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Existence and nonexistence of solutions for a singular elliptic problem with a nonlinear boundary condition

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Abstract. We consider the existence and nonexistence of solutions for the following singular quasi-linear elliptic problem with concave and convex nonlinearities:

$$\begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) + h(x)|u|^{p-2}u = g(x)|u|^{r-2}u, & x \in \Omega, \\ |x|^{-ap}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda f(x)|u|^{q-2}u, & x \in \partial\Omega, \end{cases}$$

where Ω is an exterior domain in $\mathbb{R}^N$, that is, $\Omega = \mathbb{R}^N \setminus D$, where D is a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂D $(= \partial\Omega)$, and $0 \in \Omega$. Here $\lambda > 0$, $0 \le a < (N-p)/p$, $1 < p < N$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. By the variational method, we prove the existence of multiple solutions. By the test function method, we give a sufficient condition under which the problem has no nontrivial nonnegative solutions.

1. Introduction and main results. In this paper, we consider the existence of infinitely many solutions and the nonexistence of solutions for the quasi-linear elliptic problem

$$(1.1)\qquad \begin{cases} -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) + h(x)|u|^{p-2}u = g(x)|u|^{r-2}u, & x \in \Omega, \\ |x|^{-ap}|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda f(x)|u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where Ω is an exterior domain in $\mathbb{R}^N$, that is, $\Omega = \mathbb{R}^N \setminus D$, where D is a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂D $(= \partial\Omega)$, and $0 \in \Omega$. Here $\lambda > 0$, $0 \le a < (N-p)/p$, $1 < p < N$, and $0 \le a < (N-p)/p$, and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. Problem (1.1) arises in many diverse contexts like differential geometry (e.g., the scalar curvature problem and the Yamabe problem) [K], non-Newtonian fluid mechanics [D], glaciology [PR], mathematical biology [AW], and elsewhere.

In recent years, multiplicity of solutions for elliptic equations with the p-Laplacian operator has been widely studied (see [CCD, AlCM, WT, KM,

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AB, BR, Am, P, XCH]). When $a = 0$, Averna et al. [AB] considered the Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = \lambda f(x,u), & x \in \Omega, \\ \partial u/\partial \nu = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

on a bounded domain Ω. By a critical points theorem, they proved that problem (1.2) has at least three solutions for each λ in a certain open interval. Pflüger [P] considered the following p-Laplacian equation with a nonlinear boundary condition:

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u), & x \in \Omega, \\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + b(x)|u|^{p-2} = g(x,u), & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain, $0 < a_0 < a(x) \in L^\infty(\Omega)$ and $c/(1+x)^{p-1} \le b(x) \le C/(1+x)^{p-1}$ for some $c, C > 0$. By the variational approach, he proved the existence of three solutions.

Many authors focus on the existence of infinitely many solutions (see [FIV, Aou, AsCM, Y]). For $a = g(x) = 0, h(x) = -1$, Bonder and Rossi [BR] considered a similar problem and obtained infinitely many solutions in the subcritical case via variational and topological arguments. For $a = g(x) = 0, h(x) = 1$, Faraci et al. [FIV] studied a more general p-Laplacian equation and proved the existence of infinitely many bounded solutions.

However, to the best of our knowledge, little seems to be known about the existence of infinitely many solutions for problem (1.1) on an unbounded domain Ω with $a \neq 0$. Motivated by [AB, P, BR, Aou, FIV], we consider the existence of infinitely many solutions of (1.1) by the variational method. We give two sufficient conditions under which the problem (1.1) has infinitely many solutions. Since $\Omega \subset \mathbb{R}^N$ is an unbounded domain, the loss of compactness of the Sobolev embedding renders the variational technique more delicate.

For the nonexistence of solutions for elliptic equations with p-Laplacian we refer to [CG, AP, PS, YL]. In the present paper, we will also consider the nonexistence for problem (1.1). Our method is based on the test function method, introduced by Mitidieri and Pohozaev [MP2]. We give a sufficient condition for problem (1.1) to have no nontrivial nonnegative solutions.

In Sections 2 and 3, we use the following assumptions:

- (A_1) $0 < a < pN/(N-p)$, $a \le b < a+1$, $d = a+1-b$, $p^* = pN/(N-pd)$, $\lambda > 0$;
- (A_2) $h(x) \ge 0$, $g \in L^\infty(\Omega) \cap L^\mu(\Omega,\omega)$ with $\omega(x) = |x|^{br\mu}$, $\mu = p^*/(p^*-r)$, $g^\pm(x) = \max\{\pm g(x), 0\} \not\equiv 0$;
- (A_3) $f \in L^\infty(\partial\Omega)$ and $f^+(x) = \max\{f(x), 0\} \not\equiv 0$ for $x \in \partial\Omega$.

We now introduce some weighted spaces. When $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}(\Omega, |x|^{-ap})$ to be the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{W_a^{1,p}} = \Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)^{1/p}.$$

The natural function space to study problem (1.1) is the completion X of the space of restrictions to Ω of $C_0^\infty(\mathbb{R}^N)$ functions with the norm

$$\|u\|_X = \Big(\int_\Omega (|x|^{-ap}|\nabla u|^p + h(x)|u|^p)\,dx\Big)^{1/p}. \tag{1.4}$$

For $\alpha \in \mathbb{R}$ and $r \ge 1$, let $L^r(\Omega, |x|^{-\alpha})$ be the set of Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\|u\|_{r,\alpha} = \|u\|_{L^r(\Omega,|x|^{-\alpha})} = \Big(\int_\Omega |x|^{-\alpha}|u|^r\,dx\Big)^{1/r} < \infty.$$

The following weighted Sobolev–Hardy inequality is called the *Caffarelli–Kohn–Nirenberg inequality* [CKN]. There is a constant $C_{a,b} > 0$ such that

$$\Big(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\,dx\Big)^{1/p^*} \le C_{a,b}\Big(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx\Big)^{1/p} \tag{1.5}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, where $-\infty < a < (N-p)/p$, $a \le b < a+1$, $d = a+1-b$, and $p^* = pN/(N-pd)$.

As a version of (1.5), for an exterior domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, one has

$$\Big(\int_\Omega |x|^{-bp^*}|u|^{p^*}\,dx\Big)^{1/p^*} \le S_0\Big(\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\Big)^{1/p} \tag{1.6}$$

with some $S_0 > 0$ (see [B-U, GR]).

Definition 1.1. A function $u \in X$ is said to be a *weak solution* of problem (1.1) if for any $\psi \in X$,

$$\int_\Omega (|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\psi + h|u|^{p-2}u\psi)\,dx - \int_\Omega g|u|^{r-2}u\psi\,dx - \lambda\int_{\partial\Omega} f|u|^{q-2}u\psi\,d\sigma = 0. \tag{1.7}$$

Our main results are listed below.

Theorem 1.2. *Assume* (A_1)–(A_3). *If* $p < r < q < p_* = p(N-1)/(N-p)$, *then problem* (1.1) *has infinitely many solutions* u_k *in* X *and*

$$J_\lambda(u_k) \to \infty \quad \text{as } k \to \infty.$$

Theorem 1.3. *Assume* (A_1)–(A_3). *If* $q < p$ *and* $r < p$, *then problem* (1.1) *has infinitely many solutions* u_k *in* X *such that* $J_\lambda(u_k) < 0$ *and*

$$J_\lambda(u_k) \to 0 \quad \text{as } k \to \infty.$$

In Section 4, we assume that

(A_4) $h(x) \le 0$, $f(x) \ge 0$, $g(x) \ge g_0 > 0$;

(A_5) $\dfrac{Nr}{(a+1)(r+1)+N} < p < r$, $\lambda > 0$.

Theorem 1.4. *Assume* (A_4)–(A_5). *Then problem* (1.1) *has no nontrivial nonnegative solutions.*

This paper is organized as follows. In Section 2, we give some basic definitions and lemmas. In Section 3, we consider the existence of multiple solutions for problem (1.1), and prove that (1.1) has infinitely many solutions. By the test function method, in Section 4, we prove that problem (1.1) has no nontrivial nonnegative weak solutions under appropriate conditions.

2. Preliminaries. In this section, we give some basic definitions and prove several important lemmas.

It is clear that problem (1.1) has a variational structure. Let $J_\lambda : X \to \mathbb{R}^1$ be the corresponding Euler functional, defined by

$$J_\lambda(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{r}\int_\Omega g(x)|u|^r\,dx - \frac{1}{q}\int_{\partial\Omega} \lambda f(x)|u|^q\,d\sigma. \tag{2.1}$$

We see that $J_\lambda \in C^1(X, \mathbb{R}^1)$ and for all $\psi \in X$,

$$\begin{aligned}\langle J_\lambda'(u), \psi\rangle = \int_\Omega (|x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla\psi + h(x)|u|^{p-2}u\psi)\,dx \\ - \int_\Omega g(x)|u|^{r-2}u\psi\,dx - \int_{\partial\Omega} \lambda f(x)|u|^{q-2}u\psi\,d\sigma.\end{aligned} \tag{2.2}$$

In particular, it follows from (2.2) that

$$\langle J_\lambda'(u), u\rangle = \|u\|_X^p - \int_\Omega g(x)|u|^r\,dx - \int_{\partial\Omega} \lambda f(x)|u|^q\,d\sigma, \tag{2.3}$$

where $\|u\|_X$ is defined in (1.4). It is well known that the weak solutions of problem (1.1) are precisely the critical points of $J_\lambda(u)$. Thus, to prove the existence of weak solutions for problem (1.1), it is sufficient to show that $J_\lambda(u)$ admits a sequence of critical points.

The following embedding theorem is an extension of the classical Rellich–Kondrashov compactness theorem (see [X]).

Lemma 2.1. *Assume* $\Omega \subset \mathbb{R}^N$ *is an open bounded domain with* C^1 *boundary and* $0 \in \Omega$, $N \ge 3$, $-\infty < a < pN/(N-p)$. *Then the embed-*

ding $W^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^r(\Omega,|x|^{-\alpha})$ *is continuous if* $1<r\le Np/(N-p)$ *and* $0\le\alpha\le(1+a)r+N(1-r/p)$, *and is compact if* $1\le r<Np/(N-p)$ *and* $0\le\alpha<(1+a)r+N(1-r/p)$.

Now, we give a compact embedding theorem.

LEMMA 2.2. *Assume* $1<r<p^*$. *Then the embedding* $X\hookrightarrow L^r(\Omega,g)$ *is compact.*

Proof. Let $u\in X$. By (1.6) and the Hölder inequality we have

$$\begin{aligned}\|u\|^r_{L^r(\Omega,g)} &= \int_\Omega g|u|^r\,dx \le \Big(\int_\Omega |u|^{p^*}|x|^{-bp^*}\,dx\Big)^{r/p^*}\Big(\int_\Omega \omega g^\mu\,dx\Big)^{1/\mu}\\ &\le S_0^r\|u\|_X^r\|g\|_{L^\mu(\Omega,\omega)}\end{aligned} \tag{2.4}$$

where $\omega(x)=|x|^{br\mu}$ and $\mu=p^*/(p^*-r)$. Inequality (2.4) implies that the embedding $X\hookrightarrow L^r(\Omega,g)$ is continuous. In the following we prove that the embedding is compact.

Recall that $\Omega=\mathbb{R}^N\setminus D$, where D is a bounded domain in $\mathbb{R}^N$. We can choose $R>0$ so large that $D\subset B_R=B_R(0)$. Then $\Omega_R=\mathbb{R}^N\setminus B_R=\Omega\setminus B_R\subset\Omega$. For $\mathcal{O}\subset\Omega$, we define

$$X(\mathcal{O})=\{u|_{\mathcal{O}}:u\in X\},\qquad Y(\mathcal{O})=\{u|_{\mathcal{O}}:u\in Y\}, \tag{2.5}$$

where $Y=L^r(\Omega,g)$. We divide our proof into two steps.

(i) *The embedding* $X(B_R\setminus D)\hookrightarrow Y(B_R\setminus D)$ *is compact.*

Assume $\{u_n\}$ is a bounded sequence in $X(B_R\setminus D)$. Letting $\alpha=0$ in Lemma 2.1, we see that there exist $u\in Y(B_R\setminus D)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n-u\|_{L^r(B_R\setminus D)}\to 0$ as $n\to\infty$. Since $g\in L^\infty(\Omega)$, there exists $M>0$ such that $|g|<M$ a.e. in Ω. Thus,

$$\int_{B_R\setminus D} g(x)|u_n-u|^r\,dx \le M\int_{B_R\setminus D}|u_n-u|^r\,dx, \tag{2.6}$$

which implies that $u_n\to u$ in $Y(B_R\setminus D)=L^r(B_R\setminus D,g)$.

(ii) *If* $\{u_n\}$ *is a bounded sequence in* X, *then for any* $\varepsilon>0$, *there exists* $R_\varepsilon>0$ *large enough such that* $\|u_n\|_{Y(\Omega_{R_\varepsilon})}<\varepsilon$, $n=1,2,\ldots$.

We claim that

$$\lim_{R\to\infty}\sup_{u\in X\setminus\{0\}}\frac{\|u\|_{Y(\Omega_R)}}{\|u\|_X}=0. \tag{2.7}$$

In fact, it follows from (2.4) that

$$\|u\|^r_{L^r(\Omega_R,g)}=\int_{\Omega_R} g|u|^r\,dx\le S_0^r\|u\|_X^r\|g\|_{L^\mu(\Omega_R,\omega)}. \tag{2.8}$$

Since $g \in L^\mu(\Omega,\omega)$ one has

$$\lim_{R\to\infty} \|g\|_{L^\mu(\Omega_R,\omega)} = 0. \tag{2.9}$$

Thus, we deduce from (2.8)–(2.9) that

$$\frac{\|u\|_{Y(\Omega_R)}}{\|u\|_X} \le S_0 \|g\|^{1/r}_{L^\mu(\Omega,\omega)} \to 0 \quad \text{as } R \to \infty, \tag{2.10}$$

which implies that (2.7) holds.

Since X is a reflexive Banach space and $\{u_n\}$ is bounded in X, there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ in X and $\|u_n\|_X < C_0$ for some constant $C_0 > 0$. Thus, we deduce from (2.7) that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ large enough such that

$$\|u_n\|_{Y(\Omega_{R_\varepsilon})} \le \varepsilon \quad \text{for } n = 1, 2, \ldots. \tag{2.11}$$

Since the embedding $X(B_{R_\varepsilon} \setminus D) \hookrightarrow Y(B_{R_\varepsilon} \setminus D)$ is compact by (i), there exists $N_1 > 0$ such that for $n > N_1$,

$$\|u_n - u\|_{Y(B_{R_\varepsilon}\setminus D)} < \varepsilon. \tag{2.12}$$

Consequently, (2.11)–(2.12) yield

$$\|u_n - u\|_Y \le \|u_n\|_{Y(\Omega\setminus B_{R_\varepsilon})} + \|u\|_{Y(\Omega\setminus B_{R_\varepsilon})} + \|u_n - u\|_{Y(B_{R_\varepsilon}\setminus D)} \le 3\varepsilon, \tag{2.13}$$

which implies that $\{u_n\}$ is convergent in Y. Therefore, the embedding $X \hookrightarrow L^r(\Omega, g)$ is compact. ∎

Lemma 2.3. *Assume* (A_1)–(A_3). *If* $p < r < q < p_* = p(N-1)/(N-p)$, *then* $J_\lambda(u)$ *satisfies the* $(PS)_c$ *condition in* X *for any* $c > 0$.

Proof. Let $c > 0$ and let $\{u_n\}$ be a (PS) sequence such that

$$J_\lambda(u_n) \to c, \quad J'_\lambda(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{2.14}$$

Then we can deduce from (2.14) that

$$\begin{aligned} c + \|u_n\|_X + 1 &\ge J_\lambda(u_n) - \frac{1}{r}\langle J'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_X^p + \left(\frac{1}{r} - \frac{1}{q}\right)\lambda\|u_n\|^q_{L^q(\partial\Omega,f)} \\ &\ge \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_X^p, \end{aligned} \tag{2.15}$$

which implies that $\{u_n\}$ is bounded in X. Furthermore, since X is a reflexive Banach space, there exists $u \in X$ such that $u_n \rightharpoonup u$.

In view of (2.2), a direct computation implies that

$$\begin{aligned}(2.16)\qquad &\int_\Omega |x|^{-ap}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\cdot\nabla(u_n-u_m)\,dx\\ &\quad+\int_\Omega h(x)(|u_n|^{p-2}u_n-|u_m|^{p-2}u_m)(u_n-u_m)\,dx\\ &=\int_{\partial\Omega} f(x)(|u_n|^{q-2}u_n-|u_m|^{q-2}u_m)(u_n-u_m)\,d\sigma\\ &\quad+\int_\Omega g(x)(|u_n|^{r-2}u_n-|u_m|^{r-2}u_m)(u_n-u_m)\,dx\\ &\quad+\langle J_\lambda'(u_n)-J_\lambda'(u_m),u_n-u_m\rangle.\end{aligned}$$

By the inequalities

$$(2.17)\qquad \langle|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta,\xi-\zeta\rangle\ge\begin{cases}c|\xi-\zeta|^p & \text{for } p\ge 2,\\ c|\xi-\zeta|^2(|\xi|+|\zeta|)^{p-2} & \text{for } 1<p<2,\end{cases}$$

which are a modification of the inequalities in [D], we get

$$(2.18)\qquad A_{mn}\ge\begin{cases}c_1\int_\Omega |x|^{-ap}|\nabla(u_n-u_m)|^p\,dx & \text{for } p\ge 2,\\ c_1\Big(\int_\Omega |x|^{-ap}|\nabla(u_n-u_m)|^p\,dx\Big)^{2/p} & \text{for } 1<p<2,\end{cases}$$

with some constant $c_1>0$, independent of n and m, and

$$(2.19)\qquad A_{mn}\triangleq\int_\Omega |x|^{-ap}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m)\cdot\nabla(u_n-u_m)\,dx.$$

In the following, we will prove that $\{u_n\}$ has a subsequence that converges to u strongly in X. We only give the proof for $1<p<2$, as the argument for $p\ge 2$ is similar but simpler. In fact, when $1<p<2$, it follows from (2.17) that

$$\begin{aligned}(2.20)\qquad &\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m,\nabla(u_n-u_m)\rangle\\ &\qquad\ge c|\nabla(u_n-u_m)|^2(|\nabla u_n|+|\nabla u_m|)^{p-2}.\end{aligned}$$

Hence

$$\begin{aligned}(2.21)\qquad |\nabla(u_n-u_m)|^p &\le c\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m,\nabla(u_n-u_m)\rangle^{p/2}\\ &\quad\times(|\nabla u_n|+|\nabla u_m|)^{p(2-p)/2}.\end{aligned}$$

Multiply (2.21) by $|x|^{-ap}$, integrate, and use the Hölder inequality to obtain

$$\begin{aligned}\int_\Omega |x|^{-ap}|\nabla(u_n-u_m)|^p\,dx &\le c\int_\Omega |x|^{-ap}\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m,\nabla(u_n-u_m)\rangle^{p/2}\\ &\quad\times(|\nabla u_n|+|\nabla u_m|)^{p(2-p)/2}\,dx\\ &\le c\Big(\int_\Omega |x|^{-ap}\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m,\nabla(u_n-u_m)\rangle\,dx\Big)^{p/2}\\ &\quad\times\Big(\int_\Omega |x|^{-ap}(|\nabla u_n|+|\nabla u_m|)^p\,dx\Big)^{(2-p)/2},\end{aligned}$$

which implies that there exists some constant $c_1>0$ such that

$$\int_\Omega |x|^{-ap}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m)\cdot\nabla(u_n-u_m)\,dx \ge c_1\Big(\int_\Omega |x|^{-ap}|\nabla(u_n-u_m)|^p\,dx\Big)^{2/p}.$$

Similar to the proof of (2.18), we have

$$\int_\Omega h(x)(|u_n|^{p-2}u_n-|u_m|^{p-2}u_m)(u_n-u_m)\,dx \ge \begin{cases} c_2\int_\Omega h(x)|u_n-u_m|^p\,dx & \text{for } p\ge 2,\\ c_2\Big(\int_\Omega h(x)|u_n-u_m|^p\,dx\Big)^{2/p} & \text{for } 1<p<2,\end{cases} \tag{2.22}$$

for some constant $c_2>0$, independent of n and m.

Since $0\notin\partial\Omega$, the compact trace embedding $X\hookrightarrow L^q(\partial\Omega,f)$ $(q<p_*)$ [F] and Hölder's inequality yield

$$\int_{\partial\Omega} f(x)(|u_n|^{q-2}u_n-|u_m|^{q-2}u_m)(u_n-u_m)\,d\sigma\to 0 \quad \text{as } n,m\to\infty. \tag{2.23}$$

Lemma 2.2 and the Hölder inequality imply that

$$\int_\Omega g(x)(|u_n|^{r-2}u_n-|u_m|^{r-2}u_m)(u_n-u_m)\,dx\to 0 \quad \text{as } n,m\to\infty. \tag{2.24}$$

It follows from (2.14) that

$$\langle J'_\lambda(u_n)-J'_\lambda(u_m),u_n-u_m\rangle \le (\|J'_\lambda(u_n)\|_{X^*}+\|J'_\lambda(u_m)\|_{X^*})\|u_n-u_m\|_X\to 0 \tag{2.25}$$

as $n, m \to \infty$. Therefore, it follows from (2.16), (2.18) and (2.22)–(2.25) that

$$\|u_n - u_m\|_X \to 0 \quad \text{as } n, m \to \infty, \tag{2.26}$$

so $\{u_n\}$ is a Cauchy sequence in X. Thus, there exists $u \in X$ such that $u_n \to u$ in X. ∎

Now, we introduce the Fountain Theorem, which will be used to prove multiplicity results for problem (1.1).

Let X be a reflexive and separable Banach space. It is well known that there exist $e_j \in X$ and $e_j^* \in X^*$ ($j = 1, 2, \ldots$) such that

- $\langle e_i, e_j^* \rangle = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$;
- $X = \overline{\operatorname{span}\{e_1, e_2, \ldots\}}$, $X^* = \overline{\operatorname{span}\{e_1^*, e_2^*, \ldots\}}$.

We write

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \qquad j, k = 1, 2, \ldots. \tag{2.27}$$

Lemma 2.4 (Fountain Theorem, [B]). *Assume $J_\lambda \in C^1(X, \mathbb{R})$, $J_\lambda(u) = J_\lambda(-u)$. Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$ such that*

- (A_1) $a_k = \inf_{u \in Z_k,\, \|u\|_X = \gamma_k} J_\lambda(u) \to \infty$ *as* $k \to \infty$,
- (A_2) $b_k = \sup_{u \in Y_k,\, \|u\|_X = \rho_k} J_\lambda(u) \le 0$,
- (A_3) $J_\lambda(u)$ *satisfies the* $(\mathrm{PS})_c$ *condition for every* $c > 0$.

Then J_λ has a sequence $\{u_k\}$ of critical points such that $J_\lambda(u_k) \to \infty$ as $k \to \infty$.

3. Existence of solutions. In this section, we prove the existence of multiple solutions for problem (1.1). The argument is based on the Fountain Theorem of Lemma 2.4.

Proof of Theorem 1.1. Our purpose is to verify the assumptions (A_1)–(A_3) in Lemma 2.4. Let

$$\beta_k = \sup_{u \in Z_k,\, u \neq 0} \frac{\|u\|_{L^r(\Omega, g)}}{\|u\|_X} = \sup_{u \in Z_k,\, \|u\|_X = 1} \|u\|_{L^r(\Omega, g)}, \tag{3.1}$$

$$\sigma_k = \sup_{u \in Z_k,\, u \neq 0} \frac{\|u\|_{L^q(\partial\Omega, f)}}{\|u\|_X} = \sup_{u \in Z_k,\, \|u\|_X = 1} \|u\|_{L^q(\partial\Omega, f)}. \tag{3.2}$$

Then

$$\|u\|_{L^r(\Omega, g)} \le \beta_k \|u\|_X, \qquad \|u\|_{L^q(\partial\Omega, f)} \le \sigma_k, \qquad \forall u \in Z_k. \tag{3.3}$$

Furthermore, we claim that

$$\beta_k \to 0, \qquad \sigma_k \to 0 \qquad \text{as } k \to \infty.$$

In fact, it is easy to check that $0 < \beta_{k+1} \le \beta_k$. Hence there exists $\beta_0 \ge 0$ such that $\beta_k \to \beta_0$ as $k \to \infty$. In the following, we prove that $\beta_0 = 0$.

Indeed, the definition of β_k means that there exists $u_k \in Z_k$ with $\|u_k\|_X = 1$ such that $-1/k \le \beta_k - \|u_k\|_{L^r(\Omega,g)} \le 1/k$ for all $k \ge 1$. Then there exists a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$ in X, and $\lim_{k\to\infty}\langle u_k, e_j^*\rangle = 0$ for all $j \ge 1$. Thus, $u = 0$ and $u_k \rightharpoonup 0$. It follows from Lemma 2.2 that $u_k \to 0$ in $L^r(\Omega, g)$ as $k \to \infty$, so $\beta_0 = 0$.

Similarly, we find that $\sigma_k \to 0$ as $k \to \infty$.

It follows from (2.1) and (3.3) that

(3.4)
$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|_X^p - \frac{\lambda}{q}\|u\|_{L^q(\partial\Omega,f)}^q - \frac{1}{r}\|u\|_{L^r(\Omega,g)}^r \\ &\ge \frac{1}{p}\|u\|_X^p - \frac{\lambda}{q}\sigma_k^q\|u\|_X^q - \frac{1}{r}\beta_k^r\|u\|_X^r \\ &= \frac{1}{2p}\|u\|_X^p + \left(\frac{1}{4p}\|u\|_X^p - \frac{\lambda}{q}\sigma_k^q\|u\|_X^q\right) + \left(\frac{1}{4p}\|u\|_X^p - \frac{1}{r}\beta_k^r\|u\|_X^r\right). \end{aligned}$$

Let

$$\gamma_k = \min\left\{\left(\frac{q}{4p\lambda\sigma_k^q}\right)^{1/(q-p)}, \left(\frac{r}{4p\beta_k^r}\right)^{1/(r-p)}\right\}. \tag{3.5}$$

Since $\beta_k \to 0$ and $\sigma_k \to 0$ as $k \to \infty$, we have

$$\gamma_k \to \infty \quad \text{as } k \to \infty. \tag{3.6}$$

Therefore, taking $\|u\|_X = \gamma_k$, we deduce from (3.4) and (3.6) that

$$J_\lambda(u) \ge \frac{1}{2p}\|u\|_X^p = \frac{1}{2p}\gamma_k^p \to \infty \quad \text{as } k \to \infty, \tag{3.7}$$

which implies that (A_1) holds.

Since all norms are equivalent on the finite-dimensional space Y_k, we easily infer from the assumption $p < r < q$ that (A_2) holds for large $\rho_k > 0$ with $\|u\|_X = \rho_k$. It is obvious that (A_3) holds by Lemma 2.3. Thus, the proof of Theorem 1.1 is complete. ∎

To prove Theorem 1.2, we introduce the following lemma (see [W]).

LEMMA 3.1. *Let $J_\lambda \in C^1(X, \mathbb{R})$, where X is a Banach space. Assume J_λ satisfies the* $(\mathrm{PS})_c$ *condition for every $c > 0$, $J_\lambda(-u) = J_\lambda(u)$, $J_\lambda(0) = 0$, and J_λ is bounded from below on X. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace Y_k and $\rho_k > 0$ such that*

$$\sup_{u\in Y_k,\, \|u\|_X=\rho_k} J_\lambda(u) < 0,$$

then $J_\lambda(u)$ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.

Proof of Theorem 1.2. The trace embedding $X \hookrightarrow L^q(\partial\Omega, f)$ and Lemma 2.2 mean that there exist constants $c_3, c_4 > 0$ such that

$$J_\lambda(u) = \frac{1}{p}\|u\|_X^p - \frac{\lambda}{q}\|u\|_{L^q(\partial\Omega,f)}^q - \frac{1}{r}\|u\|_{L^r(\Omega,g)}^r \geq \frac{1}{p}\|u\|_X^p - \frac{\lambda}{q}c_3\|u\|_X^q - \frac{1}{r}c_4\|u\|_X^r. \tag{3.8}$$

Let

$$h(t) = \frac{1}{p}t^p - \frac{\lambda}{q}c_3 t^q - \frac{1}{r}c_4 t^r, \quad t \geq 0.$$

It is not difficult to check that

$$h(t) \to 0 \text{ as } t \to 0^+ \quad \text{and} \quad h(t) \to \infty \text{ as } t \to \infty.$$

This together with the assumption $q < p$, $r < p$ implies that $h(t)$ attains its global minimum at some point $t_0 > 0$, and $h(t_0) < 0$. Thus, J_λ is bounded below. Similar to the proof of Lemma 2.3, we can prove that J_λ satisfies the $(\mathrm{PS})_c$ condition for every $c > 0$.

Let Y_k be defined as in (2.27). Since $r < p$ and $q < p$, and all norms on the finite-dimensional space Y_k are equivalent, we can choose ρ_k small enough such that

$$\sup_{u\in Y_k,\, \|u\|_X=\rho_k} J_\lambda(u) < 0.$$

Thus, Lemma 3.1 shows that problem (1.1) has a sequence of solutions u_k. Moreover, $J_\lambda(u_k) \to 0$ as $k \to \infty$. ■

4. Nonexistence of solutions. In this section, we prove the nonexistence of nontrivial nonnegative solutions for problem (1.1) by the test function method of Mitidieri and Pohozaev [MP2]. The approach is essentially based on a priori estimates by a careful choice of test functions without using comparison or maximum principle arguments.

For other references on this method, see [OT, LT, MP1] and the references therein.

Proof of Theorem 1.3. Define

$$\varphi_0(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq 1, \\ (n-k)^{-1}(n(2-s)^k - k(2-s)^n) & \text{for } 1 \leq s \leq 2, \\ 0 & \text{for } s \geq 2, \end{cases} \tag{4.1}$$

where $n > k > 2$. From (4.1) we deduce by a direct computation that

$$0 \leq \varphi_0(s) \leq 1, \quad 0 \leq |\varphi_0'(s)| \leq \beta_0\varphi_0^{1-1/k}(s), \quad \beta_0 = k\left(\frac{n}{n-k}\right)^{1/k}. \tag{4.2}$$

Let $\varphi(x) = \varphi_0(|x|/R)$. Take $\delta < 0$ with $|\delta|$ small enough, multiply (1.1) by $u^\delta \varphi$ and integrate to obtain

$$\begin{aligned}(4.3)\qquad \int_\Omega g(x)u^{r+\delta-1}\varphi\,dx &= \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla(u^\delta\varphi)\,dx \\ &\quad + \int_\Omega h(x)u^{p+\delta-1}\varphi\,dx - \lambda\int_{\partial\Omega} f(x)u^{q+\delta-1}\varphi\,d\sigma \\ &= \int_\Omega |x|^{-ap}|\nabla u|^{p-1}\nabla\varphi u^\delta\,dx + \delta\int_\Omega |x|^{-ap}|\nabla u|^p u^{\delta-1}\varphi\,dx \\ &\quad + \int_\Omega h(x)u^{p+\delta-1}\varphi\,dx - \lambda\int_{\partial\Omega} f(x)u^{q+\delta-1}\varphi\,d\sigma.\end{aligned}$$

Since $\delta < 0$, $f(x) > 0$ and $h(x) < 0$, (4.3) implies that

$$\begin{aligned}(4.4)\qquad \int_\Omega g(x)u^{r+\delta-1}\varphi\,dx + |\delta|\int_\Omega |x|^{-ap}|\nabla u|^p u^{\delta-1}\varphi\,dx \\ \le \int_\Omega |x|^{-ap}|\nabla u|^{p-1}\nabla\varphi u^\delta\,dx.\end{aligned}$$

By the Young inequality, we have

$$\begin{aligned}(4.5)\qquad \int_\Omega |x|^{-ap}|\nabla u|^{p-1}\nabla\varphi u^\delta\,dx &\le \varepsilon\int_\Omega |x|^{-ap}|\nabla u|^p\varphi u^{\delta-1}\,dx \\ &\quad + c(\varepsilon)\int_\Omega |x|^{-ap}u^{p+\delta-1}|\nabla\varphi|^p\varphi^{-(p-1)}\,dx.\end{aligned}$$

Using the Young inequality again, we see that for any $\eta > 0$,

$$(4.6)\qquad \int_\Omega |x|^{-ap}u^{p+\delta-1}|\nabla\varphi|^p\varphi^{-(p-1)}\,dx \le c(\eta)B + \eta\int_\Omega g(x)u^{r+\delta-1}\varphi\,dx,$$

where

$$(4.7)\qquad B = \int_\Omega |x|^{\frac{-ap(r+\delta-1)}{r-p}} g_0^{-\frac{p+\delta-1}{r-p}} |\nabla\varphi|^{\frac{p(r+\delta-1)}{r-p}} \varphi^{\frac{r-pr-p\delta}{r-p}}\,dx$$

and g_0 is given in (A_4). Let $\varepsilon, \eta > 0$ be small enough such that $\eta c(\varepsilon) < 1/2$ and $\varepsilon < |\delta|/2$. Then it follows from (4.4)–(4.6) that

$$(4.8)\qquad \frac{1}{2}\int_\Omega g(x)u^{r+\delta-1}\varphi\,dx + \frac{|\delta|}{2}\int_\Omega |x|^{-ap}|\nabla u|^p u^{\delta-1}\varphi\,dx \le c(\varepsilon)c(\eta)B.$$

Let $x = R\xi$. Noting that $\varphi(x) = \varphi_0(|x|/R)$, we obtain

$$(4.9)\qquad \frac{\partial\varphi}{\partial x_j} = \varphi_0'(|\xi|)\cdot\frac{1}{R}\cdot\frac{x_j}{|x|},\qquad |\nabla\varphi| \le \frac{N}{R}|\varphi_0'(|\xi|)| \le \frac{N}{R}\varphi_0^{1-1/k}(|\xi|)\beta_0,$$

where β_0 is defined in (4.2). Since $r > p > 1$ and $\delta < 0$, we can choose k large enough such that $kr - kp - pr > 0$. Therefore, it follows from (4.2) and (4.9) that

$$(4.10)\quad c(\varepsilon)c(\eta)B \le c_5 \int\limits_{1\le|\xi|\le 2} R^{N-\frac{p(a+1)(r+\delta-1)}{r-p}} |\xi|^{\frac{-ap(r+\delta-1)}{r-p}} \varphi_0^{\frac{kr-kp-pr-p\delta+p}{k(r-p)}}(|\xi|)\,d\xi \le c_6 R^{N-\frac{p(a+1)(r+\delta-1)}{r-p}}$$

for some constants $c_5, c_6 > 0$, with B defined in (4.7). Then (4.8) and (4.10) yield

$$(4.11)\qquad \int\limits_{\Omega} g(x)u^{r+\delta-1}\varphi\,dx \le c_6 R^{N-\frac{p(a+1)(r+\delta-1)}{r-p}}.$$

On the other hand, the assumption $\frac{Nr}{(a+1)(r+1)+N} < p$ implies that there exists $\delta < 0$ with $|\delta|$ small such that

$$N - \frac{p(a+1)(r+\delta-1)}{r-p} < 0.$$

Therefore, by virtue of (4.11), we have

$$\lim_{R\to\infty} \int\limits_{\Omega} g(x)u^{r+\delta-1} = 0,$$

that is, $u = 0$ a.e. in Ω. ■

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