_f(\Gamma_{<0})$. Let $\nu(f) \in p$ with $\nu(f) \neq \sup \mathcal{I}$.

Case 1: $\nu(f) \in \mathcal{I}$. Then there is some $\nu(g) \in \Gamma_{<0} \cap p$ with $\nu(g') = \nu(f)$. Choose $\nu(h) \in \Gamma_{<0}$ with $\nu(g) < \nu(h)$. Take $\varphi := g^{-1} \circ h$. Then $g \circ \varphi = h$.

By Proposition 4.1 we get

$$\nu(f)=\nu(g')<\nu(h')=\nu((g\circ\varphi)')=\nu((f\circ\varphi)\cdot\varphi').$$

By the same argument there is some $\nu(\widetilde{\varphi})\in\Gamma_{<0}$ with $\widetilde{\varphi}>0$ and

$$\nu(f)>\nu((f\circ\widetilde{\varphi})\cdot\widetilde{\varphi}').$$

CASE 2: $\nu(f)\in\mathcal{F}$. The argument goes similarly. Hence $s_f(\Gamma_{<0})\supsetneqq\{\nu(f)\}$ if $\nu(f)\neq\sup\mathcal{I}$.

Now let $\nu(f)=\sup\mathcal{I}$.

CASE 1: $\nu(f)\in\mathcal{I}$. Without restriction $f>0$. Let $\nu(\varphi)\in\Gamma_{<0}$; without restriction $\varphi>0$. Assume that $\nu((f\circ\varphi)\cdot\varphi')>\nu(f)$. Then by substitution

$$\int^{\infty}(f\circ\varphi)\cdot\varphi'=\int^{\infty}f=\infty.$$

So $\nu(f)\neq\sup\mathcal{I}$, a contradiction.

Assume that $\nu((f\circ\varphi)\cdot\varphi')<\nu(f)$. Then there is some $\nu(g)\in\Gamma_{<0}$ with $\nu(g')=\nu((f\circ\varphi)\cdot\varphi')$; without restriction $g>0$. For $t_0>0$ large enough we define

$$F\colon\,]t_0,\infty[\,\to\mathbb{R}_{>0},\qquad t\mapsto\int_{t_0}^{t}f(\tau)\,d\tau,$$

the antiderivative of f. Again by substitution we have

$$\int_{\varphi^{-1}(t_0)}^{\varphi^{-1}(t)}(f\circ\varphi(\tau))\cdot\varphi'(\tau)\,d\tau=F(t),$$

so $\nu((g\circ\varphi^{-1})')=\nu(f)$; but by Rosenlicht's Theorem 3.3 this is a contradiction to $\nu(f)=\sup\mathcal{I}$.

CASE 2: $\nu(f)\in\mathcal{F}$. This is similar. ∎

Assume now that $\sup\mathcal{I}$ exists. To shorten the statement below suppose that $\sup\mathcal{I}=\max\mathcal{I}$. Let $\nu(f)=\max\mathcal{I}$, and let F be the antiderivative of f. Then by Section 3 we have $0>\nu(F)>\nu(g)$ for all $\nu(g)\in\Gamma_{<0}$. But F is much smaller than all functions definable in $\mathcal{M}$, tending to ∞ at ∞. From the last theorem we get the following result:

4.4. COROLLARY. *Let $f\colon\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}$ be definable and continuous with $\nu(f)=\max\mathcal{I}$. Let $F\colon\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0}$, $t\mapsto\int_0^t f(\tau)\,d\tau$, be the antiderivative of f. Define the family*

$$H\colon\mathbb{R}_{>0}\times\mathbb{R}_{\geq0}\to\mathbb{R}_{\geq0},\qquad(c,t)\mapsto F^{-1}(cF(t)).$$

Then H is a one-parameter family of functions on $\mathbb{R}_{\geq0}$, which is definable in an o-minimal expansion of $\mathcal{M}$, and which is cofinal and coinitial in $\Gamma_{<0}$ for the given o-minimal structure.

Proof. Let φ be definable in $\mathcal{M}$ with $\nu(\varphi) < 0$; without restriction $\varphi > 0$. By the proof of the last theorem,

$$\nu(F \circ \varphi) = \nu(F),$$

so there are $c_1, c_2 > 0$ with

$$c_1 F(t) \leq F(\varphi(t)) \leq c_2 F(t) \quad \text{for all } t > 0 \text{ large.}$$

But then

$$F^{-1}(c_1 F(t)) \leq \varphi(t) \leq F^{-1}(c_2 F(t)) \quad \text{for all } t > 0 \text{ large,}$$

hence

$$\nu(F^{-1}(c_1 F)) \geq \nu(\varphi) \geq \nu(F^{-1}(c_2 F)). \ \blacksquare$$

REMARK. Obviously $\nu(F^{-1}) < \nu(F^{-1}(cF)) < \nu(F)$ for all $c > 0$.

4.5. EXAMPLE. Let $\mathcal{M}$ be polynomially bounded. Then $\max \mathcal{I} = \nu(1/t)$. We have $F(t) = \log t$ and

$$F^{-1}(cF(t)) = t^c \quad \text{for all } c > 0.$$

Finally, we prove definable asymptotic integration. Let $f\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a definable and continuous function. The question whether the function

$$F\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad (a,t) \mapsto \int_0^t f(a,\tau)\,d\tau,$$

is definable in an *o*-minimal expansion is only solved if f is definable in the *o*-minimal structure generated by restricted analytic functions: If f is definable in $\mathbb{R}_{\text{an}}$, then F is definable in $\mathbb{R}_{\text{an,exp}}$. But in the other cases it is still open. But for asymptotic integration we get a definable result:

4.6. THEOREM. *Let $f\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a definable and continuous function. Then there is an o-minimal expansion of $\mathcal{M}$ and a function $\widetilde{F}\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ definable in this expansion such that*

$$\nu((\widetilde{F}(a,\cdot))') = \nu(f(a,\cdot)) \quad \textit{for all } a \in \mathbb{R}^n.$$

The function can be chosen in such a way that

$$\widetilde{F}|\{a \in \mathbb{R}^n \mid \nu(f,\cdot) \neq \sup \mathcal{I}\}$$

is definable in $\mathcal{M}$. In particular, if $\mathcal{M}$ is closed under asymptotic integration then $\widetilde{F}$ is definable in $\mathcal{M}$.

Proof. We can assume that $\nu(f(a,\cdot)) \neq \infty$ for all $a \in \mathbb{R}^n$. Let $A := \{a \in \mathbb{R}^n \mid \nu(f(a,\cdot)) = \sup \mathcal{I}\}$. Then A is definable in $\mathcal{M}$. Fix $a_0 \in A$ and define

$$F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad t \mapsto \int_0^t f(a_0,\tau)\,d\tau.$$

Then F is definable in an o-minimal expansion $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. On A we take

$$\widetilde{F}\colon A \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \qquad (a,t) \mapsto F(t).$$

Now let $B := \mathbb{R}^n \setminus A$. We may assume that $\nu(f(a,\cdot)) \neq 0$ for all $a \in B$. (If $\nu(f(a,\cdot)) = 0$ we can take the identity function.) Again by universal boundedness of growth (compare the proof of Theorem 2.1) there are $\nu(h) \in \mathcal{I}\setminus\{\sup\mathcal{I}\}$ and $\nu(g) \in \mathcal{F}\setminus\{\sup\mathcal{I}\}$ with $\nu(f(a,\cdot)) < \nu(h)$ or $\nu(f(a,\cdot)) > \nu(g)$ for all $a \in B$. Choose now $\nu(u_{0,1}), \nu(u_{0,2}) \in \Gamma_{>0}$ according to Rosenlicht's Theorem 3.3, such that

$$\nu\Bigg(\bigg(\frac{h \cdot \big(\frac{h \cdot u_{0,1}}{u_{0,1}'}\big)}{\big(\frac{h\cdot u_{0,1}}{u_{0,1}'}\big)'}\bigg)'\Bigg) = \nu(h), \qquad \nu\Bigg(\bigg(\frac{g \cdot \big(\frac{h \cdot u_{0,2}}{u_{0,2}'}\big)}{\big(\frac{g\cdot u_{0,2}}{u_{0,2}'}\big)'}\bigg)'\Bigg) = \nu(g).$$

If we take $|\nu(u_0)| := \min\{|\nu(u_{0,1})|, |\nu(u_{0,2})|\}$, then again by the proof of that theorem (cf. [10]) we find that

$$\widetilde{F}\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}, \qquad (a,t) \mapsto \Bigg(\frac{f(a,t) \cdot \big(\frac{f(a,t)\cdot u_0(t)}{u_0'(t)}\big)}{\big(\frac{f(a,t)\cdot u_0(t)}{u_0'(t)}\big)'}\Bigg)'$$

works.

If $\sup\mathcal{I}$ exists we can give a different proof, where the solution is shorter. Again we restrict ourselves to the case that $\sup\mathcal{I} = \max\mathcal{I}$. We concentrate on the set

$$C := \{a \in \mathbb{R}^n \mid \nu(f(a,\cdot)) \neq 0, \max\mathcal{I}\}.$$

Take $g\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ definable and continuous with $\nu(g) = \max\mathcal{I}$. Let $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be definable and continuous with $\nu(h) \neq \max\mathcal{I}$. By integration by parts we have

$$\int_0^t g(\tau)\,d\tau = \int_0^t \Big(h \cdot \frac{g}{h}\Big)(\tau)\,d\tau = \Big(H \cdot \frac{g}{h}\Big)(t) - \int_0^t H \cdot \Big(\frac{g}{h}\Big)'\,d\tau,$$

where H is the antiderivative of h. Let G be the antiderivative of g and $\widetilde{H}$ be the antiderivative of $H \cdot (g/h)'$. We choose them so that $H(0) = G(0) = \widetilde{H}(0) = 0$. Then $G, H, \widetilde{H}$ live in an o-minimal expansion $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. Let $\widetilde{\Gamma}$ be the canonical value group of $\widetilde{\mathcal{M}}$, and let Γ be as usual the canonical value group of $\mathcal{M}$. By Rosenlicht's Theorem 3.3 there is a $\nu(p) \in \Gamma \setminus \{0\}$ with $\nu(p) = \nu(H)$. We get, in $\widetilde{\Gamma}$,

$$\nu(G) = \nu\Big(H \cdot \frac{g}{h} - \widetilde{H}\Big).$$

Case 1: $\nu(H \cdot g/h) \neq \nu(\widetilde{H})$. Then $\nu(G) = \min\{\nu(H \cdot g/h), \nu(\widetilde{H})\}$. Since $\nu(H \cdot g/h) = \nu(p \cdot g/h) \in \Gamma$ and $\nu(G) \notin \Gamma$ we get $\nu(G) = \nu(\widetilde{H})$, and so $\nu(g) = \nu(H \cdot (g/h)')$.

CASE 2: $\nu(H\cdot g/h)=\nu(\widetilde{H})$. Then $\nu((H\cdot g/h)')=\nu(H\cdot(g/h)')$ and as a consequence $\nu(g+H\cdot(g/h)')=\nu(H\cdot(g/h)')$. So $\nu(g)\geq\nu(H\cdot(g/h)')$.

Assume that $\nu(g)>\nu(H\cdot(g/h)')$. Since $\nu(g)=\max\mathcal{I}$ and $\nu(H\cdot(g/h)')=\nu(p(g/h)')\in\Gamma$ we get $\nu(\widetilde{H})<0$. Hence $\nu(H\cdot g/h)<0$. But

$$\nu\left(H\cdot\frac{g}{h}\right)=\nu\left(\frac{H}{h}\right)+v(g)=\nu\left(\frac{H}{H'}\right)+\nu(g)$$
$$=-\nu\left(\frac{H'}{H}\right)+v(g)=-\nu\left(\frac{p'}{p}\right)+\nu(g)\geq 0,$$

because $\nu(p'/p)\in\Psi$ and $\Psi\leq\max\mathcal{I}$ by Lemma 3.5, a contradiction. So in both cases $\nu(g)=\nu(H\cdot(g/h)')$ and therefore

$$\nu(H)=\nu\left(\frac{g}{(g/h)'}\right).$$

On C we can take now

$$F\colon B\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0},\qquad (a,t)\mapsto\frac{g(t)}{(g(t)/f(a,t))'},$$

which is definable in $\mathcal{M}$. ■

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