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Existence of three solutions to a double eigenvalue problem for the p-biharmonic equation

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Abstract. Using a three critical points theorem and variational methods, we study the existence of at least three weak solutions of the Navier problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a sufficiently smooth boundary $\partial\Omega$, $\lambda > 0$, $\mu > 0$ and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

1. Introduction and main results. Consider the following fourth-order partial differential equation coupled with Navier boundary conditions:

$$(\mathcal{P}) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a non-empty bounded open set with a sufficiently smooth boundary $\partial\Omega$, $p > \max\{1, N/2\}$, $\lambda > 0$, $\mu > 0$ and $f, g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

We recall that a function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be L^1-*Carathéodory* if

- $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$;
- $t \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$.
- for every $\varrho > 0$ there exists a function $l_\varrho \in L^1(\Omega)$ such that

$$\sup_{|t|\leq\varrho} |f(x,t)| \leq l_\varrho(x)$$

for a.e. $x \in \Omega$.

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Here and throughout, X will denote the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\| = \Big(\int_\Omega (|\Delta u(x)|^p + |\nabla u(x)|^p)\, dx \Big)^{1/p}.$$

Let

$$K := \sup_{u \in X\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{\|u\|}. \tag{1.1}$$

Since $p > \max\{1, N/2\}$, $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact, and one has $K < \infty$. As usual, a weak solution of the problem $(\mathcal{P})$ is any $u \in X$ such that

$$\begin{aligned}\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta\xi(x)\,dx + \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\xi(x)\,dx \\ = \lambda \int_\Omega f(x,u(x))\xi(x)\,dx + \mu \int_\Omega g(x,u(x))\xi(x)\,dx\end{aligned} \tag{1.2}$$

for every $\xi \in X$.

In recent years, Ricceri's three critical points theorem has been widely used to solve differential equations (see [12, 7, 17, 5, 2, 1, 8, 9, 10, 16] and references therein).

A nonlinear fourth-order equation furnishes a model to study travelling waves in suspension bridges, so it is important in physics. Several results are known concerning the existence of multiple solutions for fourth-order boundary value problems, and we refer the reader to [4, 6, 13, 14] and the references cited therein.

The aim of this paper is to establish the existence of a non-empty open interval $\Lambda \subseteq I$ and a positive real number q with the following property: for each $\lambda \in \Lambda$ and for each L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there is $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem $(\mathcal{P})$ admits at least three weak solutions whose norms in X are less than q.

For the reader's convenience, we recall the revised form of Ricceri's three critical points theorem (Theorem 1 in [15]) which is our main tool to transfer the existence of three solutions of the problem $(\mathcal{P})$ into the existence of critical points of the Euler functional.

THEOREM 1.1 ([15, Theorem 1]). *Let X be a reflexive real Banach space. Assume that $\Phi\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Φ is bounded on each bounded subset of X; $J\colon X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose*

Gâteaux derivative is compact; and $I \subseteq \mathbb{R}$ is an interval. Assume that

$$\lim_{\|u\|\to\infty} (\Phi(u) + \lambda J(u)) = \infty$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{u\in X} (\Phi(u) + \lambda(J(u) + \rho)) < \inf_{u\in X} \sup_{\lambda\in I} (\Phi(u) + \lambda(J(u) + \rho)). \tag{1.3}$$

Then there exists an open interval $\Lambda \subseteq I$ and a positive real number q with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $\Psi\colon X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

We will need the following result, which is Proposition 1.3 in [3] with J replaced by $-J$, to show the minimax inequality (1.3) of Theorem 1.1.

Proposition 1.2 ([3, Proposition 1.3]). *Let X be a non-empty set, and $\Phi : X \to \mathbb{R}$, $J : X \to \mathbb{R}$ two real functions. Assume that $\Phi(u) \ge 0$ for every $u \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$. Further, assume that there exist $u_1 \in X$ and $r > 0$ such that*

(i) $r < \Phi(u_1)$,

(ii) $\displaystyle\sup_{\Phi(u)<r} (-J(u)) < r\frac{-J(u_1)}{\Phi(u_1)}$.

Then for every $h > 1$ and every $\rho \in \mathbb{R}$ satisfying

$$\sup_{\Phi(u)<r} (-J(u)) + \frac{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r}(-J(u))}{h} < \rho < r\frac{-J(u_1)}{\Phi(u_1)}$$

one has

$$\sup_{\lambda\in I} \inf_{u\in X} (\Phi(u) + \lambda(J(u) + \rho)) < \inf_{u\in X} \sup_{\lambda\in[0,a]} (\Phi(u) + \lambda(J(u) + \rho))$$

where

$$a = \frac{hr}{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r}(-J(u))}.$$

2. Main results. Now, fix $x^0 \in \Omega$ and pick $\gamma > 0$ such that $B(x^0, \gamma) \subset \Omega$ where $B(x^0, \gamma)$ denotes the ball with center x^0 and radius γ. Put

$$Q = \int\limits_{B(x^0,\gamma)\setminus B(x^0,\gamma/2)} \left| \frac{12}{\gamma^3}|x - x^0|l - \frac{24}{\gamma^2}l + \frac{9}{\gamma}\frac{l}{|x - x^0|} \right|^p dx,$$

$$R=\frac{\pi^{N/2}}{\Gamma(N/2)}\int\limits_{(\gamma/2)^2}^{\gamma^2}\left|\frac{12(N+1)}{\gamma^3}\sqrt{t}+\frac{9(N-1)}{\gamma}\frac{1}{\sqrt{t}}-\frac{24N}{\gamma^2}\right|^p t^{N/2-1}\,dt$$

and

$$\theta=K(R+Q)^{1/p} \tag{2.1}$$

where $l=(\sum_{i=1}^N x_i^2)^{1/2}$, $|x-x^0|=(\sum_{i=1}^N(x_i-x_i^0)^2)^{1/2}$ and $m(\Omega)$ denotes the volume of Ω. We also let $F(x,t)=\int_0^t f(x,s)\,ds$ for all $(x,t)\in\Omega\times\mathbb{R}$. Our main result is formulated as follows:

THEOREM 2.1. *Assume that there exist a positive constant r and a function $w\in X$ such that*

(H1) $\|w\|^p>pr$;

(H2) $\int\limits_\Omega \sup\limits_{s\in[-K\sqrt[p]{pr},K\sqrt[p]{pr}]} F(x,s)\,dx < pr\dfrac{\int_\Omega F(x,w(x))\,dx}{\|w\|^p}$;

(H3) $pK^p m(\Omega)\limsup\limits_{|s|\to+\infty}\dfrac{F(x,s)}{|s|^p}<\dfrac{1}{r\eta}$

for almost every $x\in\Omega$ and for some η satisfying

$$\eta>\frac{1}{pr\frac{\int_\Omega F(x,w(x))\,dx}{\|w\|^p}-\int_\Omega\sup_{s\in[-K\sqrt[p]{pr},K\sqrt[p]{pr}]}F(x,s)\,dx}.$$

Then there exist a non-empty open interval $\Lambda\subseteq[0,r\eta)$ and a positive real number q with the following property: for each $\lambda\in\Lambda$ and for an arbitrary L^1-Carathéodory function $g\colon\Omega\times\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, the problem $(\mathcal{P})$ has at least three solutions whose norms in X are less than q.

Let us first present a consequence of Theorem 2.1 for a fixed test function w.

COROLLARY 2.2. *Assume that there exist positive constants c and d with $c<\theta d$ such that*

(j) $F(x,s)\ge 0$ *for a.e.* $x\in\Omega\setminus B(x^0,\gamma/2)$ *and all* $s\in[0,d]$;

(jj) $\int\limits_\Omega\sup\limits_{(x,s)\in\Omega\times[-c,c]}F(x,s)\,dx<\left(\dfrac{c}{\theta d}\right)^p\int\limits_{B(x^0,\gamma/2)}F(x,d)\,dx$;

(jjj) $c^p m(\Omega)\limsup\limits_{|s|\to+\infty}\dfrac{F(x,s)}{|s|^p}<\dfrac{1}{\eta}$

for almost every $x\in\Omega$ and for some η satisfying

$$\eta>\frac{1}{\left(\frac{c}{\theta d}\right)^p\int_{B(x^0,\gamma/2)}F(x,d)\,dx-\int_\Omega\sup_{s\in[-c,c]}F(x,s)\,dx}.$$

Then there exist a non-empty open interval $\Lambda \subseteq [0, p^{-1}(c/K)^p\eta)$ and a positive real number q with the following property: for each $\lambda \in \Lambda$ and for an arbitrary L^1-Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem $(\mathcal{P})$ has at least three solutions whose norms in X are less than q.

REMARK 2.3. We remark that the authors in [11] had already studied the problem $(\mathcal{P})$ when $\mu = 0$. Under weaker assumptions as for Theorem 1 of [11], Corollary 2.2 ensures a more precise conclusion. In fact, our condition (jjj) is weaker than the condition (A3) in Theorem 1 of [11]. For example, if F is autonomous, let $F(s) = s^p/\ln(2+s^2)$. Clearly, F satisfies our condition (jjj) but does not satisfy (A3) in Theorem 1 of [11].

The proof of Corollary 2.2 is based on the following technical lemma.

LEMMA 2.4. *Assume that c and d are positive constants with $c < \theta d$. Under assumptions* (j) *and* (jj) *of Corollary* 2.2, *there exist $r > 0$ and $w \in X$ such that $\|w\|^p > pr$ and*

$$\int_\Omega \sup_{s\in[-K\sqrt[p]{pr},K\sqrt[p]{pr}]} F(x,s)\,dx < pr\frac{\int_\Omega F(x,w(x))\,dx}{\|w\|^p}.$$

Proof. Let

$$w(x) = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(x^0,\gamma), \\ d\big(\frac{4}{\gamma^3}|x-x^0|^3 - \frac{12}{\gamma^2}|x-x^0|^2 \\ \quad + \frac{9}{\gamma}|x-x^0| - 1\big) & \text{for } x \in B(x^0,\gamma)\setminus B(x^0,\gamma/2), \\ d & \text{for } x \in B(x^0,\gamma/2), \end{cases} \tag{2.2}$$

where $r = p^{-1}(c/K)^p$. We have

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(x^0,\gamma) \cup B(x^0,\gamma/2), \\ d\big(\frac{12}{\gamma^3}|x-x^0|(x_i-x_i^0) \\ \quad - \frac{24}{\gamma^2}(x_i-x_i^0) + \frac{9(x_i-x_i^0)}{\gamma|x-x^0|}\big) & \text{for } x \in B(x^0,\gamma)\setminus B(x^0,\gamma/2) \end{cases}$$

and

$$\frac{\partial^2 w(x)}{\partial^2 x_i} = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(x^0,\gamma) \cup B(x^0,\gamma/2), \\ d\big(\frac{12}{\gamma^3|x-x^0|}(x_i-x_i^0)^2 \\ \quad - \frac{24}{\gamma^2} + \frac{9(|x-x^0|^2-(x_i-x_i^0)^2)}{\gamma|x-x^0|^3}\big) & \text{for } x \in B(x^0,\gamma)\setminus B(x^0,\gamma/2). \end{cases}$$

It is easy to verify that $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and in particular,

$$\|w\|^p = (R+Q)d^p.$$

Consequently, from (2.1) we see that

$$\|w\| = \theta d/K.$$

Moreover, by the assumption $c < \theta d$, we get

$$\frac{\|w\|^p}{p} > \frac{1}{p}\left(\frac{d\theta}{K}\right)^p > \frac{1}{p}\left(\frac{c}{K}\right)^p = r.$$

Since, $0 \le w(x) \le d$, for each $x \in \Omega$, condition (j) ensures that

$$\int\limits_{\Omega \setminus B(x^0,\gamma)} F(x, w(x))\, dx + \int\limits_{B(x^0,\gamma)\setminus B(x^0,\gamma/2)} F(x, w(x))\, dx \ge 0.$$

Hence, from (jj), $r = \frac{1}{p}\left(\frac{c}{K}\right)^p$ and the above inequality we have

$$\int\limits_{\Omega} \sup_{s\in[-K\sqrt[p]{pr},K\sqrt[p]{pr}]} F(x,s)\, dx < \left(\frac{c}{\theta d}\right)^p \int\limits_{B(x^0,\gamma/2)} F(x,d)\, dx$$
$$\le pr\frac{\int_\Omega F(x,w(x))\, dx}{\|w\|^p}. \blacksquare$$

Proof of Corollary 2.2. From Lemma 2.4 we see that assumptions (H1) and (H2) of Theorem 2.1 are fulfilled for w given in (2.2). Also, (jjj) implies that (H3) is satisfied. Hence, the conclusion follows directly from Theorem 2.1. $\blacksquare$

Remark 2.5. The statement of Corollary 2.2 mainly depends upon the choice of the test function w in Theorem 2.1. With the choice of w given in (2.2) we have the present statement of Corollary 2.2. Other candidates for w can be considered to obtain other versions of Corollary 2.2.

We end this section by giving the following example to illustrate Corollary 2.2.

Example 2.6. Consider the problem

$$\begin{cases} u^{(iv)} - u'' = \lambda f(u) + \mu g(x,u) & \text{in }]0,2\pi[, \\ u(0) = u(2\pi) = u''(0) = u''(2\pi) = 0, \end{cases} \tag{2.3}$$

where

$$f(s) = \begin{cases} s^2, & s \le 1, \\ 1/s^2, & s > 1, \end{cases}$$

and $g : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ is a fixed L^1-Carathéodory function. Choose $p = 2$, $x^0 = \pi$ and $\gamma = \pi$. Noticing that $K = 1/2\pi$ (see Proposition 2.1 of [4]), one has

$$\theta = \frac{\sqrt{15(509\pi^2 - 720\pi^2 \ln 2 + 40)}}{5\pi^2}.$$

So, we see that all the assumptions of Corollary 2.2 are satisfied by choosing, for instance $c = 10^{-3}$ and $d = 1$. Thus, for each

$$\kappa > 20^{-6}\pi^2 \cdot \frac{1}{\frac{10^{-3}\cdot 25\pi^5}{90(509\pi^2 - 720\pi^2\ln 2 + 40)} - \frac{20^{-9}\pi}{3}}$$

there exists an open interval $\Lambda \subset [0,\kappa]$ and a positive real number q such that, for each $\lambda \in \Lambda$ and for each L^1-Carathéodory function $g : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$, there is $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem (2.3) admits at least three weak solutions whose norms in $W^{2,2}([0,2\pi]) \cap W_0^{1,2}([0,2\pi])$ are less than q.

3. Proof of Theorem 2.1. For each $u \in X$, let

$$\Phi(u) = \frac{\|u\|^p}{p}, \qquad J(u) = -\int_\Omega F(x,u(x))\,dx$$

and

$$\Psi(u) = -\int_\Omega \int_0^{u(x)} g(x,s)\,ds\,dx.$$

Under the assumptions of Theorem 2.1, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, the Gâteaux derivative of Φ admits a continuous inverse on X^*; and Ψ and J are continuously Gâteaux differentiable functionals whose Gâteaux derivatives are compact. Obviously, Φ is bounded on each bounded subset of X. In particular, for each $u,\xi \in X$,

$$\Phi'(u)(\xi) = \int_\Omega |\Delta u(x)|\Delta u(x)\Delta\xi(x)\,dx + \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\xi(x)\,dx,$$

$$J'(u)(\xi) = -\int_\Omega f(x,u(x))\xi(x)\,dx,$$

$$\Psi'(u)(\xi) = -\int_\Omega g(x,u(x))\xi(x)\,dx.$$

Hence, it follows from (1.2) that the weak solutions of the problem $(\mathcal{P})$ are exactly the solutions of the equation

$$\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0.$$

Furthermore, from (H3) there exist constants $\zeta,\tau \in \mathbb{R}$ with $0 < \zeta < 1/r\eta$ such that

$$pK^p m(\Omega)F(x,s) \le \zeta|s|^p + \tau$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Fix $u \in X$. Then

$$F(x,u(x)) \le \frac{1}{pK^p m(\Omega)}(\zeta|u(x)|^p + \tau)$$

for all $x \in \Omega$. Then, for any fixed $\lambda \in \,]0,r\eta]$, since

$$\sup_{x\in\Omega}|u(x)| \le K\|u\|,$$

we get

$$\begin{aligned}\Phi(u)+\lambda J(u) &= \frac{\|u\|^p}{p}-\lambda\int_\Omega F(x,u(x))\,dx\\ &\geq \frac{\|u\|^p}{p}-\frac{r\eta}{pK^pm(\Omega)}\Big(\zeta\int_\Omega |u(x)|^p\,dx+\tau\Big)\\ &\geq \frac{1}{p}(1-\zeta r\eta)\|u\|^p-\frac{r\eta}{pK^pm(\Omega)}\tau,\end{aligned}$$

and so

$$\lim_{\|u\|\to\infty}(\Phi(u)+\lambda J(u))=\infty.$$

We claim that there exist $r>0$ and $w\in X$ such that

$$\sup_{\Phi(u)<r}(-J(u))<r\frac{-J(w)}{\Phi(w)}.$$

Note that $\sup_{x\in\Omega}|u(x)|\leq K\|u\|$ for each $u\in X$, and so

$$\begin{aligned}\{u\in X:\Phi(u)<r\} &= \{u\in X:\|u\|^p<pr\}\\ &\subseteq \{u\in X:|u(x)|<K\sqrt[p]{pr}\text{ for all }x\in\Omega\}.\end{aligned}$$

It follows that

$$\sup_{\Phi(u)<r}(-J(u))<\int_\Omega \sup_{t\in[-K\sqrt[p]{pr},K\sqrt[p]{pr}]}F(x,t)\,dx<pr\frac{\int_\Omega F(x,w(x))\,dx}{\|w\|^p},$$

from (H2), and so

$$\sup_{\Phi(u)<r}(-J(u))<r\frac{-J(w)}{\Phi(w)}.$$

Also from (H1) we have $\Phi(w)>r$. Next recall from (H3) that

$$\eta>\frac{1}{r\frac{-J(w)}{\Phi(w)}-\sup_{\Phi(u)<r}(-J(u))},$$

so

$$\sup_{\Phi(u)<r}(-J(u))+\frac{1}{\eta}<r\frac{-J(w)}{\Phi(w)}.$$

Choose

$$\nu>\eta\Big(r\frac{-J(w)}{\Phi(w)}-\sup_{\Phi(u)<r}(-J(u))\Big)$$

and note $\nu>1$ and

$$\sup_{\Phi(u)<r}(-J(u))+\frac{r\frac{-J(w)}{\Phi(w)}-\sup_{\Phi(u)<r}(-J(u))}{\nu}<r\frac{-J(w)}{\Phi(w)}.$$

Therefore, from Proposition 2.2 (with $u_0 = 0$ and $u_1 = w$) for every $\rho \in \mathbb{R}$ satisfying

$$\sup_{\Phi(u)<r}(-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{\Phi(u)<r}(-J(u))}{\nu} < \rho < r\frac{-J(w)}{\Phi(w)}$$

we have (note $\sigma = r\eta$)

$$\sup_{\lambda\in\mathbb{R}}\inf_{u\in X}(\Phi(u) + \lambda(J(u)+\rho)) < \inf_{u\in X}\sup_{\lambda\in[0,r\eta]}(\Phi(u) + \lambda(J(u)+\rho)).$$

Now, all assumptions of Theorem 1.1 are satisfied. Hence, the conclusion follows directly from Theorem 1.1. ■

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