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Whitney triangulations of semialgebraic sets

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Abstract. A compact semialgebraic set admits a semialgebraic triangulation such that the family of open simplexes forms a Whitney stratification and is compatible with a finite number of given semialgebraic subsets.

A *semialgebraic triangulation* of a compact semialgebraic set X is a pair of a simplicial complex K and a semialgebraic homeomorphism $f\colon |K| \to X$ such that for each $\sigma \in K$, $f(\operatorname{Int}\sigma)$ is a C^ω manifold and $f|_{\operatorname{Int}\sigma}$ is a C^ω diffeomorphism onto $f(\operatorname{Int}\sigma)$, where $|K|$ denotes the underlying polyhedron of K. As a consequence of the theory developed in [S_2] we have:

Theorem. *A compact semialgebraic set X admits a semialgebraic triangulation (K, f) such that $\{f(\operatorname{Int}\sigma)\colon \sigma \in K\}$ is a Whitney stratification of X and the triangulation is compatible with a finite number of given semialgebraic subsets Y_i of X.*

When we work in an o-minimal structure, let the above diffeomorphisms be of class C^r for $r \in \mathbb{N}$. Then the arguments below go through. Most of the following maps and manifolds are of class C^2. However, it is possible to work in the C^1 category. See [S_2] for that case.

A (*semialgebraic*) C^2 *tube* about a (semialgebraic) C^2 manifold $M \subset \mathbb{R}^n$ is a triple $T = (|T|, \tau, \varrho)$, where $|T|$ is a (semialgebraic) neighborhood of M in $\mathbb{R}^n$, $\tau\colon |T| \to M$ is a (semialgebraic) submersive C^2 retraction, and ϱ is a nonnegative (semialgebraic) C^2 function on $|T|$ such that $\varrho^{-1}(0) = M$ and each point x of M is a unique and nondegenerate critical point of the restriction of ϱ to $\tau^{-1}(x)$. The ambient space of M is not necessarily a Euclidean space. It may be a (semialgebraic) C^2 manifold possibly with boundary. We also define a *controlled* (*semialgebraic*) C^2 *tube system* for a (semialgebraic) C^2 stratification in the same way as in the C^∞ case (see [G-al] for the C^∞ case).

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Let $\{M_k\}_{k=1,2,\ldots}$ and $\{N_k\}$ be Whitney C^2 stratifications of sets M and N, respectively, in $\mathbb{R}^n$. Let $\{T_k^M = (|T_k^M|, \tau_k^M, \varrho_k^M)\}$ and $\{T_k^N = (|T_k^N|, \tau_k^N, \varrho_k^N)\}$ be controlled C^2 tube systems for $\{M_k\}$ and $\{N_k\}$ respectively. Let φ: $M \to N$ be a homeomorphism such that $\varphi|_{M_k}$ is a C^2 diffeomorphism onto N_k for each k. We call φ *compatible* with $\{T_k^M\}$ and $\{T_k^N\}$ if for each k, $\varphi(M \cap |T_k^M|) = N \cap |T_k^N|$ and

$$\tau_k^N \circ \varphi = \varphi \circ \tau_k^M \quad \text{and} \quad \varrho_k^N \circ \varphi = \varrho_k^M \quad \text{on } M \cap |T_k^M|.$$

Let f and f_l, $l = 1, 2, \ldots$, be semialgebraic C^0 functions on a semialgebraic set M. We say that $f_l \to f$ in the C^0 topology as $l \to \infty$ if for any positive semialgebraic C^0 function g on M there exists $l_0 \in \mathbb{Z}$ such that $|f_l - f| < g$ for $l \geq l_0$. Assume M is a semialgebraic C^2 manifold and f_l and f are semialgebraic C^2 functions. Then we say that $f_l \to f$ in the C^2 topology if $f_l \to f$ and the differentials $df_l \to df$, $ddf_l \to ddf$ in the C^0 topology. In the same way we define the C^0 topology on the set of semialgebraic C^0 maps between semialgebraic sets and the C^2 topology on the set of semialgebraic C^2 maps between semialgebraic C^2 manifolds.

The key to the proof of the theorem is the following lemma, which does not hold without the assumption of semialgebraicness nor without the one that M_k are semialgebraically diffeomorphic to Euclidean spaces (see [S_2, p. 238 and Remark II.7.4]).

Lemma (Uniqueness of controlled semialgebraic tube system, [S_2, Theorem II.7.3]). *Let $\{M_k\}$ and $\{N_k\}$ be finite semialgebraic Whitney C^2 stratifications of locally compact semialgebraic sets M and N, respectively, such that all strata are semialgebraically C^2 diffeomorphic to Euclidean spaces. Let $\{T_k^M\}$ and $\{T_k^N\}$ be controlled semialgebraic C^2 tube systems for $\{M_k\}$ and $\{N_k\}$ respectively. Let φ: $M \to N$ be a semialgebraic homeomorphism such that $\varphi|_{M_k}$ is a C^2 diffeomorphism onto N_k for each k. Then shrinking $|T_k^M|$ and $|T_k^N|$, we can modify φ to be compatible with $\{T_k^M\}$ and $\{T_k^N\}$.*

Proof of Theorem. Let $X \subset \mathbb{R}^n$. By the usual triangulation theorem for semialgebraic sets we have a finite number of semialgebraic C^0 imbeddings π_α: $B^\alpha \to X$, $\alpha \in A$, such that each B^α is the unit ball (always with center 0) in a Euclidean space, $\pi_\alpha|_{\mathrm{Int}\, B^\alpha}$ is a C^ω imbedding,

$$\pi_\alpha(\mathrm{Int}\, B^\alpha) \cap \pi_{\alpha'}(\mathrm{Int}\, B^{\alpha'}) = \emptyset \quad \text{for } \alpha \neq \alpha' \in A,$$

$\{\pi_\alpha(\mathrm{Int}\, B^\alpha)$: $\alpha \in A\}$ is a Whitney stratification of X compatible with $\{Y_i\}$, $\pi_\alpha^{-1}(\pi_{\alpha'}(\mathrm{Int}\, B^{\alpha'}))$ is a C^ω manifold for $\alpha \in A$ and $\alpha' \in A_\alpha = \{\alpha' \in A$: $\pi_{\alpha'}(B^{\alpha'}) \subset \pi_\alpha(B^\alpha)\}$, the restriction of π_α to the manifold is a C^ω imbedding, and $\{\pi_\alpha^{-1}(\pi_{\alpha'}(\mathrm{Int}\, B^{\alpha'}))$: $\alpha' \in A_\alpha\}$ is a Whitney stratification of B^α for each α. We then call $\{\pi_\alpha$: $\alpha \in A\}$ a *semialgebraic C^ω ball decomposition* of X (cf. [R-S]). Let the ambient space of B^α always be B^α itself.

Set $d_\alpha = \dim B^\alpha$ and

$$C_\alpha = \pi_\alpha(B^\alpha), \quad \partial C_\alpha = \pi_\alpha(\partial B^\alpha), \quad \operatorname{Int} C_\alpha = \pi_\alpha(\operatorname{Int} B^\alpha) \quad \text{for } \alpha \in A,$$
$$B^\alpha_{\alpha'} = \pi_\alpha^{-1}(C_{\alpha'}),\ \partial B^\alpha_{\alpha'} = \pi_\alpha^{-1}(\partial C_{\alpha'}),\ \operatorname{Int} B^\alpha_{\alpha'} = \pi_\alpha^{-1}(\operatorname{Int} C_{\alpha'}) \text{ for } \alpha' \in A_\alpha.$$

Then $\{\operatorname{Int} C_\alpha\colon \alpha \in A\}$ and $\{\operatorname{Int} B^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ for each $\alpha \in A$ are Whitney stratifications of X and B^α respectively.

Let $0 * Y$ denote the cone with vertex 0 in some B^α and base $Y \subset \partial B^\alpha$. Let $\{T_\alpha = (|T_\alpha|, \tau_\alpha, \varrho_\alpha)\colon \alpha \in A\}$ and $\{T^\alpha_{\alpha'} = (|T^\alpha_{\alpha'}|, \tau^\alpha_{\alpha'}, \varrho^\alpha_{\alpha'})\colon \alpha' \in A_\alpha\}$ for each $\alpha \in A$ be controlled semialgebraic C^2 tube systems for $\{\operatorname{Int} C_\alpha\colon \alpha \in A\}$ and $\{\operatorname{Int} B^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ respectively. Then we can choose the latter so that

$$(1)\qquad (\tau^\alpha_{\alpha'})^{-1}(x) = |T^\alpha_{\alpha'}| \cap 0 * (\partial B^\alpha \cap (\tau^\alpha_{\alpha'})^{-1}(x)) \qquad \text{for } x \in \operatorname{Int} B^\alpha_{\alpha'},\ \alpha' \in A_\alpha - \{\alpha\},\ \alpha \in A$$

by the usual method of construction of controlled tube systems (see e.g. [G-al]). Moreover, we can modify $\{\pi_\alpha\}$ so that each $\pi_\alpha\colon B^\alpha \to C_\alpha$ is compatible with $\{T^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ and $\{T_{\alpha'}\colon \alpha' \in A_\alpha\}$ by the lemma. Here $\pi_\alpha|_{\operatorname{Int} B^\alpha_{\alpha'}}$ are no longer of class C^ω but C^2. Hence $\{\pi_\alpha\colon \alpha \in A\}$ is called a *semialgebraic C^2 ball decomposition* of X. By loosening the condition of compatibility we will C^ω smooth later.

In the following arguments we subdivide $\{\operatorname{Int} C_\alpha\}$, which works by the following fact:

$(*)$ Let π_α, A, A_α, B^α, $B^\alpha_{\alpha'}$, T_α and $T^\alpha_{\alpha'}$ be given as above (i.e., $\{\pi_\alpha\}$ is a semialgebraic C^2 ball decomposition of X, (1) is satisfied and π_α are compatible). Let $\{\pi_{\widetilde\alpha}\colon B^{\widetilde\alpha} \to X\colon \widetilde\alpha \in \widetilde A\}$ be a second semialgebraic C^2 ball decomposition of X but without tube systems as yet. Define $\widetilde A_{\widetilde\alpha}$, $B^{\widetilde\alpha}_{\widetilde\alpha'}$ and $C_{\widetilde\alpha}$ in the same way. Let $m \in \mathbb{Z}$. Assume

$$A \cap \widetilde A = \{\alpha \in A\colon d_\alpha > m\} = \{\widetilde\alpha \in \widetilde A\colon d_{\widetilde\alpha} > m\},$$
$$B^\alpha = B^{\widetilde\alpha} \quad \text{for } \alpha = \widetilde\alpha \in A \cap \widetilde A,$$

the maps $\pi_\alpha\colon B^\alpha \to C_\alpha$ and $\pi_{\widetilde\alpha}\colon B^{\widetilde\alpha} \to C_{\widetilde\alpha}$ coincide for $\alpha = \widetilde\alpha \in A \cap \widetilde A$, and $\{\operatorname{Int} C_{\widetilde\alpha}\colon \widetilde\alpha \in \widetilde A,\ d_{\widetilde\alpha} \le m\}$ is compatible with $\{\operatorname{Int} C_\alpha\colon \alpha \in A,\ d_\alpha \le m\}$. Then fixing $\pi_{\widetilde\alpha}$ for $\widetilde\alpha \in \widetilde A$ with $d_{\widetilde\alpha} > m$ and $\operatorname{Im} \pi_{\widetilde\alpha}$, $d_{\widetilde\alpha} \le m$, and modifying only $\pi_{\widetilde\alpha}$, $d_{\widetilde\alpha} \le m$, we have controlled semialgebraic C^2 tube systems $\{T_{\widetilde\alpha}\colon \widetilde\alpha \in \widetilde A\}$ for $\{\operatorname{Int} C_{\widetilde\alpha}\colon \widetilde\alpha \in \widetilde A\}$ and $\{T^{\widetilde\alpha}_{\widetilde\alpha'}\colon \widetilde\alpha' \in \widetilde A_{\widetilde\alpha}\}$ for $\{\operatorname{Int} B^{\widetilde\alpha}_{\widetilde\alpha'}\colon \widetilde\alpha' \in \widetilde A_{\widetilde\alpha}\}$ for each $\widetilde\alpha \in \widetilde A$ with the same properties as $\{T_\alpha\}$ and $\{T^{\widetilde\alpha}_{\widetilde\alpha'}\}$—by (1) and compatibility of $\pi_{\widetilde\alpha}$—such that

$$(2)\qquad T_{\widetilde\alpha} = \begin{cases} T_\alpha & \text{for } \widetilde\alpha = \alpha \in A \cap \widetilde A, \\ T_\alpha|_{|T_{\widetilde\alpha}|} & \text{for } \alpha \in A,\ \widetilde\alpha \in \widetilde A \text{ with } C_{\widetilde\alpha} \subset C_\alpha,\ d_\alpha = d_{\widetilde\alpha} = m, \end{cases}$$

and for each $\widetilde{\alpha}=\alpha\in A\cap\widetilde{A}$,

$$(2)' \qquad T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}=\begin{cases} T^{\alpha}_{\alpha'} & \text{for } \widetilde{\alpha}=\alpha\in A\cap\widetilde{A},\ \widetilde{\alpha}'=\alpha'\in A_\alpha\cap\widetilde{A}_{\widetilde{\alpha}},\\ T^{\alpha}_{\alpha'}|_{|T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|} & \text{for } \widetilde{\alpha}=\alpha\in A\cap\widetilde{A},\ \alpha'\in A_\alpha,\ \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\\ & \text{with } B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\subset B^{\alpha}_{\alpha'},\ d_{\alpha'}=d_{\widetilde{\alpha}'}=m.\end{cases}$$

Proof of $(*)$. It suffices to find a controlled semialgebraic C^2 tube system $\{T_{\widetilde{\alpha}}\colon \widetilde{\alpha}\in\widetilde{A}\}$ for $\{\operatorname{Int}C_{\widetilde{\alpha}}\colon \widetilde{\alpha}\in\widetilde{A}\}$ such that (2) holds and

$$(3) \qquad \tau_{\widetilde{\alpha}}=\tau_{\widetilde{\alpha}}\circ\tau_\alpha,\qquad \varrho_{\widetilde{\alpha}}=\varrho_\alpha+\varrho_{\widetilde{\alpha}}\circ\tau_\alpha \quad \text{on } |T_{\widetilde{\alpha}}|$$
$$\text{for } \alpha\in A,\ \widetilde{\alpha}\in\widetilde{A} \text{ with } \operatorname{Int}C_{\widetilde{\alpha}}\subset\operatorname{Int}C_\alpha,\ d_{\widetilde{\alpha}}<m.$$

Indeed, assume there exists such $\{T_{\widetilde{\alpha}}\}$. Using it, we need to construct $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ for each $\widetilde{\alpha}\in\widetilde{A}$ as required in $(*)$.

For $\widetilde{\alpha}\in A\cap\widetilde{A}$ and $\widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}$, define $T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ by

$$(4) \qquad |T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|=\pi^{-1}_{\widetilde{\alpha}}(|T_{\widetilde{\alpha}'}|),\qquad \tau^{\widetilde{\alpha}}_{\widetilde{\alpha}'}=\pi^{-1}_{\widetilde{\alpha}}\circ\tau_{\widetilde{\alpha}'}\circ\pi_{\widetilde{\alpha}},\qquad \varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'}=\varrho_{\widetilde{\alpha}'}\circ\pi_{\widetilde{\alpha}}.$$

Then for each $\widetilde{\alpha}\in A\cap\widetilde{A}$, $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ is a controlled semialgebraic C^2 tube system for $\{\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$. Indeed, $|T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|$ is a semialgebraic neighborhood of $\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ in $B^{\widetilde{\alpha}}$; $\tau^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon|T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|\to\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ is a semialgebraic retraction and of class C^2 because $\pi^{-1}_{\widetilde{\alpha}}|_{\operatorname{Int}C_{\widetilde{\alpha}'}}$ is of class C^2 and because

$$\tau_{\widetilde{\alpha}'}\circ\pi_{\widetilde{\alpha}}\overset{(3)}{=}\tau_{\widetilde{\alpha}'}\circ\tau_{\alpha'}\circ\pi_\alpha\overset{\text{compatibility of }\pi_\alpha}{=}\tau_{\widetilde{\alpha}'}\circ\pi_\alpha\circ\tau^{\alpha}_{\alpha'}$$

where $\alpha\in A$ and $\alpha'\in A_\alpha$ with $\alpha=\widetilde{\alpha}$ $\operatorname{Int}C_{\widetilde{\alpha}'}\subset\operatorname{Int}C_{\alpha'}$, and because $\pi_\alpha|_{\operatorname{Int}B^{\alpha}_{\alpha'}}\colon\operatorname{Int}B^{\alpha}_{\alpha'}\to\operatorname{Int}C_{\alpha'}$ is a C^2 diffeomorphism; $\varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ is a nonnegative semialgebraic C^0 function on $|T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|$; $(\varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'})^{-1}(0)=\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$; $\varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ is of class C^2 and each $x\in\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}$ is a unique and nondegenerate critical point of $\varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'}|_{(\tau^{\widetilde{\alpha}}_{\widetilde{\alpha}'})^{-1}(x)}$ because

$$\begin{aligned}\varrho^{\widetilde{\alpha}}_{\widetilde{\alpha}'}&=\varrho_{\widetilde{\alpha}'}\circ\pi_{\widetilde{\alpha}}\overset{(3)}{=}\varrho_{\alpha'}\circ\pi_{\widetilde{\alpha}}+\varrho_{\widetilde{\alpha}'}\circ\tau_{\alpha'}\circ\pi_{\widetilde{\alpha}}\\ &=\varrho_{\alpha'}\circ\pi_\alpha+(\varrho_{\widetilde{\alpha}'}|_{\operatorname{Int}C_{\alpha'}})\circ\tau_{\alpha'}\circ\pi_\alpha\\ &\overset{\text{compatibility of }\pi_\alpha}{=}\varrho^{\alpha}_{\alpha'}+(\varrho_{\widetilde{\alpha}'}|_{\operatorname{Int}C_{\alpha'}})\circ\pi_\alpha\circ\tau^{\alpha}_{\alpha'}.\end{aligned}$$

Moreover, by (4), $\pi_{\widetilde{\alpha}}$ is compatible with $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ and $\{T_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ for each $\widetilde{\alpha}\in A\cap\widetilde{A}$, and clearly $(2)'$ is satisfied by (2) since π_α is compatible with $\{T^{\alpha}_{\alpha'}\colon \alpha'\in A_\alpha\}$ and $\{T_{\alpha'}\colon \alpha'\in A_\alpha\}$.

For $\widetilde{\alpha}\in\widetilde{A}-A$, i.e., $d_{\widetilde{\alpha}}\le m$, let $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ be an arbitrary controlled semialgebraic C^2 tube system for $\{\operatorname{Int}B^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ with (1). Then by the Lemma we can modify $\pi_{\widetilde{\alpha}}$ and assume it is compatible with $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$ and $\{T_{\widetilde{\alpha}'}\colon \widetilde{\alpha}'\in\widetilde{A}_{\widetilde{\alpha}}\}$.

Thus $\{T^{\widetilde{\alpha}}_{\widetilde{\alpha}'}\}$ together with $\{T_{\widetilde{\alpha}}\}$ fulfills the requirements in $(*)$, and hence we will construct only $\{T_{\widetilde{\alpha}}\}$ with (2) and (3).

Order the elements of $\widetilde{A}$ as $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots$ so that $d_{\widetilde{\alpha}_k} \le d_{\widetilde{\alpha}_{k+1}}$ for any k. Let $m' \in \mathbb{Z}$. Assume by induction we have a controlled semialgebraic C^2 tube system $\{T_{\widetilde{\alpha}_1}, \ldots, T_{\widetilde{\alpha}_{m'-1}}\}$ for $\{\operatorname{Int} C_{\widetilde{\alpha}_1}, \ldots, \operatorname{Int} C_{\widetilde{\alpha}_{m'-1}}\}$ with (2) and (3). Let $\alpha_k \in A$ be such that $\operatorname{Int} C_{\widetilde{\alpha}_k} \subset \operatorname{Int} C_{\alpha_k}$ for each k. Then we only need to find a semialgebraic C^2 tube $T_{\widetilde{\alpha}_{m'}}$ about $\operatorname{Int} C_{\widetilde{\alpha}_{m'}}$ such that (2) and (3) hold for $\alpha = \alpha_{m'}$ and $\widetilde{\alpha} = \widetilde{\alpha}_{m'}$, and

$$(5) \qquad \begin{aligned} \tau_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}} &= \tau_{\widetilde{\alpha}_k} \circ \tau_{\alpha_{m'}}, \\ \varrho_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}} &= \varrho_{\widetilde{\alpha}_k} \circ \tau_{\alpha_{m'}} \qquad \text{for } k < m' \end{aligned}$$

because $\{T_{\widetilde{\alpha}_{m'}}, T_{\widetilde{\alpha}_k}\}$ is controlled for each $k < m'$ as follows:

$$\begin{aligned} \tau_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} &\overset{(3)}{=} \tau_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}} \overset{(5)}{=} \tau_{\widetilde{\alpha}_k} \circ \tau_{\alpha_{m'}} \overset{(3)}{=} \tau_{\widetilde{\alpha}_k} \circ \tau_{\alpha_k} \circ \tau_{\alpha_{m'}} \\ &\overset{\text{controlledness of } \{T_\alpha\}}{=} \tau_{\widetilde{\alpha}_k} \circ \tau_{\alpha_k} \overset{(3)}{=} \tau_{\widetilde{\alpha}_k}, \\ \varrho_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} &\overset{(3)}{=} \varrho_{\widetilde{\alpha}_k} \circ \tau_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}} \overset{(5)}{=} \varrho_{\widetilde{\alpha}_k} \circ \tau_{\alpha_{m'}} \overset{(3)}{=} \varrho_{\alpha_k} \circ \tau_{\alpha_{m'}} + \varrho_{\widetilde{\alpha}_k} \circ \tau_{\alpha_k} \circ \tau_{\alpha_{m'}} \\ &\overset{\text{controlledness of } \{T_\alpha\}}{=} \varrho_{\alpha_k} + \varrho_{\widetilde{\alpha}_k} \circ \tau_{\alpha_k} \overset{(3)}{=} \varrho_{\widetilde{\alpha}_k}. \end{aligned}$$

The construction of $T_{\widetilde{\alpha}_{m'}}$ is easy. If $d_{\widetilde{\alpha}_{m'}} \ge m$, $T_{\widetilde{\alpha}_{m'}}$ is defined by (2). Then (5) is clearly satisfied. Assume $d_{\widetilde{\alpha}_{m'}} < m$. Let $\{T'_{\widetilde{\alpha}_{m'}} = (|T'_{\widetilde{\alpha}_{m'}}|, \tau'_{\widetilde{\alpha}_{m'}}, \varrho'_{\widetilde{\alpha}_{m'}}),$ $T_{\widetilde{\alpha}_k}\colon k < m'\}$ be a controlled semialgebraic C^2 tube system for $\{C_{\widetilde{\alpha}_k}\colon k \le m'\}$, whose existence is shown in the usual construction of a controlled tube system (see, e.g., [G-al]). (Here we shrink $|T_{\widetilde{\alpha}_k}|$, $k < m'$, if necessary.) Set

$$|T_{\widetilde{\alpha}_{m'}}| = |T_{\alpha_{m'}}| \cap |T'_{\widetilde{\alpha}_{m'}}|, \qquad \tau_{\widetilde{\alpha}_{m'}} = \tau'_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}}, \qquad \varrho_{\widetilde{\alpha}_{m'}} = \varrho_{\alpha_{m'}} + \varrho'_{\widetilde{\alpha}_{m'}} \circ \tau_{\alpha_{m'}}.$$

Then (3) and (5) are satisfied. Thus $(*)$ holds.

Let $\{\pi_\alpha\colon \alpha \in A\}$ be again a semialgebraic C^2 ball decomposition of X with controlled semialgebraic C^2 tube systems $\{T_\alpha\colon \alpha \in A\}$ and $\{T^\alpha_{\alpha'}\colon$ $\alpha' \in A_\alpha\}$ for each $\alpha \in A$ such that (1) holds and π_α is compatible with $\{T^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ and $\{T_{\alpha'}\colon \alpha' \in A_\alpha\}$. We show good properties of such $\{\pi_\alpha\}$, $\{T_\alpha\}$ and $\{T^\alpha_{\alpha'}\}$. Set

$$\begin{gathered} 0 *_\alpha Y = \pi_\alpha(0 * \pi_\alpha^{-1}(Y)) \qquad \text{for } Y \subset \partial C_\alpha, \\ \mathcal{J} = \{(\alpha_1, \ldots, \alpha_l) \in A^l\colon l \in \mathbb{N},\ \alpha_1 \in A_{\alpha_2}, \ldots, \alpha_{l-1} \in A_{\alpha_l},\ \alpha_1 \ne \cdots \ne \alpha_l\}, \\ U_J = \operatorname{Int} 0 *_{\alpha_l} (\cdots *_{\alpha_3} (0 *_{\alpha_2} C_{\alpha_1}) \cdots) \qquad \text{for } J = (\alpha_1, \ldots, \alpha_l) \in \mathcal{J},\ l > 1, \\ U_{x,J} = \operatorname{Int} 0 *_{\alpha_l} (\cdots *_{\alpha_3} (0 *_{\alpha_2} x) \cdots) \qquad \text{for same } J \text{ and } x \in \operatorname{Int} C_{\alpha_1}, \\ U_\alpha = \operatorname{Int} C_\alpha, \quad U_{x,\alpha} = x \qquad \text{for } \alpha \in A \text{ and } x \in \operatorname{Int} C_\alpha, \end{gathered}$$

where $0 *_{\alpha_l} (\cdots *_{\alpha_3} (0 *_{\alpha_2} C_{\alpha_1}) \cdots)$ and $0 *_{\alpha_l} (\cdots *_{\alpha_3} (0 *_{\alpha_2} x) \cdots)$ are semialgebraically homeomorphic to balls of dimension $d_{\alpha_1} + l - 1$ and $l - 1$, respectively, and Int stands for open balls. Note that U_J and $U_{x,J}$ are C^2 manifolds. For $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$, let $p_J\colon U_J \to \operatorname{Int} C_{\alpha_1}$ denote the semialgebraic C^2 submersion such that $x \in U_{p_J(x),J}$. We also define $U_J^\alpha \subset B^\alpha$, $U_{x,J}^\alpha \subset B^\alpha$ and $p_J^\alpha\colon U_J^\alpha \to \operatorname{Int} B_{\alpha_1}^\alpha$ to be $\pi_\alpha^{-1}(U_J)$, $\pi_\alpha^{-1}(U_{\pi_\alpha(x),J})$ and $\pi_\alpha^{-1} \circ p_J \circ \pi_\alpha$, respectively, for $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$, $\alpha \in A$ and $x \in \operatorname{Int} B_{\alpha_1}^\alpha$ with $\alpha_l \in A_\alpha$. Note that U_J, p_J, U_J^α and p_J^α depend only on $\{\pi_\alpha\}$ but not on $\{T_\alpha\}$, $\{T_{\alpha'}^\alpha\}$.

Then by (1) and compatibility of $\{\pi_\alpha\colon \alpha \in A\}$ we have

$$(6) \qquad p_J = \tau_{\alpha_1} \quad \text{on } U_J \cap |T_{\alpha_1}|, \qquad p_J^\alpha = \tau_{\alpha_1}^\alpha \quad \text{on } U_J^\alpha \cap |T_{\alpha_1}^\alpha|.$$

Moreover, as shown below, we can suppose the following condition (7) is satisfied, which will allow a "derived" subdivision of $\{\operatorname{Int} C_\alpha\}$ also to be a Whitney stratification.

(7) $\{U_J, \operatorname{Int} C_{\alpha_1}\}$ (resp. $\{U_J^\alpha, \operatorname{Int} B_{\alpha_1}^\alpha\}$) satisfies the Whitney condition.

Note that (7) depends on C_{α_1} but not on π_{α_1}, hence fixing C_{α_1} we can change π_{α_1} arbitrarily when (7) is satisfied and when we need to keep the property, and that (6) and (7) imply that:

(8) $\{p_J^{-1}(M_1), M_2\}$ (resp. $(\{p_J^\alpha)^{-1}(M_1), M_2\})$ is a Whitney stratification for a Whitney C^2 stratification $\{M_1, M_2\}$ in $\operatorname{Int} C_{\alpha_1}$ (resp. $\operatorname{Int} B_{\alpha_1}^\alpha$) or for a C^2 submanifold $M_1 = M_2$ of $\operatorname{Int} C_{\alpha_1}$ (resp. $\operatorname{Int} B_{\alpha_1}^\alpha$).

We modify $\{\pi_\alpha\}$ so that (7) is satisfied by downward induction. Let $0 \le m \le n$ be an integer. Assume we have a semialgebraic C^2 ball decomposition $\{\pi_\alpha\colon \alpha \in A\}$ of X and controlled semialgebraic C^2 tube systems $\{T_\alpha\colon \alpha \in A\}$ for $\{C_\alpha\colon \alpha \in A\}$ and $\{T_{\alpha'}^\alpha\colon \alpha' \in A_\alpha\}$ for $\{B_{\alpha'}^\alpha\colon \alpha' \in A_\alpha\}$ for each $\alpha \in A$ such that (1) is satisfied for any $\alpha \in A$ and $\alpha' \in A_\alpha$, any π_α is compatible with $\{T_{\alpha'}^\alpha\colon \alpha' \in A_\alpha\}$ and $\{T_{\alpha'}\colon \alpha' \in A_\alpha\}$, and (7) is satisfied for any $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$ with $l > 1$ and $d_{\alpha_1} > m$. Then we need to find a semialgebraic C^2 ball decomposition $\{\pi_{\widetilde{\alpha}}\colon \widetilde{\alpha} \in \widetilde{A}\}$ and controlled semialgebraic C^2 tube systems $\{T_{\widetilde{\alpha}}\colon \widetilde{\alpha} \in \widetilde{A}\}$ for $\{C_{\widetilde{\alpha}}\colon \widetilde{\alpha} \in \widetilde{A}\}$ and $\{T_{\widetilde{\alpha}'}^{\widetilde{\alpha}}\colon \widetilde{\alpha}' \in \widetilde{A}_{\widetilde{\alpha}}\}$ for $\{B_{\widetilde{\alpha}'}^{\widetilde{\alpha}}\colon \widetilde{\alpha}' \in \widetilde{A}_{\widetilde{\alpha}}\}$ with the same properties as $\{\pi_\alpha\}$, $\{T_\alpha\}$ and $\{T_{\alpha'}^\alpha\}$ and, moreover, such that (7) is satisfied also for $\widetilde{J} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_l) \in \widetilde{\mathcal{J}}$ with $l > 1$ and $d_{\widetilde{\alpha}_1} = m$, where $C_{\widetilde{\alpha}}$, $\widetilde{A}_{\widetilde{\alpha}}$, $B_{\widetilde{\alpha}'}^{\widetilde{\alpha}}$ and $\widetilde{\mathcal{J}}$ are defined as before. We further require, as in $(*)$,

$$A \cap \widetilde{A} = \{\alpha \in A\colon d_\alpha > m\} = \{\widetilde{\alpha} \in \widetilde{A}\colon d_{\widetilde{\alpha}} > m\},$$
$$B^\alpha = B^{\widetilde{\alpha}} \quad \text{ for } \alpha = \widetilde{\alpha} \in A \cap \widetilde{A},$$

$\pi_\alpha\colon B^\alpha \to C_\alpha$ and $\pi_{\widetilde{\alpha}}\colon B^{\widetilde{\alpha}} \to C_{\widetilde{\alpha}}$ coincide for $\alpha = \widetilde{\alpha} \in A \cap \widetilde{A}$, and $\{\operatorname{Int} C_{\widetilde{\alpha}}\colon \widetilde{\alpha} \in \widetilde{A},\ d_{\widetilde{\alpha}} \le m\}$ is compatible with $\{\operatorname{Int} C_\alpha\colon \alpha \in A,\ d_\alpha \le m\}$. The construction of such $\{\pi_{\widetilde{\alpha}}\}$, $\{T_{\widetilde{\alpha}}\}$ and $\{T_{\widetilde{\alpha}'}^{\widetilde{\alpha}}\}$ is clear by $(*)$ and the following well known fact:

FACT. *Fix* $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$ *and* $\alpha \in A$ *with* $l > 1$, $\alpha_l \in A_\alpha$ *and* $d_{\alpha_1} = m$. *The subset of* $\operatorname{Int} C_{\alpha_1}$ (*resp.* $\operatorname{Int} B^\alpha_{\alpha_1}$) *consisting of points where* $\{U_J, \operatorname{Int} C_{\alpha_1}\}$ (*resp.* $\{U^\alpha_J, \operatorname{Int} B^\alpha_{\alpha_1}\}$) *does not satisfy the Whitney condition is semialgebraic and of dimension smaller than* m.

Next we will C^ω smooth $\pi_\alpha|_{\operatorname{Int} B^\alpha}$. We assume X is the unit ball in $\mathbb{R}^n$ for simplicity of notation and without loss of generality. Preparatory to smoothing, set

$$\varepsilon *_\alpha Y = \pi_\alpha\{t\pi_\alpha^{-1}(x)\colon 1-\varepsilon \le t < 1,\, x \in Y\}$$

for $\varepsilon \in \,]0,1[$, $\alpha \in A$, $Y \subset \partial C_\alpha$,

$$U_J(\varepsilon) = \{\varepsilon_l *_{\alpha_l} (\cdots *_{\alpha_3} (\varepsilon_2 *_{\alpha_2} \operatorname{Int} C_{\alpha_1})\cdots)\colon \\ \varepsilon_2, \dots, \varepsilon_l \in \,]0,1[,\, \varepsilon_2 + \dots + \varepsilon_l = \varepsilon\}$$

for $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J},\ l > 1$,

$$U_\alpha(\varepsilon) = \operatorname{Int} C_\alpha,$$

$$W_{\alpha_1}(\varepsilon) = \bigcup_{J=(\alpha_1,\dots,\alpha_l)\in\mathcal{J}} U_J(\varepsilon) \quad \text{for each } \alpha_1 \in A,$$

and define naturally $U^\alpha_J(\varepsilon)$ and $W^\alpha_{\alpha_1}(\varepsilon)$ in B^α. Then for each $\alpha \in A$ and $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$ with $\alpha_l \in A_\alpha$, we have $U_J(\varepsilon) \subset U_J(\varepsilon')$ (resp. $U^\alpha_J(\varepsilon) \subset U^\alpha_J(\varepsilon')$) for $0 < \varepsilon < \varepsilon' < 1$, $\bigcup_{\varepsilon\in]0,1[} U_J(\varepsilon) = U_J$ (resp. $\bigcup_{\varepsilon\in]0,1[} U^\alpha_J(\varepsilon) = U^\alpha_J$), $W_{\alpha_1}(\varepsilon)$ (resp. $W^\alpha_{\alpha_1}(\varepsilon)$) is a neighborhood of $\operatorname{Int} C_{\alpha_1}$ in X (resp. $\operatorname{Int} B^\alpha_{\alpha_1}$ in B^α) called the ε-*neighborhood* of $\operatorname{Int} C_{\alpha_1}$ (resp. $\operatorname{Int} B^\alpha_{\alpha_1}$), and the map $q_{\alpha_1}\colon W_{\alpha_1}(\varepsilon) \to \operatorname{Int} C_{\alpha_1}$ (resp. $q^\alpha_{\alpha_1}\colon W^\alpha_{\alpha_1}(\varepsilon) \to \operatorname{Int} B^\alpha_{\alpha_1}$) defined to be p_J on $U_J(\varepsilon)$ (resp. p^α_J on $U^\alpha_J(\varepsilon)$) is proper. (If we define $W_{\alpha_1}(\varepsilon)$ and q_{α_1} with U_J in place of $U_J(\varepsilon)$, then q_{α_1} is not proper for $l > 1$. This is the reason why we apply $U_J(\varepsilon)$ and not U_J.)

For smoothing we drop compatibility of π_α and weaken (6) as follows:

(6)′ For $\alpha \in A$ and $\alpha_1 \in A_\alpha$, $q_{\alpha_1}\colon W_{\alpha_1}(\varepsilon) \to \operatorname{Int} C_{\alpha_1}$ (resp. $q^\alpha_{\alpha_1}\colon W^\alpha_{\alpha_1}(\varepsilon) \to \operatorname{Int} B^\alpha_{\alpha_1}$) is a submersive C^2 retraction for some ε.

Namely, we are in the situation that $\{\pi_\alpha\colon \alpha \in A\}$ is a semialgebraic C^2 ball decomposition of X, $\{T_\alpha\colon \alpha \in A\}$ and $\{T^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ for each $\alpha \in A$ are controlled semialgebraic C^2 tube systems for $\{C_\alpha\colon \alpha \in A\}$ and $\{B^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$, respectively, and (1), (6)′ and (7) are satisfied, where the definitions of U_J etc. are not changed. We saw (8) under the conditions (6) and (7). But we can replace (6) by (6)′ there. Hence (8) is now also satisfied.

Next we can assume $\pi_\alpha|_{\operatorname{Int} B^\alpha}$ are of class C^ω as follows. First choose $\{\pi_\alpha\}$ so that $\operatorname{Int} C_\alpha$ are C^ω manifolds, which is possible by the above arguments. Let $m \in \mathbb{Z}$. Inductively suppose $\pi|_{\operatorname{Int} B^\alpha}$ are of class C^ω for $d_\alpha > m$, and let $\alpha_0 \in A$ with $d_{\alpha_0} = m$. By the approximation theorem of [S1], $\pi_{\alpha_0}|_{\operatorname{Int} B^{\alpha_0}}$ can be approximated by a semialgebraic C^ω map $\widehat{\pi}_{\alpha_0}\colon \operatorname{Int} B^{\alpha_0} \to \operatorname{Int} C_{\alpha_0}$ in the

C^2 topology. Then $\widehat{\pi}_{\alpha_0}$ is a diffeomorphism (see [S$_1$]). Extend it to ∂B^{α_0} by setting $\widehat{\pi}_{\alpha_0} = \pi_{\alpha_0}$ there. Then $\widehat{\pi}_{\alpha_0}\colon B^{\alpha_0} \to C_{\alpha_0}$ is a homeomorphism by the definition of the C^0 topology, and $\{\pi_\alpha\colon \alpha \in A, \alpha \neq \alpha_0\} \cup \{\widehat{\pi}_{\alpha_0}\}$, $\{T_\alpha\colon \alpha \in A\}$ and $\{T^\alpha_{\alpha'}\colon \alpha' \in A_\alpha\}$ still satisfy (1), (6)$'$ and (7) as shown below.

(1) is clear because we do not change $\{T^\alpha_{\alpha'}\}$; (6)$'$ is also trivial if $C_{\alpha_1} \not\subset \partial C_{\alpha_0}$ (resp. $B^\alpha_{\alpha_1} \not\subset \partial B^\alpha_{\alpha_0}$).

Assume $C_{\alpha_1} \subset \partial C_{\alpha_0}$ in (6)$'$, and that (6)$'$ holds for ε. Let $\widehat{p}_J$, $\widehat{q}_\alpha$ and $\widehat{W}_\alpha(\varepsilon)$ be defined by $\{\pi_\alpha\colon \alpha \in A, \alpha \neq \alpha_0\} \cup \{\widehat{\pi}_{\alpha_0}\}$ in the same way as p_J, q_α and $W_\alpha(\varepsilon)$, and let $\widehat{\varepsilon} \in]0,1[$ be so close to 0 that $\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \subset W_{\alpha_1}(\varepsilon)$. Then

$$\{x \in \widehat{W}_{\alpha_1}(\widehat{\varepsilon})\colon q_{\alpha_1}(x) \neq \widehat{q}_{\alpha_1}(x)\} \subset W_{\alpha_0}(\widehat{\varepsilon})\ (= \widehat{W}_{\alpha_0}(\widehat{\varepsilon})).$$

Hence it suffices to see that the map

$$(q_{\alpha_1} - \widehat{q}_{\alpha_1})|_{W_{\alpha_0}(\widehat{\varepsilon}) \cap \widehat{W}_{\alpha_1}(\widehat{\varepsilon})}\colon W_{\alpha_0}(\widehat{\varepsilon}) \cap \widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \to \mathbb{R}^n$$

is close to the zero map in the C^2 topology. By the definitions of q_{α_1} and $\widehat{q}_{\alpha_1}$ we have

$$q_{\alpha_1} \circ q_{\alpha_0} = q_{\alpha_1},\ \widehat{q}_{\alpha_1} \circ \widehat{q}_{\alpha_0} = \widehat{q}_{\alpha_1} \quad \text{on } W_{\alpha_0}(\widehat{\varepsilon}) \cap \widehat{W}_{\alpha_1}(\widehat{\varepsilon}), \qquad \widehat{q}_{\alpha_0} = q_{\alpha_0}.$$

Hence

$$(q_{\alpha_1} - \widehat{q}_{\alpha_1})|_{W_{\alpha_0}(\widehat{\varepsilon}) \cap \widehat{W}_{\alpha_1}(\widehat{\varepsilon})} = (q_{\alpha_1} - \widehat{q}_{\alpha_1})|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}} \circ q_{\alpha_0}|_{W_{\alpha_0}(\widehat{\varepsilon}) \cap \widehat{W}_{\alpha_1}(\widehat{\varepsilon})}.$$

Therefore, we only need to see that

$$(q_{\alpha_1} - \widehat{q}_{\alpha_1})|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}}\colon \widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0} \to \mathbb{R}^n$$

is close to the zero map in the C^2 topology since q_{α_0} is a proper semialgebraic C^2 retraction (for a proper semialgebraic C^2 map between semialgebraic C^2 manifolds $\varphi\colon M_1 \to M_2$, the pull back by φ: {semialgebraic C^2 functions on M_2} $\to$ {semialgebraic C^2 functions on M_1} is continuous). However, that is clear because

$$q_{\alpha_1}|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}} = \pi_{\alpha_0} \circ q^{\alpha_0}_{\alpha_1} \circ \pi^{-1}_{\alpha_0}|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}},$$
$$\widehat{q}_{\alpha_1}|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}} = \pi_{\alpha_0} \circ q^{\alpha_0}_{\alpha_1} \circ \widehat{\pi}^{-1}_{\alpha_0}|_{\widehat{W}_{\alpha_1}(\widehat{\varepsilon}) \cap \operatorname{Int} C_{\alpha_0}},$$

which holds by the definitions of q_{α_1} and $\widehat{q}_{\alpha_1}$, though π_{α_0} or $\widehat{\pi}_{\alpha_0}$ is not necessarily compatible with $\{T^\alpha_{\alpha'}\}$ and $\{T_{\alpha'}\}$.

In the same way, we see that also $q^\alpha_{\alpha_1}$ is a submersive C^2 retraction when $B^\alpha_{\alpha_1} \subset \partial B^\alpha_{\alpha_0}$. (Note that after replacing π_{α_0} by $\widehat{\pi}_{\alpha_0}$ we cannot preserve (6) nor compatibility of π_α.) By the same reason (7) is kept. Therefore, we assume $\pi_\alpha|_{\operatorname{Int} B^\alpha}$ are all of class C^ω.

Now we define a semialgebraic triangulation (K, f) of X, as required in the theorem. Let the vertices of K correspond to elements of A and be denoted by $\{\sigma_\alpha\colon \alpha \in A\}$, and let vertices $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_l}$ span a simplex of K

if and only if $(\alpha_{\gamma(1)}, \dots, \alpha_{\gamma(l)}) \in \mathcal{J}$ for some permutation γ of $\{1, \dots, l\}$. Then K is a well defined simplicial complex called the *dual complex* of $\{C_\alpha\}$ (see [R-S]). For $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$, let σ_J denote the simplex spanned by $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_l}$. Thus $K = \{\sigma_J \colon J \in \mathcal{J}\}$. Let $m \in \mathbb{Z}$ and set

$$\mathcal{J}_m = \{(\alpha_1, \dots, \alpha_l) \in \mathcal{J} \colon d_{\alpha_l} \le m\}, \qquad K_m = \{\sigma_J \colon J \in \mathcal{J}_m\}.$$

Note K_m is a subcomplex of K. By induction assume we already have a semialgebraic triangulation (K_m, f_m) of $\bigcup_{d_\alpha \le m} C_\alpha$ such that $f_m|_{\mathrm{Int}\,\sigma_J}$ is a C^ω diffeomorphism onto $U_{\pi_{\alpha_1}(0),J}$ for each $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}_m$. Then we need to extend (K_m, f_m) to a semialgebraic triangulation (K_{m+1}, f_{m+1}) of $\bigcup_{d_\alpha \le m+1} C_\alpha$. For $J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}_{m+1} - \mathcal{J}_m$, i.e., $d_{\alpha_l} = m+1$, define $f_{m+1}|_{\sigma_J} \colon \sigma_J \to \mathrm{Cl}\, U_{\pi_{\alpha_1}(0),J}$ by $f_{m+1}(\sigma_{\alpha_l}) = \pi_{\alpha_l}(0)$ and

$$\begin{aligned} &f_{m+1}(t_1\sigma_{\alpha_1} + \dots + t_l\sigma_{\alpha_l}) \\ &\quad = \pi_{\alpha_l}((1-t_l)\pi_{\alpha_l}^{-1}(f_m(t_1\sigma_{\alpha_1}/(1-t_l) + \dots + t_{l-1}\sigma_{\alpha_{l-1}}/(1-t_l)))) \\ &\qquad\qquad \text{for } t_1, \dots, t_l \in [0,\,1] \text{ with } t_1 + \dots + t_l = 1 \text{ and } t_l \ne 1. \end{aligned}$$

Then (K_{m+1}, f_{m+1}) a semialgebraic triangulation of $\bigcup_{d_\alpha \le m+1} C_\alpha$, and we obtain a semialgebraic triangulation (K, f) as required.

It remains to prove that $\{U_{\pi_{\alpha_1}(0),J} \colon J = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}\}$ is a Whitney stratification. We show that, moreover,

(9) $\{U_{\pi_{\alpha_1}(0),J}, U_{\pi_{\alpha'_1}(0),J'}\}$ (resp. $\{U^\alpha_{\pi_{\alpha_1}(0),J}, U^\alpha_{\pi_{\alpha'_1}(0),J'}\}$) satisfies the Whitney condition for $J = (\alpha_1, \dots, \alpha_l)$, $J' = (\alpha'_1, \dots, \alpha'_{l'}) \in \mathcal{J}$ and $\alpha \in A$ with $l > 1$, $U_{\pi_{\alpha_1}(0),J} \subset \mathrm{Cl}\, U_{\pi_{\alpha'_1}(0),J'} - U_{\pi_{\alpha'_1}(0),J'}$ and $\alpha_l, \alpha'_{l'} \in A_\alpha$.

Here we also argue by induction. Let $m \in \mathbb{Z}$. Assume inductively (9) is proved for $d_{\alpha'_{l'}} < m$, and let $d_{\alpha'_{l'}} = m$. There are two possibilities $\alpha_l = \alpha'_{l'}$ or $\alpha_l \ne \alpha'_{l'}$.

Case of $\alpha_l = \alpha'_{l'}$. Set $J_0 = (\alpha_1, \dots, \alpha_{l-1})$ and $J'_0 = (\alpha'_1, \dots, \alpha'_{l'-1})$. Then $U_{\pi_{\alpha_1}(0),J_0} \subset \mathrm{Cl}\, U_{\pi_{\alpha'_1}(0),J'_0} - U_{\pi_{\alpha'_1}(0),J'_0}$ and $d_{\alpha'_{l'-1}} < m$. Hence $\{U_{\pi_{\alpha_1}(0),J_0}, U_{\pi_{\alpha'_1}(0),J'_0}\}$ (resp. $\{U^\alpha_{\pi_{\alpha_1}(0),J_0}, U^\alpha_{\pi_{\alpha'_1}(0),J'_0}\}$) satisfies the Whitney condition by hypothesis when $l - 1 > 1$, and trivially when $l - 1 = 1$. In particular, so does $\{U^{\alpha_l}_{\pi_{\alpha_1}(0),J_0}, U^{\alpha_l}_{\pi_{\alpha'_1}(0),J'_0}\}$. Then $\{U^{\alpha_l}_{\pi_{\alpha_1}(0),J}, U^{\alpha_l}_{\pi_{\alpha'_1}(0),J'}\}$ is a Whitney stratification by (1). Moreover, (9) holds since $\pi_{\alpha_l}|_{\mathrm{Int}\, B^{\alpha_l}} \colon \mathrm{Int}\, B^{\alpha_l} \to \mathrm{Int}\, C_{\alpha_l}$ is a C^ω diffeomorphism and $U^{\alpha_l}_{\pi_{\alpha_1}(0),J}$, $U^{\alpha_l}_{\pi_{\alpha'_1}(0),J'} \subset \mathrm{Int}\, B^{\alpha_l}$.

Case of $\alpha_l \ne \alpha'_{l'}$. Set $J'_0 = (\alpha'_1, \dots, \alpha'_{l'-1})$ as above. Then

$$J = J'_0 \quad \text{or} \quad U_{\pi_{\alpha_1}(0),J} \subset \mathrm{Cl}\, U_{\pi_{\alpha'_1}(0),J'_0} - U_{\pi_{\alpha'_1}(0),J'_0},$$

and hence $U_{\pi_{\alpha_1}(0),J}$ is equal to $U_{\pi_{\alpha'_1}(0),J'_0}$ or $\{U_{\pi_{\alpha_1}(0),J}, U_{\pi_{\alpha'_1}(0),J'}\}$ (resp.

$\{U^{\alpha}_{\pi_{\alpha_1}(0),J}, U^{\alpha}_{\pi_{\alpha'_1}(0),J'}\})$ is a Whitney stratification. In both cases, (9) follows from (8). This completes the proof. ∎

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