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The degree at infinity of the gradient of a polynomial in two real variables

by MACIEJ SĘKALSKI (Kielce)

In memory of Stanisław Łojasiewicz

Abstract. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial mapping with a finite number of critical points. We express the degree at infinity of the gradient ∇f in terms of the real branches at infinity of the level curves $\{f(x,y) = \lambda\}$ for some $\lambda \in \mathbb{R}$. The formula obtained is a counterpart at infinity of the local formula due to Arnold.

1. Main result. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping with a finite fibre over $(0,0)$. We define the degree at infinity $\deg_\infty F$ to be the topological degree of the Gauss mapping $S_R \ni (x,y) \mapsto F(x,y)/\|F(x,y)\| \in S_1$, where S_R is a circle (with radius R centred at $(0,0)$) around the fibre $F^{-1}(0,0)$ and S_1 is the unit circle.

Our paper deals with $\deg_\infty F$ for the mapping $F = \nabla f = (\partial f/\partial X, \partial f/\partial Y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial mapping with a finite number of critical points.

To formulate the main result we introduce the notion of critical values of a polynomial f at infinity. Namely, define

$$J_f(X,Y) = Y\frac{\partial f}{\partial X}(X,Y) - X\frac{\partial f}{\partial Y}(X,Y).$$

The set $\{J_f(x,y) = 0\}$ is unbounded, because it consists of points at which the polynomial f restricted to the big circles S_R has an extremum. The real number λ is a *critical value of f at infinity* if there exists a parametrization $p(t)$ meromorphic at infinity (see Section 2) of a branch of the curve $\{J_f(x,y) = 0\}$ such that $f(p(t)) \to \lambda$ as $t \to \infty$. We assume here that $J_f(x,y) \not\equiv 0$ in $\mathbb{R}^2$. The set of critical values of f at infinity will be denoted by $\Lambda(f)$. If $J_f(x,y) \equiv 0$, then by definition f has no critical values at infinity, that is, $\Lambda(f) = \emptyset$.

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Since $\Lambda(f)$ is finite we can write $\Lambda(f) = \{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_1 < \cdots < \lambda_n$. Put $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$. Then $\mathbb{R} \setminus \Lambda(f) = \bigcup_{i=1}^{n+1} (\lambda_{i-1}, \lambda_i)$ (if $\Lambda(f) = \emptyset$ then $n = 0$). Moreover, let $r_\infty(f)$ denote the number of real branches at infinity of the curve $\{f(x,y) = 0\}$ (see Section 2).

Under the above notation we have

THEOREM 1. *The function* $\mathbb{R} \ni \lambda \mapsto r_\infty(f - \lambda)$ *is constant on every connected component of* $\mathbb{R} \setminus \Lambda(f)$. *Let* $r_i = r_\infty(f - \lambda)$ *for* $\lambda \in (\lambda_{i-1}, \lambda_i)$, $i = 1, \ldots, n+1$. *Then*

$$\deg_\infty \nabla f = 1 + \sum_{\lambda \in \Lambda(f)} r_\infty(f - \lambda) - \sum_{i=1}^{n+1} r_i. \tag{1}$$

The proof of Theorem 1 will be given in Section 4. Now let us record

COROLLARY. *If* $\Lambda(f) = \emptyset$ *then* $\deg_\infty \nabla f = 1 - r_\infty(f)$.

The formula from the corollary is a counterpart at infinity of the well known local result due to Arnold (see [A]). Namely, let f be an analytic function of two real variables near $(0,0) \in \mathbb{R}^2$ such that $f(0,0) = 0$. Suppose that $(0,0)$ is an isolated solution of the equation $\nabla f(x,y) = (0,0)$. If $\deg_0 \nabla f$ denotes the local degree of ∇f at $(0,0)$ and $r_0(f)$ is the number of branches of the curve $\{f(x,y) = 0\}$ near $(0,0)$ then

$$\deg_0 \nabla f = 1 - r_0(f).$$

REMARK. Theorem 1 and its Corollary remain valid for polynomials f with compact fibre $(\nabla f)^{-1}(0,0)$.

2. Branches at infinity of an algebraic set. In this section we give the description of branches at infinity of an unbounded algebraic set in $\mathbb{R}^2$.

Let Ω and Δ be neighbourhoods of infinity in $\mathbb{R}^2$ and $\mathbb{R}$ respectively. We have the following

PROPOSITION. *Let S be an unbounded algebraic set in $\mathbb{R}^2$. Then there exists a neighbourhood of infinity Ω in $\mathbb{R}^2$ such that $S \cap \Omega$ is the union of finitely many pairwise disjoint analytic curves. Each curve* (*branch*) *is homeomorphic to an open neighbourhood of infinity Δ under a homeomorphism* $(x(t), y(t))$ (*meromorphic at infinity*) *which is given by a Laurent series*

$$(x(t), y(t)) = \Big(\sum_{i=-\infty}^{k} a_i t^i, \sum_{i=-\infty}^{k} b_i t^i \Big),$$

with $a_k \neq 0$ *or* $b_k \neq 0$ *and* $k > 0$.

Proof. See [S1, Lemma 1].

If $S = \{f(x,y) = 0\}$ for a polynomial f then the number of branches at infinity of the set S will be denoted by $r_\infty(f)$.

Example. If $S \subset \mathbb{R}^2$ is given by the equation $x^2y - 1 = 0$ then $S \cap \Omega$ consists of two branches at infinity. The mappings $t \mapsto (t, 1/t^2)$ and $t \mapsto (1/t, t^2)$ for $t \neq 0$ are their parametrizations.

3. Auxiliary lemmas. In order to prove the main result we need some lemmas.

Lemma 1. *For any polynomial mapping f whose set of critical points is finite there exists $A \in \mathbb{R}$ such that if we set $f_A(X,Y) = f(AX,Y)$ then $\nabla J_{f_A}(x,y) \neq (0,0)$ on the curve $\{J_{f_A}(x,y) = 0\}$ in a neighbourhood of infinity.*

Proof. The set $(\nabla f)^{-1}(0,0)$ is finite, so suppose that $\partial f/\partial X \not\equiv 0$ in a neighbourhood of infinity. Consider the function

$$\mathbb{R}^2 \setminus \left\{ y\,\frac{\partial f}{\partial X}(x,y) = 0 \right\} \ni (x,y) \mapsto \frac{x\frac{\partial f}{\partial Y}(x,y)}{y\frac{\partial f}{\partial X}(x,y)} \in \mathbb{R}.$$

Let $A^2 \neq 0$ be a positive regular value of this mapping. Then

$$\nabla\left(\frac{X\frac{\partial f}{\partial Y}}{Y\frac{\partial f}{\partial X}}\right)(x,y) = \left[\frac{1}{Y\frac{\partial f}{\partial X}}\nabla\left(X\,\frac{\partial f}{\partial Y} - A^2Y\,\frac{\partial f}{\partial X}\right)\right](x,y) \neq (0,0)$$

on the curve $\{(X\frac{\partial f}{\partial Y} - A^2Y\frac{\partial f}{\partial X})(x,y) = 0\}$. Since

$$\nabla\left(X\,\frac{\partial f}{\partial Y} - A^2Y\,\frac{\partial f}{\partial X}\right)(Ax,y) = A\nabla\left(X\,\frac{\partial f_A}{\partial Y} - Y\,\frac{\partial f_A}{\partial X}\right)(x,y)$$

we get $\nabla J_{f_A}(x,y) \neq (0,0)$ for $J_{f_A}(x,y) = 0$. This ends the proof.

For a function h of one real variable, meromorphic at infinity, we use the following convention:

$$\deg_\infty h = \frac{\operatorname{sgn} h(t^+) - \operatorname{sgn} h(t^-)}{2},$$

where the numbers t^- and t^+ are taken close enough to $-\infty$ and $+\infty$ respectively. Under the above convention we have

Lemma 2. *If the real polynomial mapping $G = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ has a finite fibre over $(0,0)$ and $\nabla g_1(x,y) \neq (0,0)$ on the curve $\{g_1(x,y) = 0\}$ in a neighbourhood of infinity then*

$$\deg_\infty G = \sum_{i=1}^{k} \deg_\infty(g_2(p_i(t)) \cdot \det[\nabla g_1(p_i(t)), p_i'(t)]),$$

where p_i, $i = 1, \ldots, k$, are parametrizations of the real branches at infinity of the curve $\{g_1(x,y) = 0\}$.

Proof. The proof can be found in [S1].

The following corollary to Lemma 2 will be useful.

Corollary. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$ *be a polynomial mapping such that* $\nabla g(x,y) \neq (0,0)$ *for* $g(x,y) = 0$ *near infinity. Then* $r_\infty(g) = \deg_\infty(g, J_g)$.

The local counterpart of the corollary has been proven in [FAS] and [Sz].

Proof. The mapping $(g, J_g) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the assumptions of Lemma 2. Let p_i, $i = 1, \ldots, k$, be parametrizations, meromorphic at infinity, of the branches of the curve $g = 0$, and $\langle \cdot, \cdot \rangle$ be the scalar product in $\mathbb{R}^2$. Then Lemma 2 gives

$$\begin{aligned}\deg_\infty(g, J_g) &= \sum_{i=1}^{k} \deg_\infty(J_g(p_i(t)) \det[\nabla g(p_i(t)), p_i'(t)]) \\ &= \sum_{i=1}^{k} \deg_\infty(\det[\nabla g(p_i(t)), p_i(t)] \cdot \det[\nabla g(p_i(t)), p_i'(t)]) \\ &= \sum_{i=1}^{k} \deg_\infty(\|\nabla g(p_i(t))\|^2 \langle p_i(t), p_i'(t)\rangle) = \sum_{i=1}^{k} 1 = r_\infty(g).\end{aligned}$$

Below we collect some simple properties of the degree. One can easily check them by using for instance the "Poincaré argument principle" (cf. [S2]):

Proposition (Properties of the degree). *Let* $F = (f_1, f_2), G = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$ *be polynomial mappings such that the sets* $F^{-1}(0,0)$ *and* $G^{-1}(0,0)$ *are finite. Then*

- *the mapping* $F \cdot G = (f_1g_1 - f_2g_2, f_1g_2 + f_2g_1)$ *has a finite fibre over* $(0,0)$ *and*
$$\deg_\infty(F \cdot G) = \deg_\infty F + \deg_\infty G,$$
- $\deg_\infty(f_1, f_2) = -\deg_\infty(f_2, f_1)$ (*antisymmetry*),
- $\deg_\infty(f_1, -f_2) = -\deg_\infty(f_1, f_2)$,
- $\deg_\infty(X, Y) = 1$.

4. Proof of the main result. Without loss of generality (according to Lemma 1) we can assume that $\nabla J_f(x,y) \neq (0,0)$ on the curve $\{J_f(x,y) = 0\}$ near infinity. Consider a sequence $\lambda_0', \ldots, \lambda_n'$ such that

$$(2) \qquad -\infty = \lambda_0 < \lambda_0' < \lambda_1 < \lambda_1' < \cdots < \lambda_n < \lambda_n' < \lambda_{n+1} = +\infty,$$

where λ_i, $i = 1, \ldots, n$, are the critical values of the polynomial f at infinity (in the sense of the definition from Section 1). Thus we have $r_i = r_\infty(f - \lambda_i')$ for $i = 0, \ldots, n$. We will calculate the sum

$$S = \sum_{i=1}^{n} r_\infty(f - \lambda_i) - \sum_{i=0}^{n} r_\infty(f - \lambda_i').$$

By using the Corollary to Lemma 2 and antisymmetry of the degree we get

$$S=\sum_{i=1}^{n}\deg_\infty(f-\lambda_i,J_f)-\sum_{i=0}^{n}\deg_\infty(f-\lambda_i',J_f)$$
$$=\sum_{i=0}^{n}\deg_\infty(J_f,f-\lambda_i')-\sum_{i=1}^{n}\deg_\infty(J_f,f-\lambda_i).$$

Let us split the set of all parametrizations at infinity of the curve $\{J_f(x,y)=0\}$ into two subsets G^+ and G^-, where G^+ consists of those parametrizations p for which $f(p(t))\to\infty$ as $t\to\infty$ and the remaining parametrizations are contained in G^-, i.e. if $p\in G^-$ then $f(p(t))\to\lambda\in\Lambda(f)$ as $t\to\infty$. To shorten our formulas we set $w_p(t)=\det[\nabla J_f(p(t)),p'(t)]$. Moreover we will omit the variable t and write w_p, $f(p)$ instead of $w_p(t)$, $f(p(t))$. According to Lemma 2 we have

$$S=\sum_{i=0}^{n}\sum_{p\in G^+\cup G^-}\deg_\infty((f(p)-\lambda_i')w_p)-\sum_{i=1}^{n}\sum_{p\in G^+\cup G^-}\deg_\infty((f(p)-\lambda_i)w_p)$$
$$=\sum_{p\in G^+\cup G^-}\Big[\deg_\infty((f(p)-\lambda_0')w_p)+\sum_{i=1}^{n}[\deg_\infty((f(p)-\lambda_i')w_p)-\deg_\infty((f(p)-\lambda_i)w_p)]\Big]. \tag{3}$$

Note that if $p\in G^+$ then $\deg_\infty((f(p)-\lambda)w_p)$ does not depend on λ. In this case we have

$$\sum_{i=1}^{n}[\deg_\infty((f(p)-\lambda_i')w_p)-\deg_\infty((f(p)-\lambda_i)w_p)]=0. \tag{4}$$

If $p\in G^-$ then $f(p(t))\to\lambda_p\in\Lambda(f)$ as $t\to\infty$. Then for $\lambda_i\neq\lambda_p$ we have

$$\deg_\infty((f(p)-\lambda_i')w_p)=\deg_\infty((f(p)-\lambda_i)w_p),$$

hence

$$\sum_{i=1}^{n}[\deg_\infty((f(p)-\lambda_i')w_p)-\deg_\infty((f(p)-\lambda_i)w_p)]=\deg_\infty((f(p)-\lambda_p')w_p)-\deg_\infty((f(p)-\lambda_p)w_p). \tag{5}$$

Here λ_p' denotes the next number after λ_p in the sequence (2).

From (3)–(5) we get

$$S = \sum_{p\in G^+} \deg_\infty((f(p)-\lambda_0')w_p) + \sum_{p\in G^-} [\deg_\infty((f(p)-\lambda_0')w_p) + \deg_\infty((f(p)-\lambda_p')w_p) - \deg_\infty((f(p)-\lambda_p)w_p)].$$

But the inequalities $\lambda_0' < \lambda_p < \lambda_p'$ imply that the numbers $f(p)-\lambda_0'$ and $f(p)-\lambda_p'$ have opposite signs for t large, hence

$$\deg_\infty((f(p)-\lambda_0')w_p) + \deg_\infty((f(p)-\lambda_p')w_p) = 0,$$

so we get the equality

$$S = \sum_{p\in G^+} \deg_\infty((f(p)-\lambda_0')w_p) - \sum_{p\in G^-} \deg_\infty((f(p)-\lambda_p)w_p). \tag{6}$$

Observe that for $p\in G^+$,

$$\operatorname{sgn}(f(p(t))-\lambda_0') = \operatorname{sgn}((f(p(t)))'\cdot t) = \operatorname{sgn}(\langle \nabla f(p(t)), p(t)\rangle),$$

while for $p\in G^-$,

$$\operatorname{sgn}(f(p(t))-\lambda_p) = -\operatorname{sgn}(f(p(t))'\cdot t) = -\operatorname{sgn}(\langle \nabla f(p(t)), p(t)\rangle).$$

In fact, from the equality $J_f(p(t)) = \big(Y\frac{\partial f}{\partial X} - X\frac{\partial f}{\partial Y}\big)\circ p(t) = 0$ we see that the vectors $\nabla f(p(t))$ and $p(t)$ are parallel, hence we have

$$f(p(t))'t = \langle \nabla f(p(t)), p'(t)\rangle t = \frac{\langle \nabla f(p(t)), p(t)\rangle}{\|p(t)\|^2}\langle p(t), p'(t)\rangle t$$

and the above equalities follow because the quotient $\langle p(t), p'(t)\rangle t/\|p(t)\|^2$ is positive in a neighbourhood of infinity in $\mathbb{R}$.

The above two equalities applied to (6), Lemma 2 and the properties of the degree give

$$\begin{aligned} S &= \sum_{p\in G^+} \deg_\infty(\langle \nabla f(p(t)), p(t)\rangle w_p) + \sum_{p\in G^-} \deg_\infty(\langle \nabla f(p(t)), p(t)\rangle w_p) \\ &= \deg_\infty\left(J_f, X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y}\right) = \deg_\infty\left(X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y}, X\frac{\partial f}{\partial Y} - Y\frac{\partial f}{\partial X}\right) \\ &= \deg_\infty(\nabla f\cdot(X,-Y)) = \deg_\infty \nabla f - 1. \end{aligned}$$

We are done.

We end this section with a simple example of calculation of the degree by using the main theorem.

Example. Let $f(X,Y) = \prod_{i=1}^k (Y(X^2+i)-1)$ (see [D]). One can check that the only critical value at infinity is zero, that is, $\Lambda(f) = \{0\}$. We have $r_\infty(f-1) + r_\infty(f+1) = 2$ and $r_\infty(f) = k$, thus

$$\deg_\infty \nabla f = 1 + k - 2 = k - 1.$$

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Department of Mathematics
Technical University
Al. 1000LPP 7
25-314 Kielce, Poland
E-mail: matms@tu.kielce.pl

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