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A normality criterion for meromorphic functions having multiple zeros

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Abstract. We prove a normality criterion for a family of meromorphic functions having multiple zeros which involves sharing of a non-zero value by the product of functions and their linear differential polynomials.

1. Introduction, definitions and results. Let f and g be two meromorphic functions in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$ the functions f and g have the same set of a-points ignoring multiplicities, we say that f and g *share the value a IM* (ignoring multiplicities).

In 1959 W. K. Hayman [6] proposed the following:

Theorem A. *If f is a transcendental meromorphic function in $\mathbb{C}$, then $f^n f'$ assumes every finite non-zero complex value infinitely often for any positive integer n.*

Hayman [6] himself proved Theorem A for $n \geq 3$, and $n \geq 2$ if f is entire. Further it was proved by E. Mues [12] for $n \geq 2$ and by J. Clunie [3] for $n \geq 1$ if f is entire; also by W. Bergweiler and A. Eremenko [1] and by H. H. Chen and M. L. Fang [2] for $n = 1$. Thus Theorem A is completely established.

In relation to Theorem A, Hayman [7] proposed the following conjecture on normal families.

Theorem B (Hayman's Conjecture). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, n be a positive integer and a be a non-zero finite complex number. If $f^n f' \neq a$ in D for each $f \in \mathfrak{F}$, then $\mathfrak{F}$ is normal.*

Theorem B was proved by L. Yang and G. Zhang [19, 20] (for $n \geq 5$ and $n \geq 2$ for a family of holomorphic functions), by Y. X. Gu [5] (for $n = 3, 4$), by I. B. Oshkin [13] (for holomorphic functions and $n = 1$; see also [9]) and

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by X. C. Pang (for $n \geq 2$). It is indicated in [14] that the case $n = 1$ is a consequence of the theorem of Chen–Fang [2].

In 2009 Q. Lu and Y. X. Gu [10] considered the general order derivative in Theorem B for $n = 1$. Their result can be stated as follows:

THEOREM C. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, k be a positive integer and a be a finite non-zero complex number. If for each $f \in \mathfrak{F}$, the zeros of f have multiplicities at least $k+2$ and f satisfies $ff^{(k)} \neq a$ for $z \in D$, then $\mathfrak{F}$ is normal.*

Recently J. Xu and W. Cao [18] improved Theorem C by including meromorphic functions having zeros with multiplicities at least $1+k$.

In 2011 D. W. Meng and P. C. Hu [11] improved the result of J. Xu and W. Cao [18] by including the possibility when $ff^{(k)}$ is allowed to assume the value a. The following is the result of Meng and Hu [11].

THEOREM D ([11]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, k be a positive integer and a be a finite non-zero complex number. If for each $f \in \mathfrak{F}$, the zeros of f have multiplicities at least $1+k$, and for each pair of functions $f, g \in \mathfrak{F}$, $ff^{(k)}$ and $gg^{(k)}$ share the value a IM, then $\mathfrak{F}$ is normal.*

Let f be a meromorphic function in $D \subset \mathbb{C}$ and k be a positive integer. A *linear differential polynomial* $L(f)$ is defined as

$$L(f) = a_1 f^{(1)} + \cdots + a_k f^{(k)},$$

where $a_1, \ldots, a_k$ $(\neq 0)$ are constants.

In the paper we investigate the situation when in Theorem D, $ff^{(k)}$ and $gg^{(k)}$ are respectively replaced by $fL(f)$ and $gL(g)$. The following is our main result.

THEOREM 1.1. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that $L(f) \not\equiv 0$ for $f \in \mathfrak{F}$, k be a positive integer and a be a finite non-zero complex number. If for each $f \in \mathfrak{F}$, the zeros of f have multiplicities at least $1+k$, and for each pair of functions $f, g \in \mathfrak{F}$, $fL(f)$ and $gL(g)$ share the value a IM, then $\mathfrak{F}$ is normal.*

Since the zeros of f^{k+1} have multiplicities at least $k+1$, the following is a simple consequence of Theorem 1.1.

COROLLARY 1.1. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, k be a positive integer and a be a finite non-zero complex value. If for each pair of functions $f, g \in \mathfrak{F}$, $f^{k+1}(f^{k+1})^{(k)}$ and $g^{k+1}(g^{k+1})^{(k)}$ share the value a IM, then $\mathfrak{F}$ is normal.*

The following example establishes the necessity of the hypothesis on the multiplicities of zeros.

Example 1.1 (cf. [11]). Let D be a domain containing the point $1/2$ and

$$\mathfrak{F} = \left\{ f_m : f_m(z) = mz - \frac{m}{2} + \frac{a}{m} \text{ for } m = 1, 2, \dots \right\},$$

where a is a non-zero finite complex value. For distinct positive integers m and l we have $f_m f'_m = m^2(z - 1/2) + a$ and $f_l f'_l = l^2(z - 1/2) + a$. Hence $f_m f'_m$ and $f_l f'_l$ share the value a CM. We note that each f_m has only simple zeros. Since

$$f_m^{\#}(1/2) = \frac{m^3}{m^2 + |a|^2} \to \infty \quad \text{as } m \to \infty,$$

by Marty's criterion [16, p. 75] the family $\mathfrak{F}$ is not normal in D.

2. Lemmas. In this section we present some necessary lemmas.

Lemma 2.1 ([16, p. 101], [15]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. If $\mathfrak{F}$ is not normal in D, then there exist*

(i) *a number r with $0 < r < 1$,*
(ii) *points z_j satisfying $|z_j| < r$,*
(iii) *functions $f_j \in \mathfrak{F}$,*
(iv) *positive numbers $\rho_j \to 0$ as $j \to \infty$*

such that $f_j(z_j + \rho_j\zeta) \to g(\zeta)$ as $j \to \infty$ locally spherically uniformly, where g is a non-constant meromorphic function in $\mathbb{C}$ with $g^{\#}(\zeta) \le g^{\#}(0) = 1$. In particular, g has order at most 2.

Lemma 2.2. *Let $R = A/B$ be a rational function and B be non-constant. Then $(R^{(k)})_\infty \le (R)_\infty - k$, where k is a positive integer, $(R)_\infty = \deg(A) - \deg(B)$ and $\deg(A)$ denotes the degree of A.*

Proof. By the quotient rule of differentiation we get

$$R^{(1)} = \frac{A^{(1)}B - AB^{(1)}}{B^2}$$

and so $(R^{(1)})_\infty \le \deg(A) - \deg(B) - 1 = (R)_\infty - 1$. Now the lemma follows by induction. ∎

Lemma 2.3. *Let f be a non-constant rational function, k be a positive integer and a be a non-zero finite complex number. If f has only zeros of multiplicities at least $1 + k$, then $fL(f) - a$ has at least two distinct zeros.*

Proof. We consider the following cases.

Case 1. Let $fL(f) - a$ have exactly one zero at z_0.

Subcase 1.1. Let f be a non-constant polynomial. Since f has only zeros of multiplicities at least $1 + k$, the degree of f is at least $1 + k$ (≥ 2).

So $fL(f)$ is a polynomial of degree at least $k+2$ (≥ 3). Since z_0 is the only zero of $fL(f)-a$, we can put

$$fL(f)-a = A(z-z_0)^m, \tag{2.1}$$

where $A \neq 0$, $m \geq k+2$.

We see that a zero of f is a zero of $fL(f)$ with multiplicity at least $k+2$ and so it is a zero of $(fL(f)-a)' = (fL(f))'$ with multiplicity at least $1+k$. Since $(fL(f)-a)' = Am(z-z_0)^{m-1}$ has only one zero at z_0, and f, being non-constant, must have a zero, we see that z_0 is a zero of f. This contradicts (2.1).

Subcase 1.2. Let f be a non-polynomial rational function. We put

$$f(z) = A\frac{(z-\alpha_1)^{m_1}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}\cdots(z-\beta_t)^{n_t}}, \tag{2.2}$$

where A is a non-zero constant and $m_i \geq 1+k$ ($i=1,\dots,s$) and $n_j \geq 1$ ($j=1,\dots,t$) are integers. We further put $M = m_1+\cdots+m_s$ and $N = n_1+\cdots+n_t$.

From (2.2) we get upon differentiation

$$f^{(p)}(z) = \frac{(z-\alpha_1)^{m_1-p}\cdots(z-\alpha_s)^{m_s-p}g_p(z)}{(z-\beta_1)^{n_1+p}\cdots(z-\beta_t)^{n_t+p}}, \tag{2.3}$$

where g_p is a polynomial for $p=1,\dots,k$. Hence from (2.2) and (2.3) we obtain

$$\begin{aligned} fL(f) &= \sum_{p=1}^{k}\frac{(z-\alpha_1)^{2m_1-p}\cdots(z-\alpha_s)^{2m_s-p}g_p(z)}{(z-\beta_1)^{2n_1+p}\cdots(z-\beta_t)^{2n_t+p}} \\ &= \frac{(z-\alpha_1)^{2m_1-k}\cdots(z-\alpha_s)^{2m_s-k}g(z)}{(z-\beta_1)^{2n_1+k}\cdots(z-\beta_t)^{2n_t+k}} = \frac{P}{Q}, \quad \text{say,} \end{aligned} \tag{2.4}$$

where P, Q and g are polynomials. Since $fL(f)-a$ has exactly one zero z_0, from (2.4) we get

$$fL(f) = a + \frac{B(z-z_0)^l}{(z-\beta_1)^{2n_1+k}\cdots(z-\beta_t)^{2n_t+k}} = \frac{P}{Q}, \tag{2.5}$$

where l is a positive integer and B is a non-zero constant.

From (2.4) and (2.5) we get upon differentiation

$$(fL(f))' = \frac{(z-\alpha_1)^{2m_1-k-1}\cdots(z-\alpha_s)^{2m_s-k-1}G_1(z)}{(z-\beta_1)^{2n_1+k+1}\cdots(z-\beta_t)^{2n_t+k+1}} \tag{2.6}$$

and

$$(fL(f))' = \frac{(z-z_0)^{l-1}G_2(z)}{(z-\beta_1)^{2n_1+k+1}\cdots(z-\beta_t)^{2n_t+k+1}}, \tag{2.7}$$

where G_1 and G_2 are polynomials. From (2.2) and (2.3) we obtain $(f)_\infty = M-N$ and $(f^{(p)})_\infty = (M-N)-(s+t)k+\deg(g_p)$. So by Lemma 2.2 we deduce

$$\deg(g_p) \leq p(s+t-1), \tag{2.8}$$

for $p = 1, \ldots, k$.

From (2.4) and (2.5) again we get

$$(fL(f))_\infty = 2(M-N) - k(s+t) + \deg(g) \tag{2.9}$$

and

$$(fL(f)-a)_\infty = l - 2N - kt. \tag{2.10}$$

Formulas (2.6) and (2.7) lead to

$$\big((fL(f))'\big)_\infty = 2(M-N) - (k+1)(s+t) + \deg(G_1) \tag{2.11}$$

and

$$\big((fL(f))'\big)_\infty = l - 1 - 2N - (k+1)t + \deg(G_2). \tag{2.12}$$

Let $\phi_p(z) = \{(z-\alpha_1)\cdots(z-\alpha_s)\}^p$ and $\psi_q(z) = \{(z-\beta_1)\cdots(z-\beta_s)\}^q$. Then $\deg(\phi_p) = sp$ and $\deg(\psi_q) = tq$. Also by a simple calculation we see that $g(z)$ as in (2.4) is

$$g(z) = \phi_0\psi_0 g_k(z) + \phi_1\psi_1 g_{k-1}(z) + \cdots + \phi_{k-2}\psi_{k-2}g_2(z) + \phi_{k-1}\psi_{k-1}g_1(z).$$

Hence by (2.8) we get

$$\begin{aligned} \deg(g) &\leq \max\{\deg(g_k), \deg(g_{k-1}) + s + t, \ldots, \\ &\qquad\qquad \deg(g_1) + (k-1)(s+t)\} \\ &\leq \max\{(s+t-1)k, (s+t-1)k+1, (s+t-1)k+2, \ldots, \\ &\qquad\qquad (s+t-1)k + (k-1)\} \\ &= (s+t-1)k + (k-1). \end{aligned} \tag{2.13}$$

Using Lemma 2.2 from (2.9)–(2.13) we get

$$\deg(G_1) \leq (s+t-1)(k+1) + (k-1) \tag{2.14}$$

and

$$\deg(G_2) \leq t. \tag{2.15}$$

From (2.4) and (2.5) we see that $z_0 \notin \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. So (2.6) and (2.7) together imply that $(z-\alpha_1)^{2m_1-k-1}\cdots(z-\alpha_s)^{2m_s-k-1}$ is a factor of G_2. Therefore by (2.15) we get

$$2M - (k+1)s \leq \deg(G_2) \leq t. \tag{2.16}$$

Since $M \geq (k+1)s$, from (2.16) we deduce

$$s \leq \frac{t}{k+1}. \tag{2.17}$$

Suppose that $l \geq 2N + kt$. Then from (2.6), (2.7) we see that $(z-z_0)^{l-1}$ is a factor of G_1. Hence in view of (2.14) we get

$$l - 1 \leq \deg(G_1) \leq (k+1)(s+t-1) + (k-1)$$

and so $2N+kt \le (k+1)(s+t-1)+k$, which by (2.17) implies $2t \le 2N \le (k+1)s+(t-1) \le 2t-1$, a contradiction. Therefore $l < 2N+kt$.

From (2.4) and (2.5) again we find that

$$2M - ks + \deg(g) = \deg(P) = \deg(Q) = 2N + kt.$$

Hence from (2.13) we obtain

$$2N + kt \le 2M - ks + (s+t-1)k + (k-1) = 2M + kt - 1.$$

This implies, in view of (2.16),

$$2M \le (k+1)s + t \le M + N \le M + M - \frac{1}{2} = 2M - \frac{1}{2},$$

a contradiction.

CASE 2. Let $fL(f) - a$ have no zero. Then f cannot be a polynomial because in this case $fL(f)$ becomes a polynomial of degree at least $k+2$ (≥ 3). Hence f is a non-polynomial rational function. Now putting $l = 0$ in (2.5) and proceeding as in Subcase 1.2 we arrive at a contradiction. This proves Lemma 2.3. ■

A *quasi-differential polynomial* P of a meromorphic function f is defined by

$$P(z) = \sum_{i=1}^{t} \phi_i(z),$$

where

$$\phi_i(z) = \alpha_i(z) \prod_{j=0}^{p} (f^{(j)}(z))^{S_{ij}}, \qquad \alpha_i(z) \not\equiv 0$$

is a meromorphic function such that $m(r, \alpha_i) = S(r, f)$ and S_{ij}'s are non-negative integers. The number

$$\gamma_P = \max_{1 \le i \le n} \sum_{j=0}^{p} S_{ij}$$

is called the *degree* of the quasi-differential polynomial P.

LEMMA 2.4 ([4], see also [8, p. 39]). *Let f be a non-constant meromorphic function and Q_1, Q_2 be quasi-differential polynomials in f with $Q_2 \not\equiv 0$. Let n be a positive integer and $f^n Q_1 = Q_2$. If $\gamma_{Q_2} \le n$, then $m(r, Q_1) = S(r, f)$, where γ_{Q_2} is the degree of Q_2.*

LEMMA 2.5. *Let f be a transcendental meromorphic function having no zero of multiplicity less than $1+k$ such that $L(f) \not\equiv 0$. If a is a finite non-zero complex number, then $F = fL(f) - a$ has infinitely many zeros.*

Proof. Without loss of generality we may put $a = 1$. First we verify that $fL(f)$ is non-constant. If $fL(f) \equiv K$, a constant, then we see that f has neither any pole nor any zero. So there exists an entire function α such that $f = e^{\alpha}$. Hence $Q(\alpha')e^{2\alpha} \equiv K$, where Q is a differential polynomial in α'. This implies, by the first fundamental theorem, $T(r, e^{2\alpha}) = T(r, Q(\alpha')) + O(1) = S(r, e^{2\alpha})$, a contradiction. Therefore $fL(f)$ is non-constant. Since

$$F = fL(f) - 1, \tag{2.18}$$

we get

$$T(r, F) = O(T(r, f)). \tag{2.19}$$

Also

$$fh = -\frac{F'}{F}, \tag{2.20}$$

where

$$h = \frac{f'}{f}L(f) + L'(f) - L(f)\frac{F'}{F}. \tag{2.21}$$

As F is non-constant, by (2.20) we see that $h \not\equiv 0$. By Lemma 2.4 applied to (2.20) we get, in view of (2.19),

$$m(r, h) = S(r, f). \tag{2.22}$$

Since a pole of f is a simple pole of F'/F, it follows from (2.20) that a pole of f with multiplicity q (≥ 2) is a zero of h with multiplicity $q - 1$. Hence

$$N_{(2}(r, \infty; f) \leq 2N(r, 0; h), \tag{2.23}$$

where $N_{(2}(r, \infty; f)$ denotes the counting function of multiple poles of f.

If possible, we suppose that F has only a finite number of zeros. Hence

$$N(r, 0; F) = O(\log r) = S(r, f). \tag{2.24}$$

Also we deduce from (2.20) that a simple pole of f is neither a zero nor a pole of h.

A zero of f with multiplicity q ($\geq 1+k$) is a zero of $F' = f'L(f) + fL'(f)$ with multiplicity at least $2q - (k+1)$. Hence from (2.20) we see that it is not a pole of h. Therefore the poles of h are provided by the zeros of F. Hence by (2.24) we get

$$N(r, \infty; h) \leq N(r, 0; F) = S(r, f). \tag{2.25}$$

So from (2.22) and (2.25) we obtain

$$T(r, h) = S(r, f). \tag{2.26}$$

Hence (2.23) and (2.26) imply

$$N_{(2}(r, \infty; f) = S(r, f). \tag{2.27}$$

By (2.20), (2.26) and the first fundamental theorem we get

$$m(r, f) \leq m(r, 1/h) + m(r, F'/F) = S(r, f). \tag{2.28}$$

Combining (2.27) and (2.28) we obtain

$$T(r,f)=N_{1)}(r,\infty;f)+S(r,f), \tag{2.29}$$

where $N_{1)}(r,\infty;f)$ denotes the counting function of simple poles of f.

Let z_0 be a simple pole of f. Then $h(z_0)\neq 0,\infty$. Let, in some neighbourhood of z_0,

$$f(z)=\frac{c_1}{z-z_0}+c_0+O(z-z_0) \tag{2.30}$$

$$h(z)=h(z_0)+h'(z_0)(z-z_0)+O(z-z_0)^2, \tag{2.31}$$

where $c_1\neq 0$. Differentiating both sides of (2.30) we get

$$f^{(j)}(z)=\frac{(-1)^j c_1 j!}{(z-z_0)^{j+1}}+O(1), \tag{2.32}$$

for $j=1,2,\ldots$. Therefore

$$L(f)=\sum_{j=1}^{k} a_j\frac{(-1)^j c_1 j!}{(z-z_0)^{j+1}}+O(1), \tag{2.33}$$

$$L'(f)=\sum_{j=1}^{k} a_j\frac{(-1)^{j+1} c_1 (j+1)!}{(z-z_0)^{j+2}}+O(1). \tag{2.34}$$

Also from (2.20) and (2.21) we have

$$fh=f'L(f)+fL'(f)+f^2L(f)h. \tag{2.35}$$

Now from (2.30)–(2.35) we obtain

$$\begin{aligned}
&\left\{\frac{c_1}{z-z_0}+c_0+O(z-z_0)\right\}\{h(z_0)+h'(z_0)(z-z_0)+O(z-z_0)^2\}\\
&=\left\{\frac{-c_1}{(z-z_0)^2}+O(1)\right\}\left\{\sum_{j=1}^{k} a_j\frac{(-1)^j c_1 j!}{(z-z_0)^{j+1}}+O(1)\right\}\\
&\quad+\left\{\frac{c_1}{z-z_0}+c_0+O(z-z_0)\right\}\left\{\sum_{j=1}^{k} a_j\frac{(-1)^{j+1} c_1 (j+1)!}{(z-z_0)^{j+2}}+O(1)\right\}'\\
&\quad+\left\{\frac{c_1}{z-z_0}+c_0+O(z-z_0)\right\}^2\{h(z_0)+h'(z_0)(z-z_0)+O(z-z_0)^2\}\\
&\qquad\times\left\{\sum_{j=1}^{k} a_j\frac{(-1)^j c_1 j!}{(z-z_0)^{j+1}}+O(1)\right\}.
\end{aligned}$$

Comparing the coefficients of $1/(z-z_0)^{k+3}$ and $1/(z-z_0)^{k+2}$ on both sides, we respectively get

$$c_1h(z_0)=k+2 \tag{2.36}$$

and

$$\frac{c_0}{c_1} = \frac{(k+1)a_{k-1}}{k(k+3)a_k} - \frac{1}{k+3} - \frac{(k+2)h'(z_0)}{(k+3)h(z_0)}. \tag{2.37}$$

From (2.30) and (2.32) we obtain

$$\frac{f'}{f} = \frac{-1}{z-z_0} + \frac{c_0}{c_1} + O(z-z_0). \tag{2.38}$$

Also from (2.20), (2.30) and (2.31) we get

$$\begin{aligned} -\frac{F'}{F} = fh &= \{h(z_0) + h'(z_0)(z-z_0) \\ &\quad + O(z-z_0)^2\}\left\{\frac{c_1}{z-z_0} + c_0 + O(z-z_0)\right\} \\ &= c_1 h(z_0)\left\{\frac{1}{z-z_0} + \frac{h'(z_0)}{h(z_0)} + \frac{c_0}{c_1}\right\} + O(z-z_0). \end{aligned} \tag{2.39}$$

Formulas (2.36)–(2.39) lead to

$$\begin{aligned} (k+2)(k+3)\frac{f'}{f} - (k+3)\frac{F'}{F} + (k+1)(k+2)\frac{h'(z_0)}{h(z_0)} \\ = \frac{2(k+1)(k+2)a_{k-1}}{ka_k} - 2(k+2) + O(z-z_0). \end{aligned} \tag{2.40}$$

Let us put

$$g = (k+2)(k+3)\frac{f'}{f} - (k+3)\frac{F'}{F} + (k+1)(k+2)\frac{h'}{h}$$

and

$$A = \frac{2(k+1)(k+2)a_{k-1}}{ka_k} - 2(k+2).$$

If $g \equiv A$, then upon integration we get

$$f^{(k+2)(k+3)}h^{(k+1)(k+2)} = F^{k+3}e^{Az+B}, \tag{2.41}$$

where B is a constant.

Let z_1 be a zero of f with multiplicity q ($\geq k+1$). Then from (2.41) we see that z_1 is a pole of h with multiplicity p such that $q(k+2)(k+3) = p(k+1)(k+2)$ and so

$$p = \frac{k+3}{k+1}q > q.$$

Therefore z_1 is a pole of fh with multiplicity $p-q$. Since $F(z_1) = -1$, we arrive at a contradiction by (2.20). So f has no zero. Hence by the first fundamental theorem we get

$$\begin{aligned} N(r,1/L(f)) &= N(r,0;L(f)/f) \le T(r,L(f)/f) + O(1) \\ &= N(r,L(f)/f) + S(r,f) = k\overline{N}(r,\infty;f) + S(r,f) \\ &= N(r,\infty;L(f)) - N(r,\infty;f) + S(r,f) \end{aligned}$$

and so

$$N(r,\infty;f) \le N(r,\infty;L(f)) - N(r,0;L(f)) + S(r,f). \tag{2.42}$$

From (2.21) we have

$$\frac{1}{L(f)} = \frac{1}{h}\left(\frac{f'}{f} + \frac{L'(f)}{L(f)} - \frac{F'}{F}\right)$$

and so $m(r,0;L(f)) = S(r,f)$. By the first fundamental theorem this implies

$$T(r,L(f)) = N(r,0;L(f)) + S(r,f). \tag{2.43}$$

From (2.42) and (2.43) we get $N(r,\infty;f) = S(r,f)$, which contradicts (2.29). Therefore $g \not\equiv A$.

Now by (2.40) we see that $g(z_0) = A$ and so by (2.23), (2.24), (2.26) and the first fundamental theorem we get

$$\begin{aligned} N_{1)}(r,\infty;f) &\le N(r,A;g) \le T(r,g) + O(1) \le N(r,g) + S(r,f) \\ &\le \overline{N}(r,0;f) + \overline{N}(r,0;F) + \overline{N}(r,0;h) + \overline{N}(r,\infty;h) \\ &\quad + N_{(2}(r,\infty;f) + S(r,f) \\ &\le \frac{1}{k+1}N(r,0;f) + S(r,f) \le \frac{1}{k+1}T(r,f) + S(r,f), \end{aligned}$$

which contradicts (2.29). This proves Lemma 2.5. ■

3. Proof of Theorem 1.1. We suppose that $\mathfrak{F}$ is not normal in D. Then by Lemma 2.1 there exist

(i) a number r, $0 < r < 1$,
(ii) points z_j satisfying $|z_j| < r$,
(iii) functions $f_j \in \mathfrak{F}$,
(iv) positive numbers $\rho_j \to 0$

such that $f_j(z_j + \rho_j\zeta) = g_j(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$ such that $g^{\#}(\zeta) \le g^{\#}(0) = 1$. Also the order of $g(\zeta)$ is at most 2.

We note by Hurwitz's theorem that zeros of g are of multiplicities at least $k+1$. We see that

$$f_j(z_j + \rho_j\zeta)L(f_j(z_j + \rho_j\zeta)) - a = g_j(\zeta)L(g_j(\zeta)) - a \to g(\zeta)L(g(\zeta)) - a$$

as $j \to \infty$ uniformly in any compact subset of $\mathbb{C}$ which does not contain any pole of g.

We now verify that $L(g) \not\equiv 0$. If possible, let $L(g) \equiv 0$. Then g is an entire function. Also

$$a_2\frac{g^{(2)}}{g^{(1)}}+a_3\frac{g^{(3)}}{g^{(1)}}+\cdots+a_k\frac{g^{(k)}}{g^{(1)}}\equiv -a_1. \tag{3.1}$$

If $(a_1,\ldots,a_{k-1})=(0,\ldots,0)$, then from (3.1) we get $g^{(k)}\equiv 0$ and so g is a polynomial of degree at most $k-1$, which is impossible as g has no zero of multiplicity less than $k+1$. Hence $(a_1,\ldots,a_{k-1})\neq(0,\ldots,0)$.

If $k=1$, then $L(g)\equiv 0$ implies $g^{(1)}\equiv 0$, which is impossible as g is non-constant. So $k\geq 2$ and we see from (3.1) that g has no zero. Hence by Hurwitz's theorem g_j has no zero and no pole for all large values of j.

We put $g_j(\zeta)=e^{\alpha_j(\zeta)}$, where $\alpha_j(\zeta)$ is an entire function. Now $g_jL(g_j)=Q_j(\alpha_j')e^{2\alpha_j}$, where $Q_j(\alpha_j')$ is a differential polynomial in α_j'.

As $T(r,Q_j(\alpha_j'))=S(r,e^{2\alpha_j})$ and $L(g_j)\not\equiv 0$, by the second fundamental theorem we see that

$$\overline{N}(r,a;g_jL(g_j))=T(r,g_jL(g_j))+S(r,g_jL(g_j)). \tag{3.2}$$

Since $g_jL(g_j)-a\to gL(g)-a=-a$ as $j\to\infty$ uniformly in any compact subset of $\mathbb{C}$, by Hurwitz's theorem $g_jL(g_j)-a$ has no zero for all large values of j, a contradiction to (3.2). Therefore $L(g)\not\equiv 0$. Also following the reasoning given in the first paragraph of the proof of Lemma 2.5 we can verify that $gL(g)$ is non-constant.

Now by Lemmas 2.3 and 2.5 we can choose ζ_0 and ζ_0^* as two distinct zeros of $gL(g)-a$. Since zeros are isolated points, there exist two open discs D_1 and D_2 with centres at ζ_0, ζ_0^* respectively such that $D_1\cup D_2$ contains only two zeros ζ_0, ζ_0^* of $gL(g)-a$ and $D_1\cap D_2=\emptyset$.

By Hurwitz's theorem there exist two sequences $\{\zeta_j\}\subset D_1$, $\{\zeta_j^*\}\subset D_2$ converging to ζ_0, ζ_0^* respectively such that for $j=1,2,\ldots$,

$$g_j(\zeta_j)L(g_j(\zeta_j))=g_j(\zeta_j^*)L(g_j(\zeta_j^*))=a.$$

Since $f_1L(f_1)$ and $f_jL(f_j)$ share a IM for each $j=1,2,\ldots$, it follows that

$$f_1(z_j+\rho_j\zeta_j)L(f_1(z_j+\rho_j\zeta_j))=f_1(z_j+\rho_j\zeta_j^*)L(f_1(z_j+\rho_j\zeta_j^*))=a$$

for $j=1,2,\ldots$.

By (ii) and (iv), considering a subsequence if necessary, we see that there exists a point ξ, $|\xi|\leq r$, such that $z_j+\rho_j\zeta_j\to\xi$ and $z_j+\rho_j\zeta_j^*\to\xi$ as $j\to\infty$. So $f_1(\xi)L(f_1(\xi))=a$ and, since a-points are isolated, for sufficiently large values of j we get $z_j+\rho_j\zeta_j=\xi=z_j+\rho_j\zeta_j^*$. Hence $\zeta_j=(\xi-z_j)/\rho_j=\zeta_j^*$, which is impossible as $D_1\cap D_2=\emptyset$. This proves Theorem 1.1. ∎

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