ANNALES
POLONICI MATHEMATICI
110.3 (2014)

Hölder regularity of three-dimensional minimal cones in $\mathbb{R}^n$

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Abstract. We show the local Hölder regularity of Almgren minimal cones of dimension 3 in $\mathbb{R}^n$ away from their centers. The proof is almost elementary but we use the generalized theorem of Reifenberg. In the proof, we give a classification of points away from the center of a minimal cone of dimension 3 in $\mathbb{R}^n$, into types $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$. We then treat each case separately and give a local Hölder parameterization of the cone.

1. Introduction. In this paper, we prove Hölder regularity for three-dimensional minimal cones in $\mathbb{R}^n$. This is a continuation of [D] in which G. David proved the Hölder regularity for two-dimensional almost minimal sets in $\mathbb{R}^n$. The structure of two-dimensional minimal cones in $\mathbb{R}^n$ is quite clear now, as in [D], G. David has classified them into three types: $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$ (see Section 15 of [D] for the definition). For now we do not know yet the list of cones of type $\mathbb{T}$. For three-dimensional minimal cones, Almgren [Al] has showed that any cone of dimension 3 in $\mathbb{R}^4$, centered at the origin and over a smooth surface of $\mathbb{S}^3$, must be a 3-plane. But for three-dimensional minimal cones in general, the structure of their singularities is still unclear. This paper is a first step towards understanding this structure, and we hope it may help to study the structure of singularities of three-dimensional minimal sets in $\mathbb{R}^4$.

Let us first give the definition of Almgren minimal sets of dimension d in $\mathbb{R}^n$.

Definition 1.1. Let E be a closed set in $\mathbb{R}^n$ and $d \leq n-1$ be an integer. An *Almgren competitor* (Al-competitor) for E is a closed set $F \subset \mathbb{R}^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz mapping such that $W_\varphi = \{x \in \mathbb{R}^n; \varphi(x) \neq x\}$ is bounded.

An *Al-minimal set* of dimension d in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $H^d(E \cap B(0,R)) < \infty$ for every $R > 0$ and
$$H^d(E \setminus F) \leq H^d(F \setminus E)$$
for every Al-competitor F for E.

2010 *Mathematics Subject Classification*: Primary 49Q05; Secondary 49Q15.
Key words and phrases: minimal sets, Almgren minimal sets, Hausdorff measure.

DOI: 10.4064/ap110-3-2

Even if we think that the applications will be essentially in $\mathbb{R}^4$, we shall treat the problem in the general case of $\mathbb{R}^n$. So we need the following descriptions of cones of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$ of dimension d in $\mathbb{R}^n$ [DDT].

We denote by $\mathcal{P}$ the collection of d-dimensional affine planes, which we shall also call cones of type $\mathbb{P}$.

We next define the collection $\mathcal{Y}$ of cones of type $\mathbb{Y}$. We take a propeller Y in a plane, which is the union of three half-lines with the same endpoint 0 and that make 120° angles at 0. We obtain a first set of type $\mathbb{Y}$ as the product $Y_0 = Y \times V$, where V is a $(d-1)$-dimensional vector space that is orthogonal to the plane that contains Y. We shall call V the *spine* of $\mathbb{Y}_0$. Finally, $\mathcal{Y}$ is the collection of sets $\mathbb{Y}$ of the form $\mathbb{Y} = j(\mathbb{Y}_0)$, where j is an isometry of $\mathbb{R}^n$. The spine of $\mathbb{Y}$ is the image under j of the spine of $\mathbb{Y}_0$.

We now define the collection $\mathcal{T}$ of sets of type $\mathbb{T}$. The set $\mathcal{T}$ will be the collection of sets $T = g(T_0 \times V)$, where T_0 lies in a set $\mathcal{T}_0$ of 2-dimensional cones in $\mathbb{R}^{n-d+2}$ and V is the $(d-2)$-plane orthogonal to $\mathbb{R}^{n-d+2}$ in $\mathbb{R}^n$, and g is an isometry of $\mathbb{R}^n$.

Each $T_0 \in \mathcal{T}_0$ will be the cone over a set $K \subset \partial B(0,1)$, with the following properties. First, $K = \bigcup_{j \in J} C_j$ is a finite union of great circles, or closed arcs of great circles. Denote by Q the collection of extremities of the arcs C_j, $j \in J$; each point $y \in Q$ lies in exactly three C_j, y is an endpoint for each such C_j, and the three C_j make 120° angles at y. The C_j can only meet at their endpoints (and hence the full arcs of circles are disjoint from the rest of K). In addition, we choose a small constant $\eta_0 > 0$, which depends only on n, such that

$$H^1(C_j) \geq \eta_0 \quad \text{for } j \in J, \tag{1}$$

and if $y \in C_i$ and $\operatorname{dist}(y, C_j) \leq \eta_0$ for some other j, then C_i and C_j have a common extremity in $B(y, \operatorname{dist}(y, C_j))$. Finally, we exclude the case when T is a plane or a set of type $\mathbb{Y}$.

For a set $T \in \mathcal{T}$ as above, denote by $\hat{C}_j$, $j \in J$, the cone over C_j. Then we call $g(\hat{C}_j \times V)$, $j \in J$, the *d-faces* of T. We call the sets $g(0y \times V)$, $y \in Q$, the *$(d-1)$-faces* of Q. We call $g(V)$ the *spine* of T.

Finally, we set $\mathcal{Z} = \mathcal{P} \cup \mathcal{Y} \cup \mathcal{T}$.

Note that the cones of type $\mathbb{T}$ are not all minimal, but they are good enough to apply the generalized Reifenberg theorems [DDT, 1.1 and 2.2].

Although we give the descriptions for all dimensions, we need mostly the cases $d = 2$ and $d = 3$. Moreover, in [D, Section 14], G. David classifies the two-dimensional minimal cones in $\mathbb{R}^n$ into types $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$ described above, with a suitable choice of η_0 for cones of type $\mathbb{T}$.

We can now give the definition of a Hölder ball for a set $E \subset \mathbb{R}^n$.

DEFINITION 1.2. Let E be a closed set in $\mathbb{R}^n$. Suppose that $0 \in E$. We say that $B(0,r)$ is a *Hölder ball of type* $\mathbb{P}$, $\mathbb{Y}$ *or* $\mathbb{T}$ *with exponent* $1+\alpha$ if there exist a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and a minimal cone Y of dimension d, centered at the origin, of type $\mathbb{P}, \mathbb{Y}$ or $\mathbb{T}$, respectively, such that

$$|f(x) - x| \le \alpha r \quad \text{for } x \in B(0,r), \tag{1.2.1}$$

$$\begin{aligned}(1-\alpha)[|x-y|/r]^{1+\alpha} &\le |f(x)-f(y)|/r \\ &\le (1+\alpha)[|x-y|/r]^{1-\alpha} \quad \text{for } x,y \in B(0,r),\end{aligned} \tag{1.2.2}$$

$$E \cap B(0,(1-\alpha)r) \subset f(Y \cap B(0,r)) \subset E \cap B(0,(1+\alpha)r). \tag{1.2.3}$$

We then also say that E is *bi-Hölder equivalent* to Y in $B(0,r)$, with exponent $1+\alpha$.

Our main theorem is the following.

THEOREM 1. *Let E be an Al-minimal cone of dimension 3 in $\mathbb{R}^n$ and $x \in E \cap B(0,1)$. Let H be the tangent plane to $\partial B(0,1)$ at x and $E' = E \cap H$. Then for each $\alpha > 0$, there exists $r > 0$, which depends on x, such that $B(x,r)$ is a Hölder ball of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ for E' in H, with exponent $1+\alpha$.*

Our strategy is the following: for each $y \in B(x,r)$ and each radius t such that $B(y,t) \subset B(x,r)$, we shall find a minimal cone Y of dimension 2 in H such that $d_{y,t}(E',Y) \le \epsilon$ (see the beginning of Section 2 for the definition), where $\epsilon > 0$ depends on the exponent $1+\alpha$. We shall then use the generalized theorem of Reifenberg [DDT, 1.1 and 2.2] to conclude that E' is bi-Hölder equivalent to a two-dimensional minimal cone in H, with exponent $1+\alpha$.

2. Proof of Theorem 1. Let us give a list of notations that we shall use in this paper.

- H^d is the d-dimensional Hausdorff measure.
- $\theta_A(x,r) = H^d(A \cap B(x,r))/r^d$, where $A \subset \mathbb{R}^n$ is an H^d-measurable set and $x \in A$.
- $\theta_A(x) = \lim_{r\to 0} \theta_A(x,r)$ is called the *density* of A at x, if the limit exists and is finite.
- *Local Hausdorff distance* $d_H(E,F)$. Let $E, F \subset \mathbb{R}^n$ be closed sets and $H \subset \mathbb{R}^n$ be a compact set. We define

$$d_H(E,F) = \sup\{\text{dist}(x,F);\, x \in E \cap H\} + \sup\{\text{dist}(x,E);\, x \in F \cap H\},$$

when $E \cap H$ and $F \cap H$ are not empty. We use the convention that $\sup\{\text{dist}(x,F);\, x \in E \cap H\} = 0$ when $E \cap H$ is empty.

We also define

$$\begin{aligned} d_{x,r}(E,F) = \tfrac{1}{r} \sup\{&\text{dist}(z,F);\, z \in E \cap B(x,r)\} \\ &+ \sup\{\text{dist}(z,E);\, z \in F \cap B(x,r)\}, \end{aligned}$$

where E, F are closed sets which meet $B(x,r)$.

• *Convergence of a sequence of sets.* Let $U \subset \mathbb{R}^n$ be an open set, $\{E_k\}_{k=1}^{\infty}$ be a sequence of closed sets in U, and $E \subset U$. We say that $\{E_k\}$ converges to E in U, and we write $\lim_{k\to\infty} E_k = E$, if for each compact $H \subset U$,

$$\lim_{k\to\infty} d_H(E_k, E) = 0.$$

• *Blow-up limit.* Let $E \subset \mathbb{R}^n$ be a closed set and $x \in E$. A *blow-up limit* F of E at x is defined as

$$F = \lim_{k\to\infty} \frac{E-x}{r_k},$$

where $\{r_k\}$ is any sequence of positive numbers such that $\lim_{k\to\infty} r_k = 0$ and the limit exists in $\mathbb{R}^n$.

For two points $a, b \in \mathbb{R}^n$, we denote by ab the line passing through a and b, and by $\vec{ab}$ the half-line through a and b with starting point a.

Now we fix an Al-minimal cone $E \subset \mathbb{R}^n$ of dimension 3, centered at 0, and $x \in E \cap \partial B(0,1)$. For each $y \in E \cap \partial B(0,1)$, we denote by H_y the tangent plane to $\partial B(0,1)$ at y and write $E_y = E \cap H_y$. For simplicity, we set $H_x = H$. Note that since E is minimal, the density $\theta_E(y)$ always exists for all $y \in E$.

LEMMA 2.1. *Each blow-up limit of E at x is of the form $F = F' \times 0x$, where F' is a two-dimensional Al-minimal cone in H and $0x$ denotes the line from 0 through x.*

Proof. Let F be a blow-up limit of E at x. Then $F = \lim_{k\to\infty}(E-x)/r_k$ with $\lim_{k\to\infty} r_k = 0$. Let $y \in F$. We want to show that $y + 0x \subset F$. Setting $E_k = (E-x)/r_k$, as $\{E_k\}$ converges to F, we can find points $y_k \in E_k$ such that $\{y_k\}_{k=1}^{\infty}$ converges to y. Set $z_k = r_k y_k + x$; then $z_k \in E$ by definition of E_k, and z_k converges to x because r_k converges to 0. We fix $\lambda \in \mathbb{R}$ and we set $v_k = (1+\lambda r_k) z_k$. Then $v_k \in E$ as E is a cone centered at 0. We have $w_k = r_k^{-1}(v_k - x) \in E_k$. On the other hand,

$$\begin{aligned} w_k &= r_k^{-1}((1+\lambda r_k) z_k - x) = r_k^{-1}((1+\lambda r_k)(r_k y_k + x) - x) \\ &= r_k^{-1}(r_k y_k + \lambda r_k^2 y_k + \lambda r_k x) = y_k + \lambda x + \lambda r_k y_k, \end{aligned}$$

and we see that $\lim_{k\to\infty} w_k = y + \lambda x$. As $\{E_k\}$ converges to F, we see that $y + \lambda x \in F$. Now for each $y \in F$ and $\lambda \in \mathbb{R}$, we have $y + \lambda x \in F$, which implies that $F = F' \times 0x$ with $F' \subset F \cap H$. Next, as E is a minimal set and F is a blow-up limit of E at x, by [D, 7.31], F is a minimal cone centered at 0. But $F = F' \times 0x$, so by [D, 8.3], F' is a minimal cone in H, centered at x. ∎

By [D, Section 14], F' is of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ as above. Note that the classification of two-dimensional minimal cones in $\mathbb{R}^3$ was established earlier (see [He] and [Tay]). Now, since $F = F' \times \mathbb{R}$, F is also a cone of type $\mathbb{P}$, $\mathbb{Y}$

or $\mathbb{T}$ of dimension 3 in $\mathbb{R}^n$. If F is of type $\mathbb{P}$, we set $\theta_F(0) = d_P$, which is the Hausdorff measure of the three-dimensional unit ball. If F is of type $\mathbb{Y}$, we set $\theta_F(0) = d_Y$, which is the density at any point of the spine of a $\mathbb{Y}$ of dimension 3. Otherwise F is of type $\mathbb{T}$, and we deduce from [D, Section 14] that there exists a constant $d_T > d_Y$, which depends only on n, such that $\theta_F(0) \geq d_T$. Now by [D, 7.31], $\theta_E(x) = \theta_F(0)$, so we call the point $x \in E \cap \partial B(0,1)$ of type $\mathbb{P}$ if $\theta_E(x) = d_P$, of type $\mathbb{Y}$ if $\theta_E(x) = d_Y$, and finally of type $\mathbb{T}$ if $\theta_E(x) = \theta_F(0) \geq d_T$.

Lemma 2.2. *For each $\epsilon > 0$, we can find $r_x > 0$ such that if $r \leq r_x$, then there is a three-dimensional minimal cone $F(x,r)$ of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$, and whose spine passes through 0 and x, such that*

$$d_{x,r}(E, F(x,r)) \leq \epsilon.$$

Proof. Suppose that the lemma fails; then there is a sequence $\{r_k\}$; converging to 0 and such that for each minimal cone F as above,

$$d_{x,r_k}(E, F) > \epsilon. \tag{2.2.1}$$

Set $E_k = (E - x)/r_k$; without loss of generality, we may assume that $\{E_k\}$ converges in $\mathbb{R}^n$; set $\lim_{k\to\infty} E_k = M$.

Since M is a blow-up limit of E at x, by Lemma 2.1, $M = M' \times D_x$ where M' is a two-dimensional minimal cone in H centered at x, and D_x is the line $0x$. So M is a three-dimensional minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$, whose spine passes through 0 and x. Since $\{E_k\}$ converges to M, there exists $k > 0$ such that $d_{0,1}(E_k, M) \leq \epsilon$. This means that $d_{x,r_k}(E, x + M) \leq \epsilon$. But $M = M' \times D_x$, so $M = x + M$ and hence $d_{x,r_k}(E, M) \leq \epsilon$, which contradicts (2.2.1). ∎

Lemma 2.3. *For each $\delta > 0$, we can find $\epsilon > 0$ with the following properties:*

Let R be a radius. Let $I \in \mathbb{R}^n$ with $d(0,I) > 100R$ and C be a minimal cone of dimension 3 centered at I with the property that for each $y \in C \cap B(I,R)$ and each $y' \in 0y \cap B(I,R)$, there exists $z' \in C$ such that

$$d(y', z') < \epsilon R. \tag{2.3.1}$$

Then there exists a three-dimensional minimal cone Y_C, of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$, whose spine contains 0 and I, such that $d_{I,R/2}(C, Y_C) \leq \delta$.

Proof. Suppose that the lemma fails. By homogeneity, we can fix I such that $d(0,I) = 1000$. Then there exist a sequence $\epsilon_k \to 0$, radii $R_k < 10$ and minimal cones C_k centered at I such that each C_k satisfies the hypothesis corresponding to ϵ_k in the ball $B(I, R_k)$ but does not satisfy the conclusion. That is, for each minimal cone Y as above, $d_{I,1}(C_i, Y) > \delta$. Now we can find a subsequence $\{C_{i_j}\}_{j=1}^{\infty}$ which converges to a set E. Since each C_{i_j} is a minimal cone centered at I, so is E. We shall show that

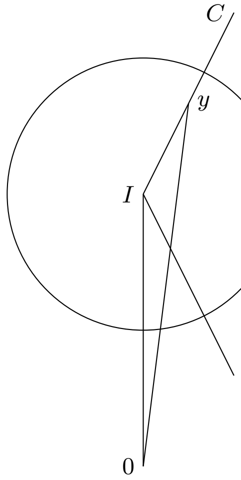

Fig. 1. Minimal cone C

(2.3.2) E is a three-dimensional minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ whose spine contains 0 and I.

We consider two cases.

CASE 1: $\limsup_{j\to\infty} R_{i_j} > 0$. In that case, without loss of generality, we may assume that $R_{i_j} = 1$ for all j. Now take $u \in E \cap B(I, 1/2)$; as $\{C_{i_j}\}$ converges to E, there exist $u_{i_j} \in C_{i_j}$, $j \geq 1$, such that $\{u_{i_j}\}$ converges to u.

If $u' \in 0u \cap B(I, 1/2)$, we take $y'_{i_k} \in 0u_{i_k}$ such that $|0y'_{i_k}|/|0u_{i_k}| = |0u'|/|0u|$, where $|AB|$ denotes the length of the segment AB. Then $|y'_{i_k}u'| = (|0u'|/|0u|).|u_{i_k}u|$, by Thales' theorem, and we deduce that $\lim_{j\to\infty} y'_{i_j} = u'$.

But for each j, there exists $u'_{i_j} \in C_{i_j}$ such that $d(u'_{i_j}, y'_{i_j}) < \epsilon_{i_j}$, by (2.3.1). So $\{u'_{i_j}\}$ converges to u' and thus $u' \in E$. Now

$$0u \cap B(I, 1/2) \subset E \quad \text{for each } u \in E \cap B(I, 1/2). \tag{2.3.3}$$

In particular $0I \cap B(I, 1/2) \subset E$.

In addition, E is a cone centered at I, so $\vec{Iu} \subset E$, where $\vec{Iu}$ denotes the half-line from I and passing through u. Now if u does not lie on the line $0I$, let $u_1 \in P \cap B(I, 1/2)$, where P is the open half-plane with boundary $0I$ and containing u. We take $u_2 \in [Iu_1]$, where $[AB]$ denotes the segment with endpoints A and B, which is close to I so that the half-line $0\vec{u}_2$ intersects the segment $[Iu]$. Set $u_3 = 0\vec{u}_2 \cap [Iu]$; then $u_3 \in E$ since E is a cone centered at I and $u \in E$. By (2.3.3), $u_2 \in 0u_3 \cap B(I, 1/2)$ belongs to E too. Finally, we use the fact that E is a cone centered at I to conclude that $u_1 \in E$.

So for each $u \in E \cap B(I, 1/2) \setminus 0I$, we have $P \cap B(I, 1/2) \subset E$, where P is the open half-plane with boundary $0I$ containing u. Since E is closed, we also have $0I \subset E$. We deduce that $E = E' \times 0I$, where E' is a two-dimensional set in the hyperplane orthogonal to $0I$ and passing through I. Since E is a minimal cone centered at I, so is E', by [D, 8.3]. Since E' is a two-dimensinal minimal cone, by [D, Section 14], E' is of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ and so is E, as $E = E' \times 0I$. We thus have (2.3.2) in this case.

CASE 2: $\limsup_{j\to\infty} R_{i_j} = 0$. In this case we want to show that

(2.3.4) for each $u \in E \cap B(I, 1/2) \setminus 0I$, we have $B(u, |Iu|/4) \cap l_u \subset E$, where l_u is the line passing through u and parallel to $0I$.

Indeed, we take a sequence $\{u_{i_j} \in C_{i_j}\}$ which converges to u as above. Let $u'_{i_j} = 0u_{i_j} \cap B(I, R_{i_j}/2)$. Then by (2.3.1), for each $z \in 0u'_{i_j} \cap B(u'_{i_j}, R_{i_j}/4)$, there exists $w \in C_{i_j}$ such that $d(z, w) \le \epsilon_{i_j} R_{i_j}$. Let $l_{u_{i_j}}$ be the line passing through u_{i_j} and parallel to $0u'_{i_j}$. Since C_{i_j} is a cone centered at I, by homothety for each $z' \in l_{u_{i_j}} \cap B(u_{i_j}, |Iu_{i_j}|/2)$, there exists $w' \in C_{i_j}$ such that

$$d(z', w') \le \epsilon_{i_j} |Iu_{i_j}| \le \epsilon_{i_j}. \tag{2.3.5}$$

Since $\limsup_{j\to\infty} R_{i_j} = 0$, the lines $l_{u_{i_j}}$ converge to the line l_u in $\mathbb{R}^n$. Next, if j is large enough, then $B(u, |Iu|/4) \subset B(u_{i_j}, |Iu_{i_j}|/2)$, and so for each $v \in B(u, |Iu|/4) \cap l_u$, there exists a sequence $v_{i_j} \in C_{i_j}$ which converges to v. We deduce $v \in E$ and we have (2.3.4).

Now for each $u \in E \cap B(I, 1/2) \setminus 0I$, by repeating this argument for the two endpoints of the segment $B(u, |Iu|/4) \cap l_u$, we can conclude that $l_u \cap B(I, 1/2) \subset E$. We want to show next that

$$l_u \subset E. \tag{2.3.6}$$

For this, take any point $v \in l_u$. Let $v' = Iv \cap B(I, 1/4)$ and let $u' \in Iu$ be such that the line $u'v'$ is parallel to l_u. Clearly $v' \in l_{u'} \cap B(I, 1/2)$, where $l_{u'}$ is defined just as l_u, and $u' \in E$ since E is the cone centered at I. So by (2.3.4), $v' \in E$ and hence $v \in E$, so that (2.3.6) follows.

Since E is closed, we deduce that $0I \subset E$; together with (2.3.6) we then see that E is of the form $E = E' \times 0I$, where E' is a two-dimensional set in the hyperplane orthogonal to $0I$ and passing through I. By the same arguments as above, we deduce that E is a three-dimensional minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ whose spine contains $0I$. We also have (2.3.2) in this case.

As $\lim_{j\to\infty} C_{i_j} = E$, there exists an integer $l > 0$ such that $d_{I,1}(C_{i_l}, E) < \delta/2$, which is a contradiction as E is a minimal cone of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ whose spine contains $0I$. ∎

We now want to use Lemma 2.3 to control the distance in the ball $B(x, r)$ between E and a three-dimensional minimal cone $C(x, r)$ of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ whose spine passes through 0 and x.

Lemma 2.4. *For each $\delta > 0$, we can find $\epsilon > 0$ such that the following properties hold. Suppose that $0 < r < 1/100$ satisfies*

$$|\theta_E(x,r) - \theta_E(x)| \leq \epsilon; \tag{2.4.1}$$

then there exists a minimal cone Y of dimension 3, of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$ and whose spine contains 0 and x, such that $d_{x,r/8}(E,Y) \leq \delta$. In addition, the type of Y is exactly the type of x.

Proof. By [D, 7.1], for each $\epsilon_1 > 0$ very small, to be chosen later, we can find $\epsilon > 0$ such that if (2.4.1) holds then there exists a minimal cone C of dimension 3 centered at x, such that

$$d_{x,r/2}(E,C) \leq \epsilon_1. \tag{2.4.2}$$

We now check the conditions of Lemma 2.3 for the cone C.

Since $d_{x,r/2}(E,C) \leq \epsilon_1$, whenever $z \in C \cap B(x,r/3)$, there exists $y \in E \cap B(x,r/2)$ such that $d(z,y) \leq \epsilon_1 r/2$. Because E is a cone centered at 0, the half-line $0y$ lies in E. Now if $z' \in 0z \cap B(x,r/3)$, take the point $y' \in 0y$ such that $y'z'$ is parallel to yz; then $d(y',z') \leq \epsilon_1 r$, and clearly $y' \in E \cap B(x,r/2)$. By (2.4.2), there exists $u \in C$ such that $d(u,y') \leq \epsilon_1 r$. Then $d(z',u) \leq 2\epsilon_1 r$. So the cone C satisfies the assumptions of Lemma 2.3 with radius $r/3$ and with constant $8\epsilon_1$; here x stands for I. Lemma 2.3 shows that for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that there exists a three-dimensional minimal cone Y, of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$, whose spine passes through 0 and x, such that

$$d_{x,r/4}(C,Y) \leq \epsilon_2. \tag{2.4.3}$$

From (2.4.2) and (2.4.3) we have

$$\begin{aligned} d_{x,r/8}(E,Y) &\leq 2(d_{x,r/4}(E,C) + d_{x,r/4}(C,Y)) \leq 2(2\epsilon_1 + \epsilon_2) \\ &= 4\epsilon_1 + 2\epsilon_2. \end{aligned} \tag{2.4.4}$$

For each $\delta > 0$, we can find $\epsilon > 0$ such that $4\epsilon_1 + 2\epsilon_2 \leq \delta$. So from (2.4.4) we have $d_{x,r/8}(E,Y) \leq \delta$, which we wanted to prove. ∎

Lemma 2.5. *Let C and C_1 be two cones centered at 0, and $\epsilon > 0$ be a small constant. Let $r \leq 1/100$ be a small radius, $y \in C \cap \partial B(0,1)$ and H_y be the hyperplane which is tangent to $\partial B(0,1)$ at y, $C' = C \cap H_y$, $C_1' = C_1 \cap H_y$. If $z \in C \cap B(y,r/2) \cap H_y$ and $t \leq r$ are such that $d_{z,t}(C,C_1) \leq \epsilon$, then*

$$d_{z,t/2}(C',C_1') \leq 2(1+r)\epsilon.$$

Proof. For each $w \in C' \cap B(z,t/2)$, there exists $w_1' \in C_1 \cap B(y,r)$ such that $d(w,w_1') \leq \epsilon t$ since $d_{z,t}(C,C_1) \leq \epsilon$. Now let w_1 be the intersection of the half-line $0w_1'$ with H_y. Then $w_1 \in C_1 \cap H_y$. We shall estimate the distance $d(w,w_1)$. By the triangular inequality, we have

$$d(w,w_1) \leq \frac{d(w,w_1')}{\sin(\widehat{ww_1w_1'})} = \frac{d(w,w_1')}{\sin(\widehat{ww_10})};$$

here $\widehat{xyz} \in [0, \pi[$, where x, y, x are points in $\mathbb{R}^n$, denotes the angle between the half-lines yx and yz. Next,

$$\sin(\widehat{ww_1 0}) = \frac{\operatorname{dist}(0, ww_1)}{d(0, w_1)} \geq \frac{1}{1+r},$$

since $\operatorname{dist}(0, ww_1) \geq \operatorname{dist}(0, H_y) = 1$ and $d(0, w_1) \leq d(0, y) + d(y, w_1) \leq 1 + r$.

So $d(w, w_1) \leq (1+r)\epsilon t$ for each $w \in C' \cap B(z, t/2)$, and it is clear that $w_1 \in B(z, t)$. By the same arguments, for each $w_1 \in C_1' \cap B(z, t/2)$, there exists $w \in C' \cap B(z', t)$ such that $d(w_1, w) \leq (1+r)\epsilon t$. ∎

We shall now prove Theorem 1. We consider three cases: where x is of type $\mathbb{P}$, $\mathbb{Y}$ or $\mathbb{T}$.

Hölder regularity near a point of type $\mathbb{P}$

THEOREM 2.6. *Suppose that x is a point of type $\mathbb{P}$. Then for each $\tau > 0$, we can find $\epsilon > 0$ such that if the radius $r > 0$ satisfies*

$$\theta_E(x, 2^8 r) - \theta_E(x) \leq \epsilon, \tag{2.6.1}$$

then $B(x, r)$ is a Hölder ball for E_x, with exponent $1 + \tau$.

We remark first that for each $\epsilon > 0$, we can find $r > 0$ such that (2.6.1) holds. Our ϵ does not depend on x, just on τ.

Proof of Theorem 2.6. The main idea is to show that for $y \in E_x \cap B(x, r)$ and $t \leq r$, we can find a 2-plane $P'(y, t)$ in H such that $d_{y,t}(E_x, P'(y, t)) \leq \delta$, where δ is a very small constant, to be chosen later. Then we can use [DDT, Theorem 1.1] to conclude that for each $\tau > 0$, we can find $\delta > 0$ such that E_x is bi-Hölder equivalent to a 2-plane in $B(x, r)$.

Now we start the proof. By Lemma 2.4, for each $\delta > 0$ very small, to be chosen later, we can find $\epsilon > 0$ such that if (x, r) satisfies (2.6.1), then there exists a 3-plane P which passes through 0 and x, such that

$$d_{x, 2^5 r}(E, P) \leq \delta. \tag{2.6.2}$$

Consider a point $y \in E_x \cap B(x, r)$. By [D, 16.43], for each $\delta_1 > 0$ very small, we can choose $\delta > 0$ such that if (2.6.2) holds for δ, then

$$\begin{aligned} H^3(E \cap B(y, 2^4 r)) &\leq H^3(P \cap B(y, (1+\delta_1) 2^4 r)) + \delta_1 (2^4 r)^3 \\ &\leq d_P((1+\delta_1) 2^4 r)^3 + \delta_1 (2^4 r)^3. \end{aligned} \tag{2.6.3}$$

We deduce that $\theta_E(y, 2^4 r) - d_P \leq \delta_1$ or $\theta_E(y, 2^4 r) \leq d_P + \delta_1$. But we know that $\theta_E(y) = d_P$, d_Y or d_T and by [D, 5.16], $\theta_E(y, \cdot)$ is a nondecreasing function. So if δ_1 is small enough, we have $\theta_E(y) = d_P$. Since $\theta_E(y, \cdot)$ is nondecreasing, $d_P \leq \theta_E(y, t) \leq d_P + \delta_1$ for $0 < t \leq 2^4 r$. With $\theta_E(y) = d_P$, we have

$$\theta_E(y, t) - \theta_E(y) \leq \delta_1 \quad \text{for } 0 < t \leq 2^4 r. \tag{2.6.4}$$

By Lemma 2.4, for each $\delta_2 > 0$, we can choose $\delta_1 > 0$ such that there exists a 3-plane $P(y,t)$ which passes through 0 and y, such that

$$d_{y,t}(E, P(y,t)) \le \delta_2 \quad \text{for } 0 < t \le 2r. \tag{2.6.5}$$

Set $P'(y,t) = P(y,t) \cap H$. Applying Lemma 2.5 for two cones E and $P(y,t)$ centered at 0, we have

$$d_{y,t/2}(E_x, P'(y,t)) \le 6\delta_2 \quad \text{for } 0 < t \le 2r. \tag{2.6.6}$$

But now $P'(y,t)$ is a 2-plane in H, so for each $y \in E_x \cap B(x,r)$ and each $t \le r$, there exists a 2-plane $P_1(y,t)$ in H such that

$$d_{y,t}(E_x, Y_1(y,t)) \le 6\delta_2. \tag{2.6.7}$$

By [DDT, Theorem 1.1], we conclude that, for each $\tau > 0$, we can choose $\delta_2 > 0$, and then $\epsilon > 0$, such that if (2.6.7) holds, then E_x is bi-Hölder equivalent to a 2-plane P in H, with Hölder exponent $1 + \tau$. ∎

Hölder regularity near a point of type $\mathbb{Y}$

PROPOSITION 2.7. *Let $y \in \partial B(0,1)$ and $r < 1/2$. For each $\tau > 0$ we can find $\epsilon > 0$ such that if $Y(y,r)$ is a minimal cone of type $\mathbb{Y}$ of dimension 3, whose spine passes through 0 and y, which satisfies*

$$d_{y,r}(E, Y(y,r)) \le \epsilon, \tag{2.7.1}$$

then there exists a $\mathbb{Y}$-point of E_y in $B(y,\tau r)$. Here, a $\mathbb{Y}$-point of E_y is a $\mathbb{Y}$-point of E which belongs to E_y.

Proof. We first take $\epsilon > 0$ very small, to be chosen later. Suppose that the proposition fails; then there exist a radius $0 < r < 1/2$ and a three-dimensional minimal cone $Y(y,r)$ of type $\mathbb{Y}$, whose spine passes through 0 and y, such that

$$d_{y,r}(E, Y(y,r)) \le \epsilon, \tag{2.7.2}$$

$$\text{for each } z \in E_y \cap B(y,\tau r),\ z \text{ is not a } \mathbb{Y}\text{-point}. \tag{2.7.3}$$

We take a point $z \in E_y \cap B(y,\tau r)$. Since $B(z,r/4) \subset B(y,r)$ and $d_{y,r}(E, Y(y,r)) \le \epsilon$, we have $d_{z,r/4}(E, Y(y,r)) \le 4 d_{y,r}(E, Y(y,r)) \le 4\epsilon$. So by [D, 16.43], for each $\delta > 0$ we can find $\epsilon > 0$ such that

$$\begin{aligned} H^3(E \cap B(z,r/4)) &\le H^3(Y(y,r) \cap B(z,(1+\delta)r/4)) + \delta(r/4)^3 \\ &\le d_Y((1+\delta)r/4)^3 + \delta(r/4)^3. \end{aligned} \tag{2.7.4}$$

So if we take δ small enough, we have $H^3(E \cap B(z,r/4)) < d_T(r/4)^3$, thus $\theta_E(z,r/4) < d_T$. Next, $\theta_E(z) \le \theta_E(z,r/4) < d_T$, since E is a minimal cone. So z cannot be a $\mathbb{T}$-point, and since it is not a $\mathbb{Y}$-point either,

$$z \text{ is a } \mathbb{P}\text{-point}. \tag{2.7.5}$$

Let L be the spine of $Y(y,r)$. Then L is a 2-plane through 0 and y. Let F_1, F_2, F_3 be three half-planes of dimension 3 which form $Y(y,r)$. Then

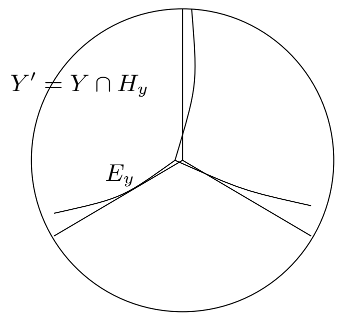

Fig. 2. Intersection of E with H_y

F_1, F_2, F_3 have L as common boundary, and the angle between any two of them is $120°$. Set $Y'(y,r) = Y(y,r) \cap H_y$ and $w_i = F_i \cap \partial B(y, \tau r/2) \cap H_y$, $i = 1,2,3$. Then $Y'(y,r)$ is a two-dimensional minimal cone of type $\mathbb{Y}$ in H_y, centered at y, with spine $L' = L \cap H_y$. Then $\operatorname{dist}(w_i, L') = d(w_i, y) = \tau r/2$ for $1 \le i \le 3$. Next, $d_{y,r}(E, Y(y,r)) \le \epsilon$, so by Lemma 2.5, $d_{y,r/2}(E_y, Y'(y,r)) \le 2(1+r)\epsilon \le 6\epsilon$. Thus for each $1 \le i \le 3$, there is $z_i \in E_y$ such that $d(z_i, w_i) \le 3\epsilon r$. It is clear that $z_i \in B(y, 5\tau r/8)$ if we choose ϵ small enough. Now

$$d_{z_i,\tau r/4}(E, Y(y,r)) \le \frac{4}{\tau} d_{y,r}(E, Y(y,r)) \le 4\epsilon/\tau \tag{2.7.6}$$

for $1 \le i \le 3$. By [D, 16.43], for each $\delta_1 > 0$, we can choose $\epsilon > 0$ such that if (2.7.6) holds for ϵ, then

$$\begin{aligned} H^3(E \cap B(z_i, \tau r/8)) &\le H^3(Y(y,r) \cap B(z_i, (1+\delta_1)\tau r/8)) + \delta_1(\tau r/8)^3 \\ &= H^3(F_i \cap B(z_i, (1+\delta_1)\tau r/8)) + \delta_1(\tau r/8)^3 \\ &\le d_P(\tau r/8)^3 + C\delta_1(\tau r/8)^3, \end{aligned} \tag{2.7.7}$$

since $\operatorname{dist}(w_i, L) = \tau r/4$, so $\operatorname{dist}(z_i, L) \ge \operatorname{dist}(w_i, L) - d(w_i, z_i) \ge \tau r/4 - \epsilon r$, so $B(z_i, (1+\delta_1)\tau r/8)$ does not meet L. Then, in $B(z_i, (1+\delta_1)\tau r/8)$, $Y(y,r)$ coincides with F_i, which is a half-plane of dimension 3.

From (2.7.7) we have

$$\theta_E(z_i, \tau r/8) \le d_P + C\delta_1, \tag{2.7.8}$$

which implies that z_i is a $\mathbb{P}$-point for $1 \le i \le 3$ if we take δ_1 small enough. By Theorem 2.6, for each $\alpha > 0$, we can choose $\delta_1 > 0$ such that if (2.7.8) holds, then

(2.7.9) for $1 \le i \le 3$, the set E_y is bi-Hölder equivalent to a 2-plane P_i in $B(z_i, \tau r/2^{11}) \cap H_y$, with Hölder exponent $1+\alpha$.

Now as each $z \in E_y \cap B(y, \tau r)$ is a $\mathbb{P}$-point, by the proof of Theorem 2.6, there is a radius $r_z \le \tau r/8$ such that E is bi-Hölder equivalent to a 3-plane P_z in the ball $B(z, r_z)$, with Hölder exponent $1+\alpha$.

We see that the set E_y satisfies the following conditions:

(i) The minimal cone $Y'(y,r)$ of dimension 2 of type $\mathbb{Y}$ centered at y, which is $Y(y,r) \cap H_y$, satisfies
$$d_{y,r/2}(E_y, Y'(y,r)) \leq 6\epsilon.$$

(ii) Let L' be the spine of $Y'(y,r)$ and F_i', $1 \leq i \leq 3$, the three half-planes of dimension 2 which form $Y'(y,r)$. Then there are three points w_i, $1 \leq i \leq 3$, such that for each i, $\mathrm{dist}(z_i, L') = \tau r/4$, $w_i \in F_i'$ and w_1, w_2, w_3 lie in the same plane of dimension 2 which is orthogonal to L'. Next, there are three points $z_i \in E_y$, $1 \leq i \leq 3$, such that $d(z_i, w_i) \leq 3\epsilon r$, and in the ball $B(z_i, \tau r/2^{11})$, E_y is bi-Hölder equivalent to a 2-plane P_i in H_y, with Hölder exponent $1+\alpha$.

(iii) For each $z \in E_y$, there is a radius $r_z \leq \tau r/2^{11}$ such that in the ball $B(z, r_z)$, E_y is bi-Hölder equivalent to a 2-plane P_z, with Hölder exponent $1+\alpha$.

We can adapt the techniques of [D, Section 17]. G. David showed there that if a two-dimensional almost minimal set F in $\mathbb{R}^n$ and a cone Y of type $\mathbb{Y}$ of dimension 2 whose spine passes through a point x satisfy $d_{x,r}(F, Y) \leq \epsilon$, then there must be a $\mathbb{Y}$-point of F in $B(x, r/1000)$. To prove this, G. David supposes that in $B(x, r/1000)$, F contains only $\mathbb{P}$-points; then he shows that the set $F_1 = F \cap B(x, r/1000)$ has the same properties (i)–(iii). He next shows that it is not possible for a set F_1 to have those properties.

We can now use the same techniques for our set E_y, and conclude that it is not possible for E_y to satisfy (i)–(iii). Proposition 2.7 follows. ∎

THEOREM 2.8. *Suppose that x is a point of type $\mathbb{Y}$. Then for each $\alpha > 0$ there exists $\epsilon > 0$ such that if the radius $r > 0$ satisfies*

$$\theta_E(x, 2^{11} r) - \theta_E(x) < \epsilon, \tag{2.8.1}$$

$$2^{11} r < \epsilon, \tag{2.8.2}$$

then in the ball $B(x,r)$, E_x is bi-Hölder equivalent to a two-dimensional minimal cone Y of type $\mathbb{Y}$ in H_x and centered at x, with Hölder exponent $1+\alpha$.

The proof uses the fact that for each $\delta > 0$, we can choose $\epsilon > 0$ such that if (2.8.1) and (2.8.2) hold, then for each $y \in E_x \cap B(x,r)$ and for each $0 < t \leq r$, there exists a two-dimensional minimal cone $Z(y,t)$ in H_x such that $d_{y,t}(E_x, Z(y,t)) \leq \delta$. We remark that for each $\epsilon > 0$, we can choose $r > 0$ such that (2.8.1) and (2.8.2) hold.

Proof of Theorem 2.8. By Lemma 2.4, for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if the radius r satisfies (2.8.1), then there exists a minimal cone $Y(x, 2^8 r)$ of dimension 3, of type $\mathbb{Y}$ and whose spine passes through 0 and x, such that

$$d_{x,2^8r}(E, Y(x, 2^8r)) \leq \epsilon_1. \tag{2.8.3}$$

We take a point $y \in E_x \cap B(x, 2^7r)$. We have

$$d_{y,2^7r}(E, Y(x, 2^8r)) \leq 2d_{x,2^8r}(E, Y(x, 2^8r)) \leq 2\epsilon_1.$$

Then by [D, 16.43], for each $\epsilon_2 > 0$, we can choose $\epsilon_1 > 0$ such that

$$\begin{aligned} H^3(E \cap B(y, 2^7r)) &\leq H^3(Y(x, 2^8r) \cap B(y, (1 + \epsilon_2)2^7r)) + \epsilon_2(2^7r)^3 \\ &\leq d_Y(2^7r)^3 + C\epsilon_2(2^7r)^3, \end{aligned} \tag{2.8.4}$$

which implies

$$\theta_E(y, 2^7r) \leq d_Y + C\epsilon_2. \tag{2.8.5}$$

So if ϵ_2 is small enough, we have $\theta_E(y) \leq \theta_E(y, 2^7r) \leq d_Y + C\epsilon_1 < d_T$. So by the classification preceding Lemma 2.2, y can only be of type $\mathbb{P}$ or $\mathbb{Y}$. We consider two cases.

Case 1: *y is of type* $\mathbb{Y}$. We have $\theta_E(y) = d_Y$. By (2.8.5), $\theta_E(y, 2^7r) - \theta_E(y) \leq C\epsilon_1$. Since E is a minimal set, the function $\theta_E(y, \cdot)$ is nondecreasing, so $0 \leq \theta_E(y, t) - \theta_E(y) \leq C\epsilon_1$ for $0 \leq t \leq 2^7r$. By Lemma 2.4, for each $\epsilon_2 > 0$, we can choose $\epsilon_1 > 0$ such that for each $t \leq 2^4r$, there exists a three-dimensional minimal cone $Y(y, t)$, of type $\mathbb{Y}$, whose spine passes through 0 and y, and satisfies

$$d_{y,t}(E, Y(y, t)) \leq \epsilon_2. \tag{2.8.6}$$

Set $Y_1(y, t) = Y(y, t) \cap H_x$; then $Y_1(y, t)$ is the union of three half-planes of dimension 2 with common boundary a line L'. We see that $L' = L \cap H_x$ where L is the spine of $Y(y, t)$. Since $y \in B(x, 2^7r)$ and $Y(y, t)$ is a $\mathbb{Y}$ of dimension 3 whose spine passes through y and 0, there is a two-dimensional minimal cone $Y'(y, t)$ in H_x with the same spine L' such that

$$d_{y,1}(Y_1(y, t), Y'(y, t)) \leq Cr \leq C\epsilon. \tag{2.8.7}$$

Now by Lemma 2.5, $d_{y,t/2}(E_x, Y_1(y, t)) \leq 2(1 + t)d_{y,t}(E, Y(y, t)) \leq 4\epsilon_2$ for $t \leq 2^4r$. This fact together with (2.8.7) gives

$$\begin{aligned} d_{y,t/4}(E_x, Y'(y, t)) &\leq 2\big(d_{y,t/2}(E_x, Y_1(y, t)) + d_{y,t/2}(Y_1(y, t), Y'(y, t))\big) \\ &\leq C_1(\epsilon + \epsilon_2) \end{aligned} \tag{2.8.8}$$

for $t \leq 2^4r$. Set $\epsilon_3 = C_1(\epsilon + \epsilon_2)$; then by (2.8.8), for each $t \leq 2^4r/4 = 4r$ and for each $\mathbb{Y}$-point $y \in E_x \cap B(x, r)$, there is a two-dimensional minimal cone $Y'(y, t) \in H_x$ of type $\mathbb{Y}$ such that

$$d_{y,t}(E_x, Y'(y, t)) \leq \epsilon_3. \tag{2.8.9}$$

This is what we need for $\mathbb{Y}$-points in $E_x \cap B(x, 2^7r)$. We then note that, for each $\mathbb{Y}$-point $y \in B(x, 2^7r)$ and $t \leq 2^4r$, $Y(y, t)$ is the minimal cone as in (2.8.6), and for $t \leq 4r$, $Y'(y, t)$ is the minimal cone as in (2.8.9).

Case 2: *y is of type $\mathbb{P}$.* Here we consider only the case $y \in B(x,r)$. We set $E_Y = \{z \in E_x \cap B(x,4r) : z$ is a $\mathbb{Y}$-point$\}$. By the proof of Theorem 2.6, there is a radius $r_y > 0$ such that in the ball $B(y,r_y)$, there are only $\mathbb{P}$-points of E; as a consequence, we have $\mathrm{dist}(y,E_Y) > 0$. Set $d = \mathrm{dist}(y,E_Y)$; then $d \le d(y,x) \le r$. We take $u \in E_Y$ such that $d(y,u) \le 11d(y)/10$; then it is clear that $u \in B(x,2^3 r)$.

We take $Y(u,2d(y))$ as in (2.8.6) and denote by L the spine of $Y'(y,2d(y))$. We want to show that

$$\mathrm{dist}(y,L) \ge d(y)/10. \tag{2.8.10}$$

Suppose that (2.8.10) does not hold. Then there is a point $w \in L'$ such that $d(y,w) < d(y)/10$. Next, $d(w,u) \le d(w,y)+d(y,u) \le 11d(y)/10+d(y)/10 \le 3d(y)/2$ and so $B(w,d(y)/10) \subset B(u,2d(y))$. Thus

$$\begin{aligned} d_{w,d(y)/10}(E,Y(u,2d(y))) &\le \frac{2d(y)}{d(y)/10} d_{u,2d(y)}(E,Y(u,2d(y))) \\ &\le 20\epsilon_2. \end{aligned} \tag{2.8.11}$$

Since w belongs to the spine of $Y(u,2d(y))$, we can apply Proposition 2.7 for E and w for $\tau = 1/100$. So we can find $\epsilon_2 > 0$ such that if (2.8.11) holds, then there is a $\mathbb{Y}$-point ξ of E in the ball $B(w,d(y)/100)$ and then $d(\xi,y) \le d(\xi,w)+d(w,y) < d(y)/3$. Let ξ' be the intersection of the half-line 0ξ with E_x. Because E is a cone centered at 0 and ξ is a $\mathbb{Y}$-point, it is clear that $\xi' \in E_Y$ and $d(\xi',y) \le 2d(\xi,y) < 2d(y)/3$, which is a contradiction. We have thus proved (2.8.10).

Next, since $d_{u,2d(y)}(E,Y(u,2d(y))) \le \delta_2$ and $B(y,d(y)/20) \subset B(u,2d(y))$, by [D, 16.43], for each $\epsilon_4 > 0$ we can find $\epsilon_2 > 0$ such that

(2.8.12)
$$\begin{aligned} \theta_E(y,d(y)/20) &= (d(y)/20)^{-3} H^3(E \cap B(y,d(y)/20)) \\ &\le (d(y)/20)^{-3}[H^3(Y(u,2d(y)) \cap B(y,(1+\epsilon_4)d(y)/20)) \\ &\qquad\qquad + \epsilon_4(d(y)/20)^3] \\ &\le d_P + C\epsilon_4 = \theta_E(y) + C\epsilon_4. \end{aligned}$$

We explain the last line: since $\mathrm{dist}(y,L) \ge 11d(y)/10$, it follows that $B(y,(1+\epsilon_4)d(y)/20)$ does not meet the spine L of $Y(u,2d(y))$, so that in the ball $B(y,(1+\epsilon_4)d(y)/20)$, $Y(u,2d(y))$ coincides with a 3-plane P, and then

$$\begin{aligned} H^3(Y(u,2d(y)) \cap B(y,(1+\epsilon_4)d(y)/20)) &= H^3(P \cap B(y,(1+\epsilon_4)d(y)/20)) \\ &\le d_P((1+\epsilon_4)d(y)/20)^3. \end{aligned}$$

Since $\theta_E(y,\cdot)$ is nondecreasing, we deduce from (2.8.12) that

$$0 \le \theta_E(y,t) - \theta_E(y) \le C\epsilon_4 \quad \text{for } t \le d(y)/20. \tag{2.8.13}$$

By Lemma 2.4, for each $\epsilon_5 > 0$, we can choose $\epsilon_4 > 0$ such that if (2.8.13) holds, then there is a 3-plane $P(y,t)$ which passes through 0 and y, such that

$$d_{y,t/8}(E, P(y,t)) \leq \epsilon_5 \quad \text{for } t \leq d(y)/20. \tag{2.8.14}$$

If we set $P'(y,t)$ to be the intersection of $P(y,t)$ with H_x, then $P'(y,t)$ is a 2-plane, and satisfies, by Lemma 2.5,

$$d_{y,t/16}(E_x, P'(y,t)) \leq 4\epsilon_5 \quad \text{for } t \leq d(y)/20. \tag{2.8.15}$$

Now consider the case when $d(y)/320 \leq t \leq r$. We keep the same point u as above, that is, $u \in E_Y$ such that $d(u,y) \leq 11d(y)/10$. We now have $t + 2d(y) \leq 4r$, so we can take the cone $Y'(u, t+2d(y))$ as in (2.8.9), thus

$$\begin{aligned} &d_{y,t}(E_x, Y'(u, t+2d(y))) \\ &\qquad \leq \frac{t+2d(y)}{t} d_{u,t+2d(y)}(E_x, Y'(u, t+2d(y))) \leq 700\epsilon_3. \end{aligned} \tag{2.8.16}$$

Now (2.8.9), (2.8.15) and (2.8.16) together show that for each $\epsilon_6 > 0$, we can choose $\epsilon > 0$ such that for each $y \in E_x \cap B(x,r)$ and each $t \leq r$, there is a minimal cone $P'(y,t) \subset H_x$ of dimension 2, of type $\mathbb{P}$ or $\mathbb{Y}$, such that

$$d_{y,t}(E_x, Y'(y,t)) \leq \epsilon_6. \tag{2.8.17}$$

By [DDT, Theorem 1.1], for each $\alpha > 0$, we can find $\epsilon_6 > 0$ such that if (2.8.17) holds, then E_x is bi-Hölder equivalent to a minimal cone of dimension 2, of type $\mathbb{Y}$, in the ball $B(x,r)$, with Hölder exponent $1+\alpha$. ∎

Hölder regularity near a point of type $\mathbb{T}$

THEOREM 2.9. *Suppose that x is a point of type $\mathbb{T}$. Then for each $\alpha > 0$, we can find $\epsilon > 0$ such that if the radius $r > 0$ satisfies*

$$\theta_E(x, 2^{14}r) - \theta_E(x) \leq \epsilon, \tag{2.9.1}$$

$$2^{14}r \leq \epsilon, \tag{2.9.2}$$

then in the ball $B(x,r)$, E_x is bi-Hölder equivalent to a minimal cone T' of dimension 2, of type $\mathbb{T}$, in the plane H_x and centered at x.

We note that for each $\epsilon > 0$, we can always find $r > 0$ which satisfies (2.9.1) and (2.9.2). Our strategy will be the same as in Theorem 2.8: we show that for each $\delta > 0$, we can choose $\epsilon > 0$ such that if (2.9.1) and (2.9.2) hold, then for each $y \in E_x \cap B(x,r)$ and for each $0 < t \leq r$, there exists a two-dimensional minimal cone $Z(y,t)$ in H_x such that $d_{y,t}(E_x, Z(y,t)) \leq \delta$.

Proof of Theorem 2.9. Since $\theta_E(x,\cdot)$ is nondecreasing, we have $0 \leq \theta_E(x,t) - \theta_E(x) \leq \epsilon$ for $0 < t \leq 2^{14}r$. By Lemma 2.4, for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that for each $t \leq 2^{11}r$, there exists a three-dimensional minimal cone $T(x,t)$ of type $\mathbb{T}$, whose spine passes through 0 and x, such that

$$d_{x,t}(E, T(x,t)) \le \epsilon_1. \tag{2.9.3}$$

Consider a point $y \in E_x \cap B(x, 2^{10}r)$ with $y \ne x$. Set $\eta = 2^{-12}\eta_0/10$, where η_0 is the constant in (1) (before Definition 1.2). Then $(1+2\eta)|x-y| \le 2^{11}r$ and so we can take the cone $T(x, (1+2\eta)|x-y|)$ to satisfy (2.9.3). Next, since $B(y, \eta|x-y|) \subset B(x, (1+2\eta)|x-y|)$, we have

$$\begin{aligned} &d_{y,\eta|x-y|}(E, T(x,(1+2\eta)|x-y|)) \\ &\quad \le \frac{(1+2\eta)|x-y|}{\eta|x-y|} d_{x,(1+2\eta)|x-y|}(E, T(x,(1+2\eta)|x-y|)) \le 2\eta^{-1}\epsilon_1. \end{aligned} \tag{2.9.4}$$

We want to show that

(2.9.5) $T(x, (1+2\eta)|x-y|)$ coincides with a cone Y_y of type $\mathbb{Y}$ in the ball $B(y, \eta_0|x-y|/10)$.

To see this, it suffices to show that

(2.9.6) $T' = T(x, (1+2\eta)|x-y|) \cap H_x$ coincides with a two-dimensional cone of type $\mathbb{Y}$ in $B(y, \eta_0|x-y|/5) \cap H_x$.

But now since the spine of $T(x, (1+2\eta)|x-y|)$ passes through 0 and x, T' is a two-dimensional minimal cone of type $\mathbb{T}$ in H_x and centered at x. So by the same arguments as in [D, (16.61)], we have (2.9.6), and hence (2.9.5).

Now (2.9.4) gives us

$$d_{y,\eta|x-y|}(E, Y_y) \le 2\eta^{-1}\epsilon_1. \tag{2.9.7}$$

By the same arguments as for (2.8.4), for each $\epsilon_2 > 0$ we can find $\epsilon_1 > 0$ such that if (2.9.7) holds, then

$$\theta_E(y) \le \theta_E(y, \eta|x-y|/2) \le d_Y + C\epsilon_2. \tag{2.9.8}$$

So if we take ϵ_2 small enough, we have, for each $y \in E_x \cap B(x, 2^{10}r)$ and $y \ne x$, $\theta_E(y) \le d_Y + C\epsilon_2 < d_T$, and hence y can only be a point of type $\mathbb{P}$ or $\mathbb{Y}$. Since E is a cone centered at the origin, each $z \in E \cap B(x, 2^9 r)$ with $z \ne x$ can only be a point of type $\mathbb{P}$ or $\mathbb{Y}$. We consider two cases.

CASE 1: *y is of type* $\mathbb{Y}$. By (2.9.7), $\theta_E(y, \eta|x-y|/2) \le d_Y + C\epsilon_2 = \theta_E(y) + C\epsilon_2$. As $\theta_E(y,\cdot)$ is nondecreasing, we have $\theta_E(y,t) \le \theta_E(y) + C\epsilon_2$ for $0 < t \le \eta|x-y|/2$. By Lemma 2.4, for each $\epsilon_3 > 0$, we can find $\epsilon_2 > 0$ such that there exists a three-dimensional minimal cone $Y(y,t)$ of type $\mathbb{Y}$, whose spine passes through 0 and y, such that

$$d_{y,t}(E, Y(y,t)) \le \epsilon_3 \quad \text{for } 0 < t \le \eta|x-y|/16. \tag{2.9.9}$$

Set $Y_1(y,t) = Y(y,t) \cap H_x$; then $Y_1(y,t)$ is a two-dimensional cone centered at y in the plane H_x. Since E and $Y(y,t)$ are cones centered at 0, by Lemma 2.5,

$$d_{y,t/2}(E_x, Y_1(y,t)) \le 4\epsilon_3 \quad \text{for } 0 < t \le \eta|x-y|/16. \tag{2.9.10}$$

Next, since $Y(y,t)$ is a three-dimensional minimal cone of type $\mathbb{Y}$ whose spine passes through 0 and y, and $y \in B(x, 2^{10}r)$, there exists a two-dimensional minimal cone $Y'(y,t) \subset H_x$ of type $\mathbb{Y}$ centered at y such that

$$d_{y,1}(Y_1(y,t), Y'(y,t)) \leq Cr. \tag{2.9.11}$$

By (2.9.10) and (2.9.11) we have

$$\begin{aligned} d_{y,t/4}(E_x, Y'(y,t)) &\leq 2[d_{y,t/2}(E_x, Y_1(y,t)) + d_{y,t/2}(Y_1(y,t), Y'(y,t))] \\ &\leq 2(4\epsilon_3 + Cr) \leq C_1(\epsilon_3 + \epsilon) \end{aligned} \tag{2.9.12}$$

for $0 < t \leq \eta|x-y|/16$.

We consider the case when $r \geq t \geq \eta|x-y|/64$, and we only consider $y \in B(x,r)$ in this case. By (2.9.3), we can take the cone $T(x, 2(t+|x-y|))$ whose spine is $0x$ and which satisfies $d_{x,2(t+|x-y|)}(E, T(x, 2(t+|x-y|))) \leq \epsilon_1$. Set $T'(x, 2(t+|x-y|)) = H_x \cap T(x, 2(t+|x-y|))$. Then $T'(x, 2(t+|x-y|))$ is a two-dimensional minimal cone of type $\mathbb{T}$. By Lemma 2.5,

$$d_{x,t+|x-y|}(E_x, T'(x, 2(t+|x-y|))) \leq 4\epsilon_1. \tag{2.9.13}$$

Since $B(y,t) \subset B(x, t+|x-y|)$, we have

$$\begin{aligned} d_{y,t}(E_x, T'(x, t+\rho(y))) &\leq \frac{t+\rho(y)}{t} d_{x,t+\rho(y)}(E_x, T'(x, t+\rho(y))) \\ &\leq \frac{t+|x-y|}{t} 4\epsilon_1 \leq \frac{2^{10}}{\eta}\epsilon_1. \end{aligned} \tag{2.9.14}$$

From (2.9.12) and (2.9.14), for $y \in B(x,r)$ and $0 < t \leq r$, there exists a two-dimensional minimal cone $Z'(y,t) \subset H_x$ of type $\mathbb{Y}$ or $\mathbb{T}$ such that

$$d_{y,t}(E_x, Z'(y,t)) \leq \epsilon_4, \tag{2.9.15}$$

with $\epsilon_4 = \max\{C_1(\epsilon_3+\epsilon), (2^{10}/\eta)\epsilon_1\}$.

Case 2: *y is of type* $\mathbb{P}$. Recall that each $z \in E_x \cap B(x, 2^{10}r)$, $z \neq x$, can only be of type $\mathbb{P}$ or $\mathbb{Y}$. Let E_Y be the set of $\mathbb{Y}$-points of E in $B(x,4r)$, and $d = \min\{\operatorname{dist}(y, E_Y), |x-y|\}$.

We have two subcases:

Subcase 1: $d > \eta|x-y|$. Let T_2 be the union of the 2-faces of $T(x, 2|x-y|)$. We want to show that

$$\operatorname{dist}(y, T_2) > d/10. \tag{2.9.16}$$

Otherwise there is a 2-face L of $T(x, 2|x-y|)$ (see the definition of d-face prior to Definition 1.2) and a point $z \in L$ such that $d(y,z) = \operatorname{dist}(y,L) < d/10$. Next, $|z-x| > |x-y| - d/10 > |x-y|/2$ and so $\operatorname{dist}(z, 0x) > \operatorname{dist}(y, 0x) - d/10 > |x-y|/2$. By the same argument as for (2.9.5), in the ball $B(z, \eta d/2)$, $T(x, 2|x-y|)$ coincides with a three-dimensional minimal cone Y_z of type $\mathbb{Y}$

and whose spine passes through 0 and z. In addition

$$d_{z,\eta d/2}(E,Y_z) \le \frac{2|x-y|}{\eta d/2} d_{x,2|x-y|}(E,T(x,2|x-y|)) \le \frac{4}{\eta^2}\epsilon_1. \tag{2.9.17}$$

By Proposition 2.7, we can choose ϵ_1 small enough such that if (2.9.17) holds, there exists a $\mathbb{Y}$-point $z' \in B(z,\eta d/4)$. We deduce that $d(y,z') \le d(y,z)+d(z,z') < d/5$. But $z' \in E_Y$, a contradiction. Hence (2.9.16) holds.

Now since $\operatorname{dist}(y,0x) \ge d$ and $\operatorname{dist}(y,T_2) > d/10$, by a simple geometrical argument, in $B(y,\eta d), T(x,2|x-y|)$ coincides with a 3-plane P_y whose spine passes through 0 and x. In fact, P_y is the 3-face of $T(x,2|x-y|)$ which is nearest to y. Since $B(y,\eta d) \subset B(x,2|x-y|)$, we obtain

$$d_{y,\eta d}(E,P_y) \le \frac{2|x-y|}{\eta d} d_{x,2|x-y|}(E,T(x,2|x-y|)) \le \frac{2}{\eta^2}\epsilon_1. \tag{2.9.18}$$

By [D, 16.43], for each $\epsilon_5 > 0$, we can find $\epsilon_1 > 0$ such that if (2.9.18) holds, then $\theta_E(y,\eta d/2) \le d_P + \epsilon_5$. Since E is a minimal cone, $\theta_E(y,t) \le d_P + \epsilon_5$ for $0 < t \le \eta d/2$. By Lemma 2.4, for each $\epsilon_6 > 0$, we can find $\epsilon_5 > 0$ such that there exists a 3-plane $P(y,t)$ which passes through 0 and y, such that

$$d_{y,t/8}(E,P(y,t)) \le \epsilon_6 \quad \text{for } 0 < t \le \eta d/2. \tag{2.9.19}$$

By Lemma 2.5, the 2-plane $P'(y,t) = P(y,t) \cap H_x$ satisfies

$$d_{y,t/16}(E_x,P'(y,t)) \le 4 d_{y,t/8}(E,P(y,t)) \le 4\epsilon_6 \quad \text{for } t \le \eta d/2. \tag{2.9.20}$$

For $|x-y|/2 \ge t > \eta d/32$, let $T'(x,2|x-y|) = H_x \cap T(x,2|x-y|)$. Then $T'(x,2|x-y|)$ is a two-dimensional minimal cone of type $\mathbb{T}$ in H_x and centered at x. By Lemma 2.5, we have $d_{x,|x-y|}(E_x,T'(x,2|x-y|)) \le 4\epsilon_1$. Since $B(y,t) \subset B(x,|x-y|)$, we obtain

$$\begin{aligned} d_{y,t}(E_x,T'(x,2|x-y|)) &\le \frac{|x-y|}{t} d_{x,|x-y|}(E_x,T'(x,2|x-y|)) \\ &\le \frac{128}{\eta^2}\epsilon_1. \end{aligned} \tag{2.9.21}$$

For $r \ge t > |x-y|/2$, we set $T'(x,2t+|x-y|) = T(x,2t+|x-y|) \cap H_x$. Then $T'(x,2t+|x-y|)$ is a two-dimensional minimal cone of type $\mathbb{T}$ in H_x. As above, we have $d_{x,t+|x-y|/2}(E_x,T'(x,2t+|x-y|)) \le 4\epsilon_1$. Since $B(y,t) \subset B(x,t+|x-y|/2)$, we obtain

$$\begin{aligned} &d_{y,t}(E_x,T'(x,2t+|x-y|)) \\ &\quad \le \frac{t+|x-y|/2}{t} d_{x,t+|x-y|/2}(E_x,T'(x,2t+|x-y|)) \le 8\epsilon_1/\eta. \end{aligned} \tag{2.9.22}$$

Now (2.9.20)–(2.9.22) are all that we need for Subcase 1.

Subcase 2: $d \le \eta|x-y|$. By definition of E_Y, there exists a $\mathbb{Y}$-point such that $d(y,z) < 2d$. This implies $z \in B(x,3|x-y|/2)$ and $d(z,x) > |x-y|/2$ and $\operatorname{dist}(z,0x) > \operatorname{dist}(y,0x) - 2d > |x-y|/2$. By the same arguments as for (2.9.5), $T(x, 2|x-y|)$ coincides with a three-dimensional minimal cone Y_z of type $\mathbb{Y}$ in $B(z, 2^{12}\eta|x-y|)$. This is clear that the spine of Y_z passes through 0. Since $B(z, 2^{12}\eta|x-y|) \subset B(x, 2|x-y|)$, we have

$$
\begin{aligned}
(2.9.23) \qquad d_{z,2^{12}\eta|x-y|}(E, Y_z) &= d_{z,2^{12}\eta|x-y|}(E, T(x, 2|x-y|)) \\
&\leq \frac{2|x-y|}{2^{12}\eta|x-y|} d_{x,2|x-y|}(E, T(x, 2|x-y|)) \\
&\leq \frac{1}{2^{11}\eta}\epsilon_1.
\end{aligned}
$$

Since z is a $\mathbb{Y}$-point, by [D, 16.43], for each $\epsilon_7 > 0$, we can find $\epsilon_1 > 0$ such that if (2.9.23) holds then

$$
(2.9.24) \qquad \theta_E(x, 2^{11}\eta|x-y|) - \theta_E(z) \leq \epsilon_7.
$$

In addition

$$
(2.9.25) \qquad 2^{12}\eta|x-y| \leq r \leq \epsilon.
$$

We see that (2.9.24) and (2.9.25) are the hypotheses of Theorem 2.8, with radius $\eta|x-y|$ and with constant $\epsilon_8 = \max\{\epsilon, \epsilon_1\}$. As in the proof of Theorem 2.8, for each $\epsilon_9 > 0$, we can find $\epsilon_8 > 0$ such that for $t \leq 2\eta|x-y|$, there is a two-dimensional minimal cone $Y'(y,t) \subset H_x$ of type $\mathbb{P}$ or $\mathbb{Y}$ such that

$$
(2.9.26) \qquad d_{y,t}(E_x, Y'(y,t)) \leq \epsilon_9.
$$

The case $r \geq t > 2\eta|x-y|$ is the same as (2.9.22). We now have all that we need for Subcase 2.

Now we can conclude that, for each $\epsilon_{10} > 0$, we can find $\epsilon > 0$ such that for $y \in E_x \cap B(x,r)$ and for $t \leq r$, there exists a two-dimensional minimal cone $Y'(y,t) \subset H_x$ such that $d_{y,t}(E_x, Y'(y,t)) \leq \epsilon_{10}$. By [DDT, 2.2], for each $\alpha > 0$, we can find $\epsilon > 0$ such that if (2.9.1) and (2.9.2) hold, then $B(x,r)$ is a Hölder ball of type $\mathbb{T}$ for E_x, with exponent $1+\alpha$. ∎

Acknowledgments. I wish to thank Guy David for a number of helpful suggestions on this paper.

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Received 4.5.2012
and in final form 27.12.2012 (2787)