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Entire solutions of q-difference equations and value distribution of q-difference polynomials

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Abstract. We investigate the existence and uniqueness of entire solutions of order zero of the nonlinear q-difference equation of the form $f^n(z) + L(z) = p(z)$, where $p(z)$ is a polynomial and $L(z)$ is a linear differential-q-difference polynomial of f with small growth coefficients. We also study the zeros distribution of some special type of q-difference polynomials.

1. Introduction. Let f be a meromorphic function in the whole complex plane. We assume the reader is familiar with the standard notations and results of Nevanlinna's value distribution theory such as the proximity function $m(r,f)$, counting function $N(r,f)$, characteristic function $T(r,f)$, the first and second main theorems, the lemma on the logarithmic derivative etc. (see, e.g., [H2, L]). In this paper, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside of a set of upper logarithmic density 0, i.e., outside of a set E such that

$$\limsup_{r\to\infty} \frac{\int_{E\cap[1,r]} \frac{dt}{t}}{\log r} = 0.$$

Moreover, we denote by $\varepsilon(r,f)$ any quantity satisfying $\varepsilon(r,f) = o(T(r,f))$ for all r outside of a possible exceptional set of finite logarithmic measure. A meromorphic function α is said to be a *small function* of f if $T(r,\alpha) = S(r,f)$.

Meromorphic solutions of complex difference equations have become a subject of great interest recently, due to applications of Nevanlinna's value distribution theory to difference expressions [BIY2, CF, HK1, HK2]. In particular, Halburd and Korhonen [HK2, Theorem 3.1] gave a difference analogue of the Clunie lemma, which was developed by Laine and Yang [LY1, Theorem 2.3] as follows.

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Theorem A. *Let $f(z)$ be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$, $P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg U(z,f)$ *of* $U(z,f)$ *in* $f(z)$ *and its shifts* $f(z+c_1),\dots,f(z+c_k)$ *is n, and* $\deg Q(z,f) \le n$. *Moreover, assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon > 0$,*

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + \varepsilon(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Using Theorem A, Laine and Yang [LY2] studied some special types of nonlinear difference equations:

Theorem B. *Let p, q be polynomials. Then a nonlinear difference equation*

$$f^2(z) + q(z)f(z+1) = p(z)$$

has no transcendental entire solutions of finite order.

Theorem C. *Let $n \ge 4$ be an integer, $M(z,f)$ be a linear differential-difference polynomial of f, not vanishing identically, and h be a meromorphic function of finite order. Then the differential-difference equation*

$$f^n(z) + M(z,f) = h$$

has at most one admissible transcendental entire solution of finite order such that all coefficients of $M(z,f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

In this paper, we will study some q-difference equations and the value distribution of related q-difference polynomials.

2. q-difference equations. The non-autonomous Schröder q-difference equation

$$f(qz) = R(z, f(z)), \tag{2.1}$$

where the right-hand side is rational in both arguments, has been widely studied during the last decades ([B-M, IY1, IY2]). For the classical developments, the reader is invited to see [V]. Gundersen et al. [G-Y] considered the order of growth of meromorphic solutions of (2.1); their results imply a q-difference analogue of the classical Malmquist theorem [M]: if the q-difference equation (2.1) admits a meromorphic solution of order zero, then (2.1) reduces to a q-difference Riccati equation, i.e. $\deg_f R = 1$.

Bergweiler et al. [BIY1] treated the functional equation

$$\sum_{j=0}^{n} a_j(z) f(c^j z) = Q(z), \tag{2.2}$$

where $0 < |c| < 1$ is a complex number, and $a_j(z)$ $(j = 0, 1, \ldots, n)$ and $Q(z)$ are rational functions with $a_0(z) \not\equiv 0$, $a_1(z) \equiv 1$. They concluded that all meromorphic solutions of (2.2) satisfy $T(r, f) = O((\log r)^2)$. This means that all meromorphic solutions of (2.2) are of zero order of growth.

Barnett et al. [B-M] investigated the properties of $f(qz)$ of *order zero*. A key result, which is an analogue of the logarithmic derivative lemma, reads as follows.

LEMMA 2.1. *Let $f(z)$ be a non-constant meromorphic function of order zero, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m(r, f(qz)/f(z)) = S(r, f).$$

Using the above lemma, Zhang and Korhonen [ZK] got the relation between the characteristics of $f(z)$ and $f(qz)$:

LEMMA 2.2. *Let $f(z)$ be a non-constant meromorphic function of order zero, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = T(r, f(z)) + S(r, f). \tag{2.3}$$

In this paper, we get the following result corresponding to Theorem B:

THEOREM 2.3. *Let p, r be polynomials. Then a nonlinear difference equation*

$$f^2(z) + r(z) f(qz) = p(z) \tag{2.4}$$

has no transcendental meromorphic solutions of order zero.

Proof. Suppose that f is a transcendental meromorphic solution of order zero to equation (2.4). Without loss of generality, we assume that $r(z)$ does not vanish identically. From (2.4), we conclude by Lemma 2.2 that

$$\begin{aligned} 2T(r, f(z)) = T(r, f^2(z)) &= T(r, p(z) - r(z) f(qz)) \\ &\le T(r, f(qz)) + O(\log r) = T(r, f(z)) + S(r, f), \end{aligned}$$

which is a contradiction. ∎

If f is supposed to be an entire function, we get a similar result. To formulate it, take a linear q-difference polynomial

$$L(z) = \sum_{j=1}^{k} b_j(z) f(q_j z),$$

where at least one of the arguments q_j is non-zero, and the coefficients are small functions of f.

Theorem 2.4. *Let $p(z)$ be a polynomial and let $L(z)$ be a linear differential-q-difference polynomial of f with small growth coefficients. Then a nonlinear q-difference equation*

$$f^n(z) + L(z) = p(z) \tag{2.5}$$

has no transcendental entire solutions of order zero, where $n \geq 2$ is an integer.

If (2.5) has a transcendental entire solution f of order zero, then $T(r,f) = m(r,f)$. The proof of Theorem 2.4 is similar to that of Theorem 2.3, just using Lemma 2.1 instead of Lemma 2.2 and the classical logarithmic derivative lemma. We omit the details.

In Theorem 2.4, the right side of (2.5) is a polynomial, which is a small function with respect to a transcendental entire function f. A natural question is what happens if $p(z)$ in (2.5) is replaced by a general meromorphic function h. Concerning this, we obtain the following result.

Theorem 2.5. *Let $n \geq 4$ be an integer, $L(z,f)$ be as in Theorem* 2.4, *not vanishing identically, and h be a meromorphic function. Then the differential-difference equation*

$$f^n(z) + L(z,f) = h \tag{2.6}$$

has at most one admissible transcendental entire solution of order zero. If such a solution f exists, then the order of h is zero.

Proof. Some ideas here are from [LY2]. Suppose that f is a transcendental zero-order entire solution to (2.6). We deduce from Lemma 2.2 and (2.6) that $T(r,h) = O(T(r,f))$. Then $\rho(h) = \rho(f) = 0$.

Assume to the contrary that f and g are two distinct transcendental entire solutions of order zero to (2.6). We get

$$f^n(z) + L(z,f) = g^n(z) + L(z,g).$$

Then

$$f^n(z) - g^n(z) = L(z,g) - L(z,f) = L(z,g-f).$$

Denote $F = \frac{f^n - g^n}{f-g}$. Obviously, F is an entire function. From the above equation, we have

$$F = \prod_{j=1}^{n-1}(f - \omega_j g) = -\frac{L(z,f-g)}{f-g}, \tag{2.7}$$

where $\omega_j \neq 1$ $(j = 1,\dots,n-1)$ are distinct roots of unity. From Lemma 2.1 and the logarithmic derivative lemma, we obtain

$$\begin{aligned} T(r,F) = m(r,F) &= m\left(r, \frac{L(z,f-g)}{f-g}\right) \\ &= S(r,f-g) \leq S(r,f) + S(r,g) =: S(r). \end{aligned}$$

An immediate observation now is that

$$\sum_{j=1}^{n-1} N\left(r, \frac{1}{f-\omega_j g}\right) = N(r, 1/F) = S(r),$$

and therefore

$$N\left(r, \frac{1}{f-\omega_j g}\right) = S(r) \quad \text{for all } j = 1, \dots, n-1.$$

Since $\frac{1}{f/g-\omega_j} = \frac{g}{f-\omega_j g}$, we conclude that $N\big(r, \frac{1}{f/g-\omega_j}\big) = S(r)$ $(j=1,\dots,n-1)$. Denote $\phi = f/g$. As $n \geq 4$, the second main theorem implies that $T(r,\phi) = S(r)$. Then

$$T(r,f) = T(r,g) + S(r). \tag{2.8}$$

Again by (2.7), we deduce that

$$F = \prod_{j=1}^{n-1}(f-\omega_j g) = g^{n-1}\prod_{j=1}^{n-1}(\phi-\omega_j).$$

From this and (2.8), provided that $\phi \neq \omega_j$ $(j = 1, \dots, n-1)$, we get

$$\begin{aligned}(n-1)T(r,f) &= (n-1)T(r,g) + S(r)\\ &\leq T(r,F) + T\left(r, \frac{1}{\prod_{j=1}^{n-1}(\phi-\omega_j)}\right) + S(r) = S(r),\end{aligned}$$

which means $T(r,f) + T(r,g) = S(r,f) + S(r,g)$, a contradiction. Hence, $\phi = \omega_j$ for some $j = 1,\dots,n-1$. But then $f = \omega_j g$, and $f^n = g^n$, $L(z,f) = L(z,g)$. On the other hand, since L is linear, we have $L(z,f) = \omega_j L(z,g)$. This is a contradiction since $\omega_j \neq 1$. ∎

3. Value distribution of difference polynomials. Let f be a transcendental entire function and n be a positive integer. Hayman [H1] and Clunie [C] proved that $f^n f'$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often. From Section 1, it is easy to find that $f(z+c)$ plays a parallel role in the value distribution theory of difference polynomials to f' in the theory of differential polynomials. Noting this, Laine–Yang [LY2] and Bergweiler–Langley [BL] investigated the distribution of zeros of $f^n f(z+c)$ and $f(z+c) - f(z)$ respectively.

Recently, Liu and Qi [LQ] studied the value distribution of q-differences and obtained the following results.

THEOREM D. *Let f be a transcendental meromorphic function of order zero with finitely many poles, $q \in \mathbb{C}\setminus\{0\}$, and let $R(z)$ be a rational function. Then $f^n(z)f(qz) - R(z)$ has infinitely many zeros for $n \in \mathbb{N}$, and $f^n(z) + f(qz) - f(z) - R(z)$ has infinitely many zeros for $n \geq 2$.*

Theorem E. *If we remove the condition 'f has finitely many poles' in Theorem D, then* $f^n(z)f(qz) - R(z)$ *has infinitely many zeros for* $n \geq 6$, *and* $f^n(z) + f(qz) - f(z) - R(z)$ *has infinitely many zeros for* $n \geq 8$, *where* $R(z)$ *is a nonzero rational function.*

It is natural to ask what happens if f has infinitely many poles and the rational function $R(z)$ is replaced by an arbitrary small function in Theorem D. Concerning this, we get the following theorem.

Theorem 3.1. *Let* f *be a transcendental meromorphic function of order zero with* $N(r,f) = S(r,f)$, $q \in \mathbb{C} \setminus \{0\}$, *and let* $\alpha(z)$ $(\not\equiv 0)$ *be a small function of* f. *Then* $f^n(z)f(qz) - \alpha(z)$ *has infinitely many zeros for* $n \geq 2$, *and* $f^n(z) + f(qz) - f(z) - \alpha(z)$ *has infinitely many zeros for* $n \geq 3$.

Proof. Denote

$$F(z) = -\frac{f(qz) - f(z) - \alpha(z)}{f^n(z)}. \tag{3.1}$$

Then $f^n(z) + f(qz) - f(z) - \alpha(z)$ has infinitely many zeros if $F(z) - 1$ does. We next show that the latter is indeed the case.

Noting that $N(r,f) = S(r,f)$, we deduce from Lemma 2.1 and (3.1) that

$$\begin{aligned} nT(r,f) &= m(r,f^n) + S(r,f) = m\left(r, f^n\frac{f(qz) - f - \alpha}{f(qz) - f - \alpha}\right) + S(r,f) \\ &\leq m(r, 1/F) + m(r, f(qz) - f) + S(r,f) \\ &\leq T(r,F) - N(r,1/F) + m(r,f) + m\left(r, \frac{f(qz)}{f(z)} - 1\right) + S(r,f) \\ &\leq T(r,F) - N(r,1/F) + m(r,f) + S(r,f). \end{aligned} \tag{3.2}$$

Applying the second main theorem to $F(z)$, we get

$$\begin{aligned} T(r,F) &= \overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}(r,1/(F-1)) + S(r,f) \\ &\leq \overline{N}(r,f(qz)) + \overline{N}(r,1/f) + \overline{N}(r,1/F) + \overline{N}(r,1/(F-1)) + S(r,f) \\ &\leq T(r,f) + \overline{N}(r,1/F) + \overline{N}(r,1/(F-1)) + S(r,f) \end{aligned}$$

Combining the last inequality with (3.2) yields

$$nT(r,f) \leq T(r,f) + m(r,f) + \overline{N}(r,1/(F-1)) + S(r,f),$$

which is

$$(n-2)T(r,f) \leq \overline{N}(r,1/(F-1)) + S(r,f),$$

and the assertion follows since $n \geq 3$.

Denote $G(z) = f^n(z)f(qz)$. By the same arguments as above, we have

$$\begin{aligned} (n+1)T(r,f) &= m(r,f^{n+1}) + S(r,f) \\ &\leq m(r,f^{n+1}/G) + m(r,G) + S(r,f) \leq T(r,G) + S(r,f). \end{aligned} \tag{3.3}$$

Applying the second main theorem for the small functions to $G(z)$, we get

$$\begin{aligned} T(r,G) &= \overline{N}(r,G) + \overline{N}(r,1/G) + \overline{N}(r,1/(G-\alpha)) + S(r,f) \\ &\le \overline{N}(r,1/G) + \overline{N}(r,1/(G-\alpha)) + S(r,f) \\ &\le 2T(r,f) + \overline{N}(r,1/(G-\alpha)) + S(r,f). \end{aligned}$$

Combining the last inequality with (3.3) gives

$$(n-1)T(r,f) \le \overline{N}(r,1/(G-\alpha)) + S(r,f).$$

Then $f^n(z)f(qz) - \alpha(z)$ has infinitely many zeros since $n \ge 2$. ■

REMARK 3.2. Using the proof Theorem 3.1 again, we deduce that Theorem E still holds if the rational function $R(z)$ is replaced by a small function $\alpha(z)$ of $f(z)$.

4. Concluding remarks. The original Theorems D and E in [LQ] concern the zeros of $f^n(z)f(qz+\eta) - R(z)$ and $f^n(z) + f(qz+\eta) - f(z) - R(z)$, where $\eta \in \mathbb{C}$. In the present paper, we use $f(qz)$ instead of $f(qz+\eta)$ for brevity. In fact, all results in Section 3 remain true for $f(qz+\eta)$, by

THEOREM 4.1. *Let $f(z)$ be a non-constant meromorphic function of order zero, $c \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz+c)) = T(r, f(z)) + S(r,f).$$

Theorem 4.1 comes immediately from Lemma 2.1 and the following theorem:

THEOREM F ([CF, Theorem 2.1]). *Let f be a meromorphic function of finite order ρ and c is a non-zero complex constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

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