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The pluricomplex Green function on some regular pseudoconvex domains

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Abstract. Let D be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$ of finite type. We prove an estimate on the pluricomplex Green function $\mathscr{G}_D(z,w)$ of D that gives quantitative information on how fast the Green function vanishes if the pole w approaches the boundary. Also the Hölder continuity of the Green function is discussed.

1. Introduction. Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a smooth boundary. We will investigate the behavior of the pluricomplex Green function $\mathscr{G}_D(\cdot,w)$, $w \in D$, of D when w tends to the boundary.

This function is defined by

$$\mathscr{G}_D(\cdot,w) := \sup\{u(z) \mid u \in P(w;D)\},$$

where $P(w;D)$ denotes the class of all negative plurisubharmonic functions on D such that $u - \log|\cdot - w|$ is bounded from above near w. It has been introduced by Klimek [Kli1], and later in hyperconvex domains in general complex manifolds by Demailly [Dem]. In both papers fundamental properties of $\mathscr{G}_D$ were proved (in particular its relationship to the Monge–Ampère operator was clarified in [Dem]).

The fact that $\mathscr{G}_D$ has a logarithmic pole at w makes it an important tool in applications of real methods in complex analysis, in particular those that are based upon the L^2-theory for the $\bar{\partial}$-operator with plurisubharmonic weight functions (see [Hör], [OhTa]). We need to know, however, how $\mathscr{G}_D(\cdot,w)$ behaves when w tends to the boundary. First results in this context were obtained in [CCW], [He], and [DiHe] (for quantitative results in special cases see [Car], [Che]).

For a domain $D \subsetneq \mathbb{C}^n$ we denote by δ_D the boundary-distance function. Our main result is

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Theorem 1.1. *Let $D \subset\subset \mathbb{C}^n$ be a smooth bounded domain, and let $w_0 \in \partial D$. Assume that there exist an open neighborhood $U_1 \ni w_0$, constants $C > 0$ and $0 < \varepsilon \le 1/2$, and a $\mathscr{C}^2$-smooth plurisubharmonic function $\Phi : D \cap U_1 \to (-1, 0)$ such that:*

(a) *The function $z \mapsto \Phi(z) - C^{-1}|z|^2$ is plurisubharmonic.*
(b) *One has $\Phi(z) \ge -C\delta_D(z)^{2\varepsilon}$ for all $z \in D \cap U_1$.*

Then there exists a constant $C > 0$, and a neighborhood $U_2 \subset\subset U_1$ of w_0, such that

$$|\mathscr{G}_D(z,w)| \le CM(z,w)^{1-1/n} \log\left(1 + C\left(\frac{\delta_D(w)^\varepsilon}{|z-w|}\right)^{1/n}\right) \tag{1.1}$$

and

$$|\mathscr{G}_D(z,w)| \le CM(z,w)^{1-1/n} \log\left(1 + C\left(\frac{\delta_D(z)^\varepsilon \delta_D(w)^\varepsilon}{|z-w|^2}\right)^{1/n}\right), \tag{1.2}$$

where

$$M(z,w) := \left|\log\log\left(1 + C\frac{\delta_D(w)^\varepsilon}{|z-w|}\right)\right| + 1 + C\log\frac{R_D}{|z-w|}$$

for all $z, w \in D \cap U_2$. Here we denote by R_D the diameter of D.

In [Car] the case of strongly pseudoconvex domains was treated, where one can find a plurisubharmonic function ψ with properties (a) and (b) for $\varepsilon = 1/2$. For a convex domain of finite type an estimate for $\mathscr{G}_D$ was established in [Che] that implies (1.2) without the factor $M(z,w)$.

The methods from [Car], [Che] do not carry over to our case since the Green function is not symmetric in general; whether or not holomorphic peak functions are available under the above comparatively weak hypotheses is also an open question.

As a corollary we obtain from Theorem 1.1:

Theorem 1.2. *Let $D \subset\subset \mathbb{C}^n$ be a smooth bounded domain, and let $w_0 \in \partial D$ be such that there exist an open neighborhood $U_1 \ni w_0$, constants $C > 0$ and $0 < \varepsilon \le 1/2$, and a $\mathscr{C}^2$-smooth plurisubharmonic function $\Phi : D \cap U_1 \to (-1, 0)$ with properties* (a) *and* (b). *Then we can choose an open neighborhood $U_2 \subset\subset U_1$ of w_0 such that for any $w \in D \cap U_2$ the sublevel set $\{\mathscr{G}_D(\cdot, w) < -1\}$ is contained in a ball about w of radius $\le C'\delta_D(w)^\varepsilon \log^n \frac{1}{\delta_D(w)}$. In particular, the Bergman metric B_D of D grows at least like*

$$B_D(w; X) \ge C_2 \frac{|X|}{\delta_D(w)^\varepsilon \log^n \frac{1}{\delta_D(w)}}$$

for all $w \in D \cap U_2$ and $X \in \mathbb{C}^n$.

The first assertion is clear. The second, concerning the growth order of the Bergman metric, follows from Proposition 4.1 from [DiHe].

This improves the estimate of Theorem 1.13 from [DiHe] insofar as the points w are not restricted to nontangential approach to the boundary point w^0 and the growth is up to a log-term exactly of order ε and not only $\varepsilon - t$ (with arbitrary $0 < t \ll 1$).

In [Bło2] and [NPT] the question of Hölder continuity of the Green function was treated for a special class of domains. With the methods applied in those papers we will show (1.1).

As a by-product we further obtain

THEOREM 1.3. *Suppose D is as in Theorem* 1.1. *Then there is a constant $C > 0$ such that*

$$|\mathscr{G}_D(z'', w) - \mathscr{G}_D(z', w)| \le C \frac{|z' - z''|^{\varepsilon^2/3n}}{M(z', z'', w)^{(1+\varepsilon)/n}} \log \frac{R_D}{M(z', z'', w)}$$

for any $z', z'' \in D \setminus \{w\}$, where

$$M(z', z'', w) = \min\{|z' - w|, |z'' - w|\}.$$

Later we will consider pseudoconvex domains that are uniformly extendable in a pseudoconvex way of some finite order $N \ge 2$. They belong to the class of pseudoconvex domains to which the above results apply.

The notion of pseudoconvex extendability is explained in the following

DEFINITION 1.4 (cf. [DiHe, Def. 1.10]). Let $D \subset\subset \mathbb{C}^n$ be pseudoconvex and smoothly bounded. We call D *uniformly extendable* of order N in a pseudoconvex way near a point $w^0 \in \partial D$ if there exist an open neighborhood $U' \ni w^0$, a constant $C_1 > 0$ and a $\mathscr{C}^2$-smooth function $\psi : U' \times U' \to \mathbb{R}$ such that:

(i) The open set $\{\psi(q, \cdot) < 0\} \cap U'$ is pseudoconvex and the surface $\{\psi(q, \cdot) = 0\} \cap U'$ is smooth and passes through q when $q \in \partial D \cap U'$.

(ii) For $x \in U'$, $q \in U' \cap \partial D$ we have the estimate

$$C_1(r(x) - |x - q|) \le \psi(q, x) \le r(x) - \frac{1}{C_1}|x - q|^N.$$

In [DiFo] it was shown that real-analytically bounded pseudoconvex domains have this property. This result was extended later in [Cho] to the larger class of smooth bounded pseudoconvex domains that are of finite type in the sense of [DA]. We will prove

LEMMA 1.5. *Assume that the domain $D \subset\subset \mathbb{C}^n$ is uniformly extendable of order N in a pseudoconvex way near a point $w^0 \in \partial D$. Then there exist an open neighborhood U_1 of w^0, a continuous plurisubharmonic function $\Phi : D \cap U_1 \to \mathbb{R}$ and constants $C_1, c_1 > 0$ such that:*

(i) *On* $D\cap U_1$ *we have* $-C_1\delta_D^{2/N}\le \Phi<0$.
(ii) *The function* $z\mapsto \Phi(z)-c_1|z|^2$ *is plurisubharmonic on* $D\cap U_1$.

Hence Theorems 1.1 and 1.2 apply with $\varepsilon=1/N$.

In conjunction with a result of [Cho] one obtains the following:

LEMMA 1.6. *Let* $D\subset\subset\mathbb{C}^n$ *be a smooth bounded domain and* $w_0\in\partial D$ *a point such that there exist an open neighborhood* $U_1\ni w_0$, *constants* $C>0$ *and* $0<\varepsilon\le 1/2$, *and a family* $(\lambda_\delta)_{0<\delta<\delta_0}$ *of plurisubharmonic functions on* U_1 *satisfying:*

(a) *For all* $\delta\in(0,\delta_0)$ *one has* $0\le\lambda_\delta\le 1$.
(b) *On the strip* $S_\delta:=\{z\in D\cap U_1\mid \delta_D(z)<\delta\}$ *the function* $z\mapsto \lambda_\delta(z)-\delta^{-2\varepsilon}|z|^2$ *is still plurisubharmonic.*
(c) *For any derivative* $D\lambda_\delta$ *of* λ_δ *of order* $k\le 2$ *one has* $|D\lambda_\delta|\le C\delta^{-k}$.

Then D *is uniformly extendable of order* $1/\varepsilon$ *near* w_0. *In particular Theorems* 1.1 *and* 1.2 *apply.*

2. Estimating the Green function in terms of the boundary distance of its first argument. Our plan is to estimate the Green function $\mathscr{G}_D(P,Q)$ in terms of $\delta_D(P)$ and $|P-Q|$, and then to compare $\mathscr{G}_D(P,Q)$ with $\mathscr{G}_D(Q,P)$.

PROPOSITION 2.1. *Let* $D\subset\subset\mathbb{C}^n$ *satisfy the hypotheses of Theorem* 1.1 *near* $w^0\in\partial D$. *Then there exist a constant* $C_5>0$ *and a radius* $R_1>0$ *such that for any* $P\in D\cap B(w^0,R_1)$ *and* $Q\in D$ *one has*

$$|\mathscr{G}_D(P,Q)|\le \frac{1}{2}\log\biggl(1+C_5\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}\biggr). \tag{2.1}$$

Proof. Let us take radii $R_1<\widetilde{R}_1<R_2$ such that $B(w_0,2R_2)\subset U_1$ and $R_2\ge 5\widetilde{R}_1$. Furthermore, we may assume that each $x\in B(w^0,3R_2/2)$ has an orthogonal projection $x^*\in\partial D\cap B(w^0,3R_2)$. We may certainly suppose that $\delta_D(P)<R_1/2$. Let us consider two cases:

CASE I: $|Q-w^0|<3R_1$. On U_1 the function

$$\Phi_1(z):=\Phi(z)-c_1|z-Q|^2-c_1|z-P^*|^2,$$

where $c_1<\frac{1}{2C}$, is negative and plurisubharmonic. Also we have

$$\frac{-\Phi_1(z)}{c_1|z-Q|^2}\ge 1+\frac{|z-P^*|^2}{|z-Q|^2}.$$

But for $z\in\partial B(w^0,R_2)$ we have, for $P\in B(w^0,R_1)$,

$$|z-P^*|\ge|z-w^0|-|w^0-P|-|P-P^*|\ge R_2-\tfrac{3}{2}R_1\ge\tfrac{7}{10}R_2$$

and

$$|z-Q|\le|z-w^0|+|w^0-Q|\le R_2+3R_1\le\tfrac{8}{5}R_2.$$

Therefore,

$$\frac{-\Phi_1(z)}{c_1|z-Q|^2} \geq 1 + \left(\frac{7}{16}\right)^2$$

and hence

$$\frac{1}{2}\log\frac{c_1|z-Q|^2}{-\Phi_1(z)} \leq -c_3 := -\frac{1}{2}\log\left(1+\left(\frac{7}{16}\right)^2\right).$$

The function

$$\Phi_2(z) := \begin{cases} \max\left\{\frac{1}{2}\log\frac{c_1|z-Q|^2}{-\Phi_1(z)}, -c_3\right\} & \text{if } z \in B(w^0, R_2)\cap D, \\ -c_3 & \text{if } z \in D\setminus(B(w^0,R_2)\cap D), \end{cases}$$

now becomes plurisubharmonic on D and thus it is a good candidate for $\mathscr{G}_D(P,Q)$. Since $|P-w^0| < R_1 < R_2$, we obtain

$$\begin{aligned} |\mathscr{G}_D(P,Q)| \leq -\Phi_2(P) &\leq \frac{1}{2}\log\frac{-\Phi_1(P)}{c_1|P-Q|^2} \\ &= \frac{1}{2}\log\frac{-\Phi(P)+c_1|P-Q|^2+c_1|P-P^*|^2}{c_1|P-Q|^2} \\ &\leq \frac{1}{2}\log\left(1+c_2\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}\right) \end{aligned}$$

with some positive constant c_2, as desired.

Case II: $|Q-w^0| \geq 3R_1$. Now we put

$$\Phi_3(z) := \Phi(z) - c_1|z-P^*|^2$$

in $D\cap U_1$. This function is plurisubharmonic and if $|z-P^*| = R_1/2$ we obtain $\Phi_3(z) \leq -c_1R_1^2/4$. Thus the function

$$\Phi_4(z) = \begin{cases} \max\{\Phi_3(z), -c_1R_1^2/4\} & \text{if } z\in B(P^*,R_1/2)\cap D, \\ -c_1R_1^2/4 & \text{if } z\in D\setminus(B(P^*,R_1/2)\cap D), \end{cases}$$

becomes well-defined and plurisubharmonic on D. Next we define an appropriate candidate for $\mathscr{G}_D(\cdot,Q)$. Let

$$\Phi_5(z) = \begin{cases} \max\left\{C_7\Phi_4(z), \log\frac{|z-Q|}{R_D}\right\} & \text{if } z\in D\setminus(B(Q,R_1/2)\cap D), \\ \log\frac{|z-Q|}{R_D} & \text{if } z\in B(Q,R_1/2)\cap D, \end{cases}$$

where $C_7 > 0$ is chosen so large that

$$C_7\Phi_4(z) \leq \log\frac{R_1}{2R_D} \quad \text{for } z\in D\cap\partial B(Q,R_1/2).$$

Note that this is possible, since for such points z one has

$$\begin{aligned}|z-P^*| &\geq |Q-P^*|-|z-Q| \geq |Q-w^0|-|w^0-P^*|-|z-Q|\\ &\geq |Q-w^0|-|w^0-P|-\delta_D(P)-|z-Q| \geq R_1,\end{aligned}$$

hence $\Phi_4(z) = -c_1R_1^2/4$.

We find that

$$\begin{aligned}|\mathscr{G}_D(P,Q)| &\leq -\Phi_5(P) \leq -C_7\Phi_4(P) = -C_7\Phi(P) + c_1C_7|P-P^*|^2\\ &\leq C_8\delta_D(P)^{2\varepsilon},\end{aligned}$$

because in our situation we have

$$|P-Q| \geq |Q-w_0|-|P-w_0| \geq 2R_1.$$

This implies also $\frac{\delta_D(P)^\varepsilon}{|P-Q|} \leq \frac{R_D^\varepsilon}{2R_1}$.

The function $V(t) := \frac{1}{t}\log(1+t)$ is decreasing on $(0,\infty)$. This yields, if we choose C_5 so large that

$$\frac{2R_1^2}{R_D^{2+2\varepsilon}}\log\left(1+C_5^2\frac{R_D^{2\varepsilon}}{4R_1^2}\right) \geq C_8,$$

the estimate

$$\begin{aligned}\frac{1}{2}\log\left(1+C_5^2\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}\right) &= \frac{1}{2}C_5^2\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}V\left(C_5^2\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}\right)\\ &\geq \frac{1}{2}C_5^2\frac{\delta_D(P)^{2\varepsilon}}{|P-Q|^2}V\left(C_5^2\frac{R_D^{2\varepsilon}}{4R_1^2}\right)\\ &\geq \frac{1}{2R_D^2}C_5^2\,V\left(C_5^2\frac{R_D^{2\varepsilon}}{4R_1^2}\right)\delta_D(P)^{2\varepsilon}\\ &= \frac{2R_1^2}{R_D^{2+2\varepsilon}}\log\left(1+C_5^2\frac{R_D^{2\varepsilon}}{4R_1^2}\right)\delta_D(P)^{2\varepsilon}\\ &\geq C_8\delta_D(P)^{2\varepsilon} \geq |\mathscr{G}_D(P,Q)|.\end{aligned}$$

The proposition is proved. ∎

3. A first Hölder estimate for the Green function. We adopt the methods from [Bło2] and [NPT]. Let D be a bounded hyperconvex domain in $\mathbb{C}^n$. Let $w \in D$ be fixed and $r = \frac{2}{3}\delta_D(w)$. We consider for $r > \eta > 0$ the family

$$\mathscr{P}_\eta := \{v \mid v \text{ plurisubharmonic on } D,\ v \leq 0 \text{ on } D,\ v \leq \log(\eta/r) \text{ on } \overline{B}(w,\eta)\}$$

and its upper envelope

$$u^\eta(z) = \sup\{v(z) \mid v \in \mathscr{P}_\eta\}.$$

Then (for details see [Kli2, Sec. 4.5]) we have:

(a) u^η is continuous, and $u^\eta(z) \to 0$ as z tends to a boundary point of D.
(b) The function u^η is maximal plurisubharmonic outside $\overline{B}(w,\eta)$.
(c) We have $u^\eta = \log(\eta/r)$ on $\overline{B}(w,\eta)$ and $u^\eta(z) \le \log(\max\{|z-w|,\eta\}/r)$ on D.
(d) For $\eta_1 \le \eta_2$ we have $u^{\eta_1} \le u^{\eta_2}$. Further $\lim_{\eta\searrow 0} u^\eta(z) = \mathscr{G}_D(z,w)$ on D.

We recall from [NPT] the following construction, associated with two given different points $P, w \in D$. Let $\mathbb{D}$ denote the unit disc in the plane. We choose a holomorphic function $F_{P,w}$ on D with values in $\mathbb{D}$ such that $F_{P,w}(w) = 0$ and $\operatorname{artgh}|F_{P,w}(P)|$ is equal to the Carathéodory distance $c_D(P,w)$ between P and w. Then we put, for $h \in \mathbb{C}^n$, $z \in D$,

$$H_{P,w,h}(z) := z + \frac{F_{P,w}(z)}{F_{P,w}(P)} h.$$

Since $\operatorname{tgh} c_D(P,Q) \ge |P-Q|/R_D$ for $P,Q \in D$, we obtain

$$|H_{P,w,h}(z) - z| \le R_D |F_{P,w}(z)| \frac{|h|}{|P-w|} \tag{3.1}$$

for $z \in D$. If $|z-w| < \delta_D(w)$, then we get

$$|F_{P,w}(z)| \le \operatorname{tgh} c_D(z,w) \le \frac{|z-w|}{\delta_D(w)},$$

and from (3.1) we see that

$$|H_{P,w,h}(z) - z| \le R_D \frac{|z-w|}{|P-w|} \frac{|h|}{\delta_D(w)}. \tag{3.2}$$

We will make use of this later.

Lemma 3.1. *Let D be a pseudoconvex domain as in Theorem* 1.1. *Then, with some constant $C_1 > 0$, for $z', z'' \in D \setminus \{w\}$ we have*

$$|\mathscr{G}_D(z',w) - \mathscr{G}_D(z'',w)| \le \log\left(1 + C_1 \frac{R_D^\varepsilon}{\delta_D(w)} \frac{|z'-z''|^\varepsilon}{M(z',z'',w)^\varepsilon}\right)$$

provided that

$$|z'-z''| \le \frac{\delta_D(w)}{8R_D} M(z',z'',w), \tag{3.3}$$

where we write $M(z',z'',w) = \min\{|z'-w|, |z''-w|\}$.

Proof. We follow an idea from [NPT]. Let $h := z'' - z'$, and consider the domain

$$D_1 := \{z \in D \mid H_{z',w,h}(z) \in D\}.$$

Then $\overline{B}(w,\eta) \subset D_1$ for small enough η, since $w \in D_1$.

If $z \in D$ and $H_{z',w,h}(z) \in \partial D$, we have

$$\delta_D(z) \le |H_{z',w,h}(z) - z| \le \frac{R_D}{|z'-w|}|h| \tag{3.4}$$

and

$$\begin{aligned} |z-w| &\ge |H_{z',w,h}(z) - w| - |H_{z',w,h}(z) - z| \\ &\ge \delta_D(w) - \frac{R_D}{|z'-w|}|h| \ge \frac{1}{2}\delta_D(w), \end{aligned} \tag{3.5}$$

using (3.3). This implies (together with Prop. 2.1)

$$\begin{aligned} u^\eta(z) \ge \mathscr{G}_D(z,w) &\ge -\log\left(1 + C_5 \frac{\delta_D(z)^\varepsilon}{|z-w|}\right) \\ &\ge -\log\left(1 + C_5 \frac{R_D^\varepsilon}{|z-w|}\,\frac{|h|^\varepsilon}{|z'-w|^\varepsilon}\right) \quad \text{by (3.4)} \\ &\ge -\log\left(1 + 2C_5 \frac{R_D^\varepsilon}{\delta_D(w)}\,\frac{|h|^\varepsilon}{|z'-w|^\varepsilon}\right) \quad \text{by (3.5).} \end{aligned}$$

For $z \in \partial D$ we even have $u^\eta(z) = 0$. In each case we see that the last displayed estimate holds for any $z \in \partial D_1$. In particular,

$$u^\eta(H_{z',w,h}(z)) \le 0 \le u^\eta(z) + \log\left(1 + 2C_5 \frac{R_D^\varepsilon}{\delta_D(w)}\,\frac{|h|^\varepsilon}{|z'-w|^\varepsilon}\right)$$

on ∂D_1. We want to prove this estimate also for $z \in \partial B(w,\eta)$.

For this purpose we take an arbitrary $z \in D$ with $|z-w| = \eta$. Then, by (3.2),

$$\begin{aligned} |H_{z',w,h}(z) - w| &\le |z-w| + \frac{R_D}{\delta_D(w)}\,\frac{|z-w|}{|z'-w|}|h| \\ &= \left(1 + \frac{R_D}{\delta_D(w)}\,\frac{|h|}{|z'-w|}\right)\eta. \end{aligned}$$

This gives

$$\begin{aligned} u^\eta(H_{z',w,h}(z)) &\le \log\frac{\max\{|H_{z',w,h}(z) - w|, \eta\}}{r} \\ &\le \log(\eta/r) + \log\left(1 + \frac{R_D}{\delta_D(w)}\,\frac{|h|}{|z'-w|}\right) \\ &\le u^\eta(z) + \log\left(1 + 2C_5 \frac{R_D^\varepsilon}{\delta_D(w)}\,\frac{|h|^\varepsilon}{|z'-w|^\varepsilon}\right), \quad \text{by (3.3).} \end{aligned}$$

Since u^η is maximal on $D_1 \setminus \overline{B}(w,\eta)$, the above estimate holds even on $D_1 \setminus \overline{B}(w,\eta)$, since it holds on $\partial(D_1 \setminus \overline{B}(w,\eta))$. We choose $z = z'$ and get, because $H_{z',w,h}(z') = z''$,

$$u^\eta(z'') \le u^\eta(z') + \log\left(1 + 2C_5 \frac{R_D^\varepsilon}{\delta_D(w)}\,\frac{|h|^\varepsilon}{M(z',z'',w)^\varepsilon}\right).$$

Letting η tend to zero and recalling the definition of h, we obtain the desired estimate

$$\mathscr{G}_D(z'',w)-\mathscr{G}_D(z',w)\le \log\biggl(1+2C_5\frac{R_D^\varepsilon}{\delta_D(w)}\,\frac{|z'-z''|^\varepsilon}{M(z',z'',w)^\varepsilon}\biggr).$$

Interchanging the roles of z' and z'' we can complete the proof. ∎

4. Proof of Theorem 1.1

4.1. Proof of estimate (1.1). We must consider two cases.

CASE 1: $\delta_D(w)^\varepsilon\le|z-w|$. The starting point is the following estimate that was obtained in [He] (based upon an inequality of [Bło1]):

$$\int_D|\mathscr{G}_D(\cdot,w)|\,d\mu_{z,\eta}\le(2\pi)^n(n!)^{1/n}\eta^{(n-1)/n}|\mathscr{G}_D(w,z)|^{1/n},\tag{4.6}$$

where $d\mu_{z,\eta}$ denotes for any $\eta>0$ the measure

$$d\mu_{z,\eta}:=(dd^c\max\{\mathscr{G}_D(\cdot,z),\,-\eta\})^n.$$

This measure is supported on the set $\{\mathscr{G}_D(\cdot,z)=-\eta\}\subset B(z,R_De^{-\eta})$, and its total mass is $(2\pi)^n$ (see [He]).

We want to apply Lemma 3.1 for $z'=z$. For this we must choose $\eta>1$ such that

$$R_De^{-\eta}\le\frac{\delta_D(w)}{8R_D}\min\{|z-w|,|z''-w|\}\tag{4.7}$$

for $|z''-z|<R_De^{-\eta}$. Now we note that

$$|z''-w|\ge|z-w|-|z''-z|\ge|z-w|-R_De^{-\eta}\ge\tfrac12|z-w|,$$

if only $\eta\ge\log\frac{2R_D}{|z-w|}$. We must choose

$$\eta\ge\log\frac{16R_D}{|z-w|\delta_D(w)}\tag{4.8}$$

in order to arrange for (4.7). Lemma 3.1 and (4.6) yield

$$\begin{aligned}(n!)^{1/n}&\eta^{(n-1)/n}|\mathscr{G}_D(w,z)|^{1/n}\\
&\ge(2\pi)^{-n}\int_D|\mathscr{G}_D(z'',w)|\,d\mu_{z,\eta}(z'')\ge(2\pi)^{-n}\int_D|\mathscr{G}_D(z,w)|\,d\mu_{z,\eta}(z'')\\
&\qquad-(2\pi)^{-n}\int_D|\mathscr{G}_D(z'',w)-\mathscr{G}_D(z,w)|\,d\mu_{z,\eta}(z'')\\
&=|\mathscr{G}_D(z,w)|-(2\pi)^{-n}\int_D|\mathscr{G}_D(z'',w)-\mathscr{G}_D(z,w)|\,d\mu_{z,\eta}(z'')\\
&\ge|\mathscr{G}_D(z,w)|-(2\pi)^{-n}\int_D\log\biggl(1+2C_5\frac{R_D^{2\varepsilon}}{\delta_D(w)}\,\frac{e^{-\varepsilon\eta}}{|z-w|^\varepsilon}\biggr)\,d\mu_{z,\eta}(z'')\\
&=|\mathscr{G}_D(z,w)|-\log(1+M_\eta e^{-\varepsilon\eta})\end{aligned}\tag{4.9}$$

with the abbreviation

$$M_\eta := (2C_5)^\varepsilon \frac{R_D^{\varepsilon^2+1}}{\delta_D(w)|z-w|^\varepsilon}.$$

We now choose

$$\eta := \left(\frac{1}{n}+\frac{1}{\varepsilon}\right)\frac{1}{\varepsilon}\log\frac{1}{|\mathscr{G}_D(w,z)|}+\frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{2C_5}{R_D^{1-\varepsilon}}+\frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{R_D}{|z-w|}. \tag{4.10}$$

Next we show that

$$\eta > \frac{1}{\varepsilon}\log M_\eta + \frac{1}{n\varepsilon}\log\frac{1}{|\mathscr{G}_D(w,z)|}. \tag{4.11}$$

By Proposition 2.1 we have

$$|\mathscr{G}_D(w,z)| \le C_5\frac{\delta_D(w)^\varepsilon}{|z-w|}.$$

This leads to

$$\begin{aligned}
&\eta - \frac{1}{\varepsilon}\log M_\eta - \frac{1}{n\varepsilon}\log\frac{1}{|\mathscr{G}_D(w,z)|}\\
&\quad = \frac{1}{\varepsilon^2}\log\frac{1}{|\mathscr{G}_D(w,z)|} + \frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{R_D}{|z-w|} + \frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{2C_5}{R_D^{1-\varepsilon}}\\
&\qquad - \log(2C_5) - \frac{1+\varepsilon^2}{\varepsilon}\log R_D + \frac{1}{\varepsilon}\log\delta_D(w) + \log|z-w|\\
&\quad \ge -\frac{1}{\varepsilon^2}\log C_5 - \frac{1}{\varepsilon}\log\delta_D(w) + \frac{1}{\varepsilon^2}\log|z-w| + \frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{R_D}{|z-w|}\\
&\qquad + \frac{1+\varepsilon^2}{\varepsilon^2}\log\frac{2C_5}{R_D^{1-\varepsilon}} - \log(2C_5) - \frac{1+\varepsilon^2}{\varepsilon}\log R_D + \frac{1}{\varepsilon}\log\delta_D(w) + \log|z-w|\\
&\quad = \frac{1}{\varepsilon^2}\log 2 > 0,
\end{aligned}$$

which yields (4.11). Further we get (4.8) from

$$\begin{aligned}
\eta - \log&\frac{16R_D}{|z-w|\delta_D(w)}\\
&\ge \frac{1}{\varepsilon}\log M_\eta - \frac{1}{n\varepsilon}\log C_5 - \log(16R_D) + \log\delta_D(w) + \log|z-w|\\
&\ge -\log(8R_D^{1-\varepsilon-1/\varepsilon}C_5^{1/n\varepsilon\,-1}) + \left(\frac{1}{\varepsilon}-1\right)\log\frac{1}{\delta_D(w)} > 0
\end{aligned}$$

using (4.11) and $|\mathscr{G}_D(w,z)| \le C_5\delta_D(w)^\varepsilon/|z-w| \le C_5$ (recall that we suppose $\delta_D(w)^\varepsilon/|z-w| \le 1$).

Finally,

$$\begin{aligned}
&\log\frac{M_\eta}{\exp(\eta^{1-1/n}|\mathscr{G}_D(w,z)|^{1/n})-1}\\
&\quad\le \log\frac{M_\eta}{\eta^{1-1/n}|\mathscr{G}_D(w,z)|^{1/n}} = \log M_\eta - \left(1-\frac{1}{n}\right)\log\eta + \frac{1}{n}\log\frac{1}{|\mathscr{G}_D(w,z)|}\\
&\quad\le \varepsilon\eta - \left(1-\frac{1}{n}\right)\log\eta \le \varepsilon\eta, \qquad \text{by (4.11)},
\end{aligned}$$

hence

$$e^{-\varepsilon\eta} \le \frac{1}{M_\eta}\big(\exp(\eta^{1-1/n}|\mathscr{G}_D(w,z)|^{1/n})-1\big)$$

and

$$\log(1+M_\eta e^{-\varepsilon\eta}) \le \eta^{1-1/n}|\mathscr{G}_D(w,z)|^{1/n}.$$

Plugging these into (4.9) we find by means of (2.1), applied to $|\mathscr{G}_D(w,z)|$,

$$\begin{aligned}
|\mathscr{G}_D(z,w)| &\le (1+(n!)^{1/n})\eta^{(n-1)/n}|\mathscr{G}_D(w,z)|^{1/n}\\
&\le \frac{2}{\varepsilon^2}(1+(n!)^{1/n})M(z,w)|\mathscr{G}_D(w,z)|^{1/n},
\end{aligned}$$

from which the claim follows.

Case 2: $\delta_D(w)^\varepsilon > |z-w|$. With a constant $\widehat{M}_\eta > 1$ to be chosen later, we consider the function

$$v(x) := \widehat{M}_\eta \log\left(\frac{1}{2}\,\frac{|x-w|}{\delta_D(w)^\varepsilon}\right)$$

on the domain

$$\Omega_r := D\cap B(w,\delta_D(w)^\varepsilon)\setminus B(w,r),$$

where the radius $r>0$ is less than $|z-w|$ and satisfies

$$\widehat{M}_\eta \log\frac{r}{\delta_D(w)^\varepsilon} \le \log\frac{r}{R_D},$$

which is equivalent to

$$\log r < \frac{\varepsilon\widehat{M}_\eta\log\delta_D(w) - \log R_D}{\widehat{M}_\eta - 1}.$$

Then $z\in\Omega_r$, and $w\notin\Omega_r$.

On Ω_r we have $v \le -\widehat{M}_\eta\log 2 < 0$.

Next let us consider v on $\partial\Omega_r$. For $x\in D\cap\partial B(w,r)$ we can estimate

$$v(x) = \widehat{M}_\eta\log\frac{r}{\delta_D(w)^\varepsilon} \le \log\frac{r}{R_D} < \mathscr{G}_D(x,w).$$

For $x\in\partial D$ we obtain

$$v(x) \le -\widehat{M}_\eta\log 2 < 0 = \mathscr{G}_D(x,w).$$

Finally, let $x \in D \cap \partial B(w, \delta_D(w)^\varepsilon)$. Then, by Case 1, because $|x - w| = \delta_D(w)^\varepsilon$,

$$|\mathscr{G}_D(x, w)| \leq 2\varepsilon^{-2}(1 + (n!)^{1/n})M(x, w)^{1-1/n}|\mathscr{G}_D(w, x)|^{1/n}.$$

But

$$|\mathscr{G}_D(w, x)| \log^{n-1} \frac{1}{|\mathscr{G}_D(w, x)|} \leq C_6$$

and

$$\log \frac{R_D}{|x - w|} = \log \frac{R_D}{\delta_D(w)^\varepsilon} \leq \log \frac{R_D}{|z - w|}.$$

(We are considering the case $\delta_D(w)^\varepsilon > |z - w|$.) This proves

$$|\mathscr{G}_D(x, w)| \leq C_7 + C_8 \left(\log \frac{R_D}{|z - w|} \right)^{1-1/n}$$

and, since $v(x) = -\widehat{M}_\eta \log 2$, we get

$$|\mathscr{G}_D(x, w)| \leq \frac{1}{\widehat{M}_\eta} \left(C_7' + C_8' \left(\log \frac{R_D}{|z - w|} \right)^{1-1/n} \right) |v(x)|.$$

Let

$$\widehat{M}_\eta := C_9 M(z, w)^{1-1/n},$$

where the constant C_9 can be chosen independently of z, w in such a way that $|\mathscr{G}_D(x, w)| \leq |v(x)|$. Hence, by the maximality of $\mathscr{G}_D(\cdot, w)$ we get $v \leq \mathscr{G}_D(\cdot, w)$ on Ω_r. This implies

$$|\mathscr{G}_D(z, w)| \leq |v(z)| \leq \widehat{M}_\eta \log \left(2 \frac{\delta_D(w)^\varepsilon}{|z - w|} \right) \leq n \widehat{M}_\eta \log \left(1 + 2 \left(\frac{\delta_D(w)^\varepsilon}{|z - w|} \right)^{1/n} \right),$$

from which the desired estimate on $\mathscr{G}_D(z, w)$ will follow. ∎

4.2. Proof of estimate (1.2). Our aim is the proof of

$$|\mathscr{G}_D(z, w)| \leq C M(z, w)^{1-1/n} \log \left(1 + C \left(\frac{\delta_D(z)^\varepsilon \delta_D(w)^\varepsilon}{|z - w|^2} \right)^{1/n} \right).$$

We fix distinct $z, w \in D \cap U_1$. Without loss of generality we may assume that they are close to the boundary so that the orthogonal projections z^*, w^* to the boundary are well-defined.

Let $c > 0$ denote a small constant such that

$$4c^{1/\varepsilon} R_D^{1/\varepsilon - 1} < 1.$$

If $\delta_D(z)^\varepsilon \ge c|z-w|$, then (1.1) yields

$$|\mathscr{G}_D(z,w)| \le nCM(z,w)^{1-1/n}\log\bigg(1 + C\bigg(\frac{\delta_D(w)^\varepsilon}{|z-w|}\bigg)^{1/n}\bigg)$$
$$\le nCM(z,w)^{1-1/n}\log\bigg(1 + Cc^{-1/n}\bigg(\frac{\delta_D(z)^\varepsilon\delta_D(w)^\varepsilon}{|z-w|^2}\bigg)^{1/n}\bigg).$$

So we suppose that $\delta_D(z)^\varepsilon \le c|z-w|$. Now we define

$$V := D \cap B(z^*, 2c^{1/\varepsilon}R_D^{1/\varepsilon-1}|z-w|)$$

and note that

$$|z-z^*| = \delta_D(z) \le (c|z-w|)^{1/\varepsilon} \le c^{1/\varepsilon}R_D^{1/\varepsilon-1}|z-w|,$$

hence $z \in V$. At the same time we have

$$|w-z^*| \ge |z-w| - |z-z^*| = |z-w| - \delta_D(z) \ge |z-w| - (c|z-w|)^{1/\varepsilon}$$
$$> 2c^{1/\varepsilon}R_D^{1/\varepsilon-1}|z-w|$$

by the choice of c. Hence $w \notin V$, and $\mathscr{G}_D(\cdot,w)$ is a maximal plurisubharmonic function on V. We define on V a plurisubharmonic comparison function v_2. For this we use

$$\psi(x) := \Phi(x) - \gamma|x-z^*|^2,$$

which is negative and for small enough $\gamma > 0$ also plurisubharmonic. Then, for any $x \in V$,

$$|x-w| \ge |z-w| - |x-z| \ge |z-w| - \delta_D(z) - |x-z^*|$$
$$\ge (1 - 3c^{1/\varepsilon}R_D^{1/\varepsilon-1})|z-w| \ge \tfrac{1}{4}|z-w|.$$

By (1.1) we have the estimate

$$\mathscr{G}_D(x,w) \ge -CM(x,w)^{1-1/n}\log\bigg(1 + \widetilde{C}\bigg(\frac{\delta_D(w)^\varepsilon}{|x-w|}\bigg)^{1/n}\bigg)$$
$$\ge -C'M(z,w)^{1-1/n}\log(1 + CM_1^{1/n})$$
$$\ge -C'M(z,w)^{1-1/n}\log\bigg(1 + CM_1^{1/n}\bigg(\frac{-\psi(x)}{\gamma|x-z^*|^2}\bigg)^{1/2n}\bigg)$$

with some constant C' and $M_1 := 4\delta_D(w)^\varepsilon/|z-w|$. Our plurisubharmonic comparison function v_2 is now defined by

$$v_2(x) := -C'M(z,w)^{1-1/n}\log\bigg(1 + C\bigg(\frac{\delta_D(w)^\varepsilon}{\gamma_1|z-w|^2}\bigg)^{1/n}(-\psi(x))^{1/2n}\bigg)$$

with a constant γ_1 that will be chosen in a moment.

It is easily verified that v_2 is plurisubharmonic on V. We compare v_2 and $\mathscr{G}_D(\cdot,w)$ on ∂V. On $V \cap \partial D$ certainly $v_2 \le 0 = \mathscr{G}_D(\cdot,w)$.

For $x \in D \cap \partial B(z^*, 2c^{1/\varepsilon} R_D^{1/\varepsilon-1}|z-w|)$ we have

$$\gamma_1|z-w| = \gamma_1 \frac{|x-z^*|}{2c^{1/\varepsilon} R_D^{1/\varepsilon-1}} = \frac{1}{4}\sqrt{\gamma}\,|x-z^*|$$

for $\gamma_1 = \frac{\sqrt{\gamma}}{2} c^{1/\varepsilon} R_D^{1/\varepsilon-1}$, and therefore

$$\begin{aligned} v_2(x) &= -C'M(z,w)^{1-1/n} \log\big(1 + CM_1^{1/n}(-\psi(x))^{1/2n}\big) \\ &\le -C'M(z,w)^{1-1/n} \log\bigg(1 + CM_1^{1/n}\bigg(\frac{-\psi(x)}{\gamma|x-z^*|^2}\bigg)^{1/2n}\bigg) \le \mathscr{G}_D(x,w). \end{aligned}$$

Hence $v_2 \le \mathscr{G}_D(\cdot, w)$ on ∂V and, by the comparison principle, $v_2 \le \mathscr{G}_D(\cdot, w)$ on V. But this gives

$$\begin{aligned} |\mathscr{G}_D(z,w)| &\le |v_2(z)| \\ &= C'M(z,w)^{1-1/n} \log\bigg(1 + C\bigg(\frac{\delta_D(w)^\varepsilon}{\gamma_1|z-w|^2}\bigg)^{1/n}(-\psi(z))^{1/2n}\bigg) \\ &\le C'M(z,w)^{1-1/n} \log\bigg(1 + \widehat{C}\bigg(\frac{\delta_D(z)^\varepsilon \delta_D(w)^\varepsilon}{|z-w|^2}\bigg)^{1/n}\bigg) \end{aligned}$$

with some new constant $\widehat{C}$. Note that

$$|\psi(z)| = |\Phi(z)| + \delta_D(z) \le (C+1)\delta_D(z)^{2\varepsilon}.$$

This finishes the proof of Theorem 1.1. ∎

4.3. Proof of Theorem 1.3. We let $h := z' - z''$ and consider two cases.

CASE 1: $|h|^{2\varepsilon/3} \le \delta_D(w)$ and $|h| \le (\delta_D(w)/8R_D)M(z', z'', w)$. Then, by Lemma 3.1 we have

$$\begin{aligned} |\mathscr{G}_D(z',w) - \mathscr{G}_D(z'',w)| &\le \log\bigg(1 + C'\frac{|h|^\varepsilon}{\delta_D(w)M(z',z'',w)^\varepsilon}\bigg) \\ &\le \log\bigg(1 + C'\frac{|h|^{\varepsilon/3}}{M(z',z'',w)^\varepsilon}\bigg), \end{aligned}$$

which proves the claimed estimate.

CASE 2: $|h|^{2\varepsilon/3} \ge \delta_D(w)$ or $|h| \ge (\delta_D(w)/8R_D)M(z', z'', w)$. Now we simply estimate

$$|\mathscr{G}_D(z',w) - \mathscr{G}_D(z'',w)| \le |\mathscr{G}_D(z',w)| + |\mathscr{G}_D(z'',w)|$$

and want to apply (1.1) to the right-hand side. For this we note that

$$\begin{aligned}M(z',w) &\le \left|\log\log\left(1+C\frac{\delta_D(w)^\varepsilon}{|z'-w|}\right)\right| + \log\frac{R_D}{M(z',z'',w)}\\ &\le C'\log\frac{R_D}{M(z',z'',w)},\end{aligned}$$

and $M(z'',w) \le C'\log\frac{R_D}{M(z',z'',w)}$. This results in

$$\begin{aligned}|\mathscr{G}_D(z',w)| &\le \widehat{C}M(z',w)^{1-1/n}\log\left(1+C\left(\frac{\delta_D(w)^\varepsilon}{|z'-w|}\right)^{1/n}\right)\\ &\le \widehat{C}M(z',w)^{1-1/n}\log\left(1+\widehat{C}\frac{|h|^{2\varepsilon^2/n}}{M(z',z'',w)^{1+\varepsilon/n}}\right)\end{aligned}$$

and

$$|\mathscr{G}_D(z'',w)| \le \widehat{C}M(z'',w)^{1-1/n}\log\left(1+\widehat{C}\frac{|h|^{2\varepsilon^2/n}}{M(z',z'',w)^{1+\varepsilon/n}}\right),$$

which in conjunction with the estimates on $M(z',w)$ and $M(z'',w)$ gives the desired Hölder estimate for $|\mathscr{G}_D(z',w)|$. ■

5. The case of pseudoconvex extendable domains. Proofs of Lemmas 1.5 and 1.6

5.1. Proof of Lemma 1.5. We assume that $N > 2$, otherwise the assertion is well-known. As in the definition of pseudoconvex extendability, let $\psi \in \mathscr{C}^2(U' \times U')$ be an extending function of order N, defined on a neighborhood U' of w^0. Then there exists a constant $C_2 > 0$ such that its Levi form $\mathscr{L}_{\psi(q,\cdot)}$ satisfies (for all $q \in \partial D \cap U'$)

$$\mathscr{L}_{\psi(q,\cdot)}(z;X) \ge -C_2\big(|\psi(q,z)|\,|X|^2 + |\langle\partial\psi(q,\cdot),X\rangle|\,|X|\big).$$

For any constant $A > 0$ and any $q \in \partial D \cap U'$, the function

$$\sigma(q,z) := \psi(q,z)e^{-A|z-q|^2} \tag{5.1}$$

also extends in a pseudoconvex way on ∂D near w^0, more explicitly

$$-C_3(-r(z)+|z-q|) \le \sigma(q,z) \le e^{-AR'^2}r(z) - c_2|z-q|^N, \tag{5.2}$$

where R' is the diameter of U' and $c_2 > 0$ is a small constant.

We choose open neighborhoods $U_1 \subset\subset U_2 \subset\subset U'$ of w^0 such that, given $z \in U_1$, its orthogonal projection z^* onto ∂D lies inside U_2. By making A very large and then shrinking U_1 we can arrange that for any $q \in \partial D \cap U_2$, the function $-(-\sigma(q,z))^{2/N}$ is plurisubharmonic on $D \cap U_1$. Now we put, for $z \in D \cap U_1$,

$$\varPhi'(z) := \sup_{q\in\partial D\cap U_2}\big(-(-\sigma(q,z))^{2/N} + \tfrac14 c_2^{2/N}|z-q|^2\big).$$

Our claim is that Φ' satisfies the estimate
$$-C_1'\delta_D^{2/N} \le \Phi' \le -C_2'\delta_D^{2/N}$$
with suitable constants $C_1', C_2' > 0$.

For this we observe that for any $t, s \ge 0$,
$$(t^{2/N} + s^{2/N})^{N/2} \le 2^N(t+s).$$
This implies
$$\begin{aligned}\big((-e^{-AR'^2}r(z))^{2/N} + c_2^{2/N}|z-q|^2\big)^{N/2} &\le 2^N(-e^{-AR'^2}r(z) + c_2|z-q|^N)\\ &\le -2^N\sigma(q,z)\end{aligned}$$
by (5.2), or
$$(-e^{-AR'^2}r(z))^{2/N} + c_2^{2/N}|z-q|^2 \le 4(-\sigma(q,z))^{2/N}.$$
This gives
$$\Phi'(z) \le -\tfrac{1}{4}e^{-2AR'^2/N}(-r(z))^{2/N}.$$
The lower estimate is easier to show. Let $z \in D \cap U_1$; then $z^* \in \partial D \cap U_2$, and we find that
$$\begin{aligned}\Phi'(z) &\ge -(-\sigma(z^*,z))^{2/N} + \tfrac{1}{4}c_2^{2/N}|z-z^*|^2\\ &\ge -(C_3(-r(z) + |z-z^*|))^{2/N} \ge -C_4\delta_D(z)^{2/N}.\end{aligned}$$

The upper semicontinuous regularization Φ of Φ' is plurisubharmonic and satisfies property (i). But also property (ii) holds, since the function $\Phi''(z) := \Phi'(z) - \frac{1}{5}c_2^{2/N}|z|^2$ is the supremum of a family of plurisubharmonic functions, and furthermore $z \mapsto \Phi(z) - \frac{1}{5}c_2^{2/N}|z|^2$ equals the upper semicontinuous regularization of Φ'' and hence is also plurisubharmonic. ∎

5.2. Proof of Lemma 1.6. We only need to recall Cho's proof. We give a sketch of this proof and then state where to modify it.

Let $\phi \in \mathscr{C}_0^\infty(B(0,2) \setminus B(0,1/4))$ be a function such that $\phi(z) = 1$ for $1/2 < |z| < 1$. Also let $\psi \in \mathscr{C}^\infty(\mathbb{C}^n)$ be a smooth function such that $\psi(z) = 1$ for $|z| \ge 2$, and $\psi(z) = 0$ if $|z| < 1$.

For some large integer $\mathscr{N}$ we put $\phi_{\mathscr{N}}(z) = \psi(2^{\mathscr{N}\varepsilon}z)$ and $\phi_k(z) = \phi(2^{k\varepsilon}z)$ for $k > \mathscr{N}$. Let $\zeta \in \partial D$. Then we consider, with a suitable small number $a > 0$, the function
$$E_\zeta(z) := \sum_{k=\mathscr{N}}^{\infty} 2^{-2k}\phi_k(z-\zeta)(\lambda_{2^{-k}a}(z) - 2).$$
The only difference between this definition for E_ζ and that of Cho's proof is the factor 2^{-2k} in front of $\phi_k(z-\zeta)(\lambda_{2^{-k}a}(z) - 2)$. In Cho's proof the factor was 2^{-4k}.

There exists $L \in \mathbb{N}$ such that for any ζ and z there are at most L integers k such that $z \in \operatorname{supp} \phi_k(\cdot - \zeta)$. Again we have

$$|DE_\zeta(z)| \le La^{-\ell} 2^{\ell-2} k$$

for any ℓth order derivative DE_ζ of E_ζ and $z \in \operatorname{supp} \phi_k(\cdot - \zeta)$. This shows that E_ζ is of class $\mathscr{C}^2$. The rest of the proof of the pseudoconvexity of the surface $\{E_\zeta = 0\}$ is completely analogous to that in [Cho]. Because of the factor 2^{-2k} instead of 2^{-4k}, now the function E_ζ extends in a pseudoconvex way to order $\le 1/\varepsilon$ instead of $2/\varepsilon$. ∎

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