ANNALES
POLONICI MATHEMATICI
110.3 (2014)

A note on the separated maximum modulus points of meromorphic functions

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Abstract. We give an upper estimate of Petrenko's deviation for a meromorphic function of finite lower order in terms of Valiron's defect and the number $p(\infty, f)$ of separated maximum modulus points of the function. We also present examples showing that this estimate is sharp.

1. Introduction. We shall use the standard notations of value distribution theory of meromorphic functions: $m(r,a,f)$ for the proximity function, $n(r,a,f)$ and $N(r,a,f)$ for the functions counting a-points, $T(r,f)$ for Nevanlinna's characteristic, $\delta(a,f)$ for Nevanlinna's defect, and λ, ρ for the lower order and order, respectively [10, 17]. In 1969 Petrenko raised a question: how will Nevanlinna's theory change if we measure the proximity of a meromorphic function f to a value a applying a different metric? He introduced the following *deviation function*:

$$\mathcal{L}(r,a,f) = \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for } a = \infty, \\ \max\limits_{|z|=r} \log^+ \left| \dfrac{1}{f(z)-a} \right| & \text{for } a \neq \infty. \end{cases}$$

The quantity

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

is called the *deviation of f from a*, and $\Omega(f) := \{a \in \overline{\mathbb{C}} : \beta(a,f) > 0\}$, the *set of positive deviations* of f [18]. The deviation $\beta(a,f)$ characterizes the proximity of f to a with a stronger metric than $\delta(a,f)$ does, and always $\delta(a,f) \le \beta(a,f)$. However, in the case of meromorphic functions of finite lower order the properties of $\beta(a,f)$ are similar to the properties of $\delta(a,f)$.

2010 *Mathematics Subject Classification*: Primary 30D35; Secondary 30D30.
Key words and phrases: meromorphic function, maximum modulus point, Valiron's defect, logarithmic density.

DOI: 10.4064/ap110-3-6

Petrenko himself obtained a sharp upper estimate of $\beta(a,f)$ and also an estimate of the sum $\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)$.

Theorem A ([18]). *If f is a meromorphic function of finite lower order λ, then for all $a\in\overline{\mathbb{C}}$ we have*

$$\beta(a,f)\le B(\lambda):=\begin{cases}\dfrac{\pi\lambda}{\sin\pi\lambda} & \text{if } \lambda\le 0.5,\\ \pi\lambda & \text{if } \lambda>0.5.\end{cases}$$

$$\sum_{a\in\overline{\mathbb{C}}}\beta(a,f)\le 816\pi(\lambda+1)^2.$$

It should be mentioned here that the conjecture that $\beta(\infty,f)\le\pi\varrho$ for entire functions of order ϱ with $0.5\le\varrho<\infty$ was stated in 1932 by Paley and proved in 1969 by Govorov [11].

In 1990 Marchenko and Shcherba presented an exact upper estimate of the sum of deviations for functions of finite lower order, which is an analogue of the estimate of the sum of Nevanlinna's defects. This way they solved the problem stated by Petrenko in his monograph [19].

Theorem B ([16]). *If f is a meromorphic function of finite lower order λ, then*

$$\sum_a\beta(a,f)\le 2B(\lambda).$$

Let $E\subset(0,\infty)$ be a measurable set. The quantites

$$\overline{\operatorname{logdens}}\,E=\limsup_{R\to\infty}\frac{1}{\ln R}\int\limits_{E\cap[1,R]}\frac{dt}{t},$$

$$\underline{\operatorname{logdens}}\,E=\liminf_{R\to\infty}\frac{1}{\ln R}\int\limits_{E\cap[1,R]}\frac{dt}{t}$$

are called, respectively, the *upper* and *lower logarithmic density* of E.

In 1998 Marchenko proved the following theorem.

Theorem C ([14]). *Let f be a meromorphic function of finite lower order λ and order ρ. For $a\in\mathbb{C}$ and $0<\gamma<\infty$ put*

$$E_1(\gamma)=\{r:\mathcal{L}(r,a,f)<B(\gamma)T(r,f)\}.$$

Then

$$\overline{\operatorname{logdens}}\,E_1(\gamma)\ge 1-\lambda/\gamma \quad \text{and} \quad \underline{\operatorname{logdens}}\,E_1(\gamma)\ge 1-\rho/\gamma.$$

Let now $f(z)$ be a meromorphic function and let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r>0$, such that $\phi(r)=o(T(r,f))$. We denote by $p_\phi(r,\infty,f)$ the number of component arcs of the set

$$\{z:|z|=r,\ \log|f(z)|>\phi(r)\}$$

containing at least one maximum modulus point of f. Moreover, let

$$p_\phi(\infty, f) = \liminf_{r\to\infty} p_\phi(r, \infty, f), \qquad p(\infty, f) = \sup_\phi p_\phi(\infty, f).$$

For $a \in \mathbb{C}$ we put $p(a, f) := p(\infty, 1/(f - a))$.

In [4] we obtained the following relationship between deviation from infinity and the number of separated maximum modulus points of a meromorphic function of finite lower order.

Theorem D. *For a meromorphic function f of finite lower order λ we have*

$$\beta(\infty, f) \le \begin{cases} \dfrac{\pi\lambda}{p(\infty, f)} & \text{if } \lambda/p(\infty, f) \ge 1/2, \\ \dfrac{\pi\lambda}{\sin \pi\lambda} & \text{if } p(\infty, f) = 1 \text{ and } \lambda < 1/2, \\ \dfrac{\pi\lambda}{p(\infty, f)} \sin \dfrac{\pi\lambda}{p(\infty, f)} & \text{if } p(\infty, f) > 1 \text{ and } \lambda/p(\infty, f) < 1/2. \end{cases}$$

The value

$$\Delta(a, f) := \limsup_{r\to\infty} \frac{m(r, a, f)}{T(r, f)}$$

is called *Valiron's defect* of f *at* a. If $\Delta(a, f) > 0$ we say that a is a *defective value* of f *in the sense of Valiron*, and we set $V(f) := \{a \in \overline{\mathbb{C}} : \Delta(a, f) > 0\}$. It easily follows from Nevanlinna's first main theorem that

$$0 \le \delta(a, f) \le \Delta(a, f) \le 1$$

and thus $N(f) \subset V(f)$, where $N(f)$ denotes the set of values defective in the sense of Nevanlinna.

An interesting issue is the relationship between the set of positive deviations and the set of Valiron's defective values. The solution of this problem was given by Shea and presented by Fuchs [8] (see also [19]).

For $\gamma \ge 0$ we put

$$B(\gamma, \Delta) := \begin{cases} \pi\gamma\sqrt{\Delta(2 - \Delta)} & \text{if } \gamma > 1/2 \text{ or } \sin(\pi\gamma/2) > \sqrt{\Delta/2}, \\ \dfrac{\pi\gamma(1 - (1 - \Delta)\cos \pi\gamma)}{\sin \pi\gamma} & \text{if } 0 \le \gamma \le 1/2 \text{ and } \sin(\pi\gamma/2) \le \sqrt{\Delta/2}. \end{cases}$$

Theorem E. *Let f be a meromorphic function of finite lower order λ. Then for each $a \in \overline{\mathbb{C}}$ we have*

$$\beta(a, f) \le B(\lambda, \Delta), \qquad \textit{where } \Delta = \Delta(a, f).$$

Corollary. *For meromorphic functions f of finite lower order, we have $\Omega(f) \subset V(f)$.*

The estimate in Theorem E is sharp. An appropriate example of a meromorphic function was given by Ryshkov [21]. The following extension of Theorem E was given in 2000 by Marchenko [15].

THEOREM F. *Let f be a meromorphic function of finite lower order λ and order ρ. For each positive number γ and $a \in \overline{\mathbb{C}}$ set*

$$E(\gamma) := \{r : \mathcal{L}(r, a, f) < B(\gamma/p_0(a, f), \varDelta(a, f))T(r, f)\}.$$

Then

$$\overline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \lambda/\gamma \quad \text{and} \quad \underline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \rho/\gamma.$$

Here $p_0(r, a, f)$ is the number of component arcs of the set $\{z : |z| = r, \log(1/|f(z) - a|) > 0\}$ containing at least one maximum modulus point of $1/(f(z) - a)$, and $p_0(a, f) = \liminf_{r\to\infty} p_0(r, a, f)$. It is easy to notice that $p_0(a, f) \leq p(a, f)$.

We now present a result which improves Theorems E and F in a certain sense and expresses the upper estimate of $\beta(a, f)$ in terms of both Valiron's defect and the number $p(a, f)$.

THEOREM 1.1. *Let f be a meromorphic function of finite lower order λ and order ρ. Let $0 < \gamma < \infty$, $a \in \overline{\mathbb{C}}$ and $p > 0$. Put*

$$E(\gamma) := \{r : \mathcal{L}(r, a, f) < B(\gamma/p(a, f), \varDelta(a, f))T(r, f)\}$$

if $p(a, f) < \infty$ and $\varDelta(a, f) > 0$, and

$$E(\gamma) := \{r : \mathcal{L}(r, a, f) < B(\gamma/p, \varDelta(a, f))T(r, f)\}$$

if $p(a, f) = \infty$ or $\varDelta(a, f) = 0$. Then

$$\overline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \lambda/\gamma \quad \text{and} \quad \underline{\operatorname{logdens}}\, E(\gamma) \geq 1 - \rho/\gamma.$$

COROLLARY. *Let f be a meromorphic function of finite lower order λ. Then for each $a \in \overline{\mathbb{C}}$ we have*

$$\beta(a, f) \leq B(\lambda/p(a, f), \varDelta(a, f)). \tag{1.1}$$

The estimate in the above corollary is sharp. We give relevant examples in the last section of this paper.

2. Auxiliary results. Let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$, such that $\phi(r) = o(T(r, f))$. We consider the function

$$u_\phi(z) = \max(\log|f(z)|, \phi(|z|)),$$

where $f(z)$ is a meromorphic function in $\mathbb{C}$. In [4] we proved the following lemma. We repeat the proof for completeness.

LEMMA 2.1. *The function u_ϕ is δ-subharmonic in $\mathbb{C}$.*

Proof. Let g_1 and g_2 be entire functions without common zeros such that $f(z) = g_1(z)/g_2(z)$. It is easy to see that

$$u_\phi(z) = \max\bigl(\log|g_1(z)|, \log|g_2(z)| + \phi(|z|)\bigr) - \log|g_2(z)|.$$

Since $\phi(r)$ is a convex function of $\log r$ for $r > 0$, $\phi(|z|)$ is a subharmonic function in $\mathbb{C}$ (see [20]). Also

$$v_1(z) := \max\bigl(\log|g_1(z)|, \log|g_2(z)| + \phi(|z|)\bigr)$$

is subharmonic in $\mathbb{C}$. Thus

$$u_\phi(z) = v_1(z) - \log|g_2(z)| : = v_1(z) - v_2(z)$$

is a δ-subharmonic function in $\mathbb{C}$. ■

For a complex number $z = re^{i\theta}$ we now put [1]

$$m^*(r,\theta,u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\phi(re^{i\varphi})\,d\varphi,$$

$$T^*(r,\theta,u_\phi) = T^*(re^{i\theta}) = m^*(r,\theta,u_\phi) + N(r,\infty,f)$$

where $r \in (0,\infty)$, $\theta \in [0,\pi]$, and $|E|$ is the Lebesgue measure of the set E. Let us write $\tilde{u}_\phi$ for the circular symmetrization of u_ϕ. The function $\tilde{u}_\phi(re^{i\varphi})$ is nonnegative and nonincreasing on $[0,\pi]$, even in φ and equimeasurable with $u_\phi(re^{i\varphi})$ for each fixed r [12]. Moreover,

$$\tilde{u}_\phi(r) = \max\Bigl(\log\max_{|z|=r}|f(z)|, \phi(r)\Bigr),$$

$$\tilde{u}_\phi(re^{i\pi}) = \max\Bigl(\log\min_{|z|=r}|f(z)|, \phi(r)\Bigr),$$

$$m^*(r,\theta,u_\phi) = \sup_{|E|=2\theta} \frac{1}{2\pi} \int_E u_\phi(re^{i\varphi})\,d\varphi = \frac{1}{\pi}\int_0^\theta \tilde{u}_\phi(re^{i\varphi})\,d\varphi.$$

From Baernstein's theorem [1] the function $T^*(r,\theta,u_\phi)$ is subharmonic on

$$D = \{re^{i\theta} : 0 < r < \infty,\ 0 < \theta < \pi\},$$

continuous on $D \cup (-\infty,0) \cup (0,\infty)$ and logarithmically convex in $r > 0$ for each fixed $\theta \in [0,\pi]$. What is more,

$$T^*(r,0,u_\phi) = N(r,\infty,f),$$
$$T^*(r,\pi,u_\phi) = T(r,f) + o(T(r,f)) \quad (r \to \infty),$$
$$\frac{\partial}{\partial\theta} T^*(r,\theta,u_\phi) = \frac{\tilde{u}_\phi(re^{i\theta})}{\pi} \quad \text{for } 0 < \theta < \pi.$$

For $\alpha(r)$ a real-valued function of a real variable r we put

$$L\alpha(r) = \liminf_{h\to 0} \frac{\alpha(re^h) + \alpha(re^{-h}) - 2\alpha(r)}{h^2}.$$

If $\alpha(r)$ is twice differentiable in r, then

$$L\alpha(r) = r\frac{d}{dr}\left(r\frac{d}{dr}\alpha(r)\right).$$

LEMMA 2.2 ([4]). *For almost all $\theta \in [0,\pi]$ and for all $r>0$ such that on the set $\{z : |z| = r\}$ the meromorphic function $f(z)$ has neither zeros nor poles, we have*

$$LT^*(r,\theta,u_\phi) \geq -\frac{p_\phi^2(r,\infty,f)}{\pi}\,\frac{\partial\tilde{u}_\phi(re^{i\theta})}{\partial\theta}.$$

Let $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r>0$, such $\phi(r) = o(T(r,f))$ and $0 < p_\phi(\infty,f) < \infty$. For $\tau > 0$ we choose numbers α and ψ such that

$$(2.1)\qquad 0<\alpha\leq\min\left(\pi,\frac{\pi p_\phi(\infty,f)}{2\tau}\right),\qquad -\frac{\pi p_\phi(\infty,f)}{2\tau} \leq \psi \leq \frac{\pi p_\phi(\infty,f)}{2\tau} - \alpha.$$

We set

$$(2.2)\qquad h_\phi(r,\tau) := \frac{p_\phi^2(\infty,f)}{\pi}\left(\tilde{u}_\phi(r)\cos\frac{\tau\psi}{p_\phi(\infty,f)} - \tilde{u}_\phi(re^{i\alpha})\cos\frac{\tau(\alpha+\psi)}{p_\phi(\infty,f)}\right) - \tau p_\phi(\infty,f)\left(\sin\frac{\tau(\alpha+\psi)}{p_\phi(\infty,f)}T^*(r,\alpha,u_\phi) - \sin\frac{\tau\psi}{p_\phi(\infty,f)}N(r,\infty,f)\right).$$

LEMMA 2.3. *Let $A(\phi,\tau) := \{r : h_\phi(r,\tau) > 0\}$. Then*

$$\tau \int\limits_{A(\phi,\tau)\cap[1,R]} \frac{dt}{t} \leq \log T(2R,f) + O(1).$$

Proof. In the course of this proof, r_0 stands for an appropriate positive number, not necessarily the same at each occurrence. We put [7, 9]

$$\sigma(r) = \int\limits_0^\alpha T^*(r,\theta,u_\phi)\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta.$$

As $T^*(r,\theta,u_\phi)$ is a convex function of $\log r$, we have $LT^*(r,\theta,u_\phi) \geq 0$. We apply Fatou's lemma to get

$$(2.3)\qquad L\sigma(r) \geq \int\limits_0^\alpha LT^*(r,\theta,u_\phi)\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta \geq 0.$$

It follows that $\sigma(r)$ is a convex function of $\log r$ so $r\sigma'_-(r)$ is an increasing function on $(0,\infty)$. Therefore for almost all $r>0$,

$$L\sigma(r) = r\frac{d}{dr}(r\sigma'_-(r)),$$

where $\sigma'_-(r)$ is the left derivative of $\sigma(r)$ at r. Lemma 2.2 and inequality (2.3) imply that for almost all $r > 0$,

$$L\sigma(r) = r\frac{d}{dr}(r\sigma'_-(r)) \geq -\int_0^\alpha \frac{p_\phi^2(r,\infty,f)}{\pi}\frac{\partial \tilde{u}_\phi(re^{i\theta})}{\partial\theta}\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta.$$

By definition $p_\phi(r,\infty,f)$ takes only integral values. Thus, $p_\phi(\infty,f) \leq p_\phi(r,\infty,f)$ for $r \geq r_0$. It follows that for almost all $r \geq r_0$,

$$r\frac{d}{dr}(r\sigma'_-(r)) \geq -\int_0^\alpha \frac{p_\phi^2(\infty,f)}{\pi}\frac{\partial \tilde{u}_\phi(re^{i\theta})}{\partial\theta}\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta.$$

If for $r > 0$ there are neither zeros nor poles of $f(z)$ on the circle $|z| = r$, the function $u_\phi(re^{i\theta})$ satisfies the Lipschitz condition in θ. Therefore $\tilde{u}_\phi(re^{i\theta})$ also satisfies the Lipschitz condition on $[0,\pi]$ [12], and hence is absolutely continuous on $[0,\pi]$. We integrate by parts twice to obtain

$$\begin{aligned}
&\int_0^\alpha \frac{p_\phi^2(\infty,f)}{\pi}\frac{\partial \tilde{u}_\phi(re^{i\theta})}{\partial\theta}\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta\\
&= \frac{p_\phi^2(\infty,f)}{\pi}\tilde{u}_\phi(re^{i\alpha})\cos\frac{\tau(\alpha+\psi)}{p_\phi(\infty,f)} - \frac{p_\phi^2(\infty,f)}{\pi}\tilde{u}_\phi(r)\cos\frac{\tau\psi}{p_\phi(\infty,f)}\\
&+ \tau p_\phi(\infty,f)\left(T^*(r,\alpha,u_\phi)\sin\frac{\tau(\alpha+\psi)}{p_\phi(\infty,f)} - N(r,\infty,f)\sin\frac{\tau\psi}{p_\phi(\infty,f)}\right) - \tau^2\sigma(r)\\
&= -h_\phi(r,\tau) - \tau^2\sigma(r).
\end{aligned}$$

As $\tilde{u}(re^{i\theta})$ is decreasing in θ we have

$$h_\phi(r,\tau) + \tau^2\sigma(r) \geq 0 \quad \text{for } r \geq r_0. \tag{2.4}$$

Thus for almost all $r \geq r_0$ we get

$$r\frac{d}{dr}r\sigma'_-(r) \geq h_\phi(r,\tau) + \tau^2\sigma(r).$$

We divide this inequality by $r^{\tau+1}$ and integrate it by parts over the interval $[r,R]$ [13] to obtain

$$\begin{aligned}
\int_r^R \frac{h_\phi(t,\tau)}{t^{\tau+1}}\,dt &\leq \int_r^R \frac{1}{t^\tau}\frac{d}{dt}(t\sigma'_-(t))\,dt + \tau^2\int_r^R \frac{1}{t^{\tau+1}}\sigma(t)\,dt\\
&\leq \left(\frac{t\sigma'_-(t)}{t^\tau} + \tau\frac{\sigma(t)}{t^\tau}\right)\Bigg|_r^R, \qquad r_0 \leq r \leq R.
\end{aligned}$$

We now apply the method of P. Barry [2, 3]. We set

$$\varPhi(r) = -\int_r^R \frac{h_\phi(t,\tau)}{t^{\tau+1}}\,dt, \qquad r_0 \leq r \leq R.$$

From the above inequality we get

$$\Phi(r) \geq -\frac{\sigma'_-(R)}{R^{\tau-1}} - \tau\frac{\sigma(R)}{R^\tau} + \frac{\sigma'_-(r)}{r^{\tau-1}} + \tau\frac{\sigma(r)}{r^\tau}.$$

We now put

$$\psi(r) = r^\tau\left[\Phi(r) + \frac{\sigma'_-(R)}{R^{\tau-1}} + \tau\frac{\sigma(R)}{R^\tau}\right].$$

Then

$$\psi(r) \geq r\sigma'_-(r) + \tau\sigma(r), \qquad r_0 \leq r \leq R.$$

Using (2.4) we get, for $r \geq r_0$,

$$r\psi'(r) = \tau\psi(r) + h_\phi(r,\tau) \geq \tau r\sigma'_-(r) + \tau^2\sigma(r) + h_\phi(r,\tau) \geq \tau r\sigma'_-(r) \geq 0.$$

The function $T^*(r,\theta,u_\phi)$ is increasing for $r > r_0$ ([9, 5]) and so $\sigma(r)$ is increasing on (r_0,R). Therefore $r\sigma'_-(r) \geq 0$ for all $r > r_0$. Moreover, $\sigma(r) > 0$ for all $r > r_0$. Thus we have, for all $r > r_0$,

$$\psi(r) \geq r\sigma'_-(r) + \tau\sigma(r) > 0.$$

If $r \in A(\phi,\tau)$ then $r\psi'(r) > \tau\psi(r) > 0$. Therefore $\psi'(r)/\psi(r) > \tau/r$. As a result, for $r \geq r_0$,

$$\tau \int\limits_{A(\phi,\tau)\cap[1,R]} \frac{dr}{r} \leq \int\limits_{A(\phi,\tau)\cap[r_0,R]} \frac{\psi'(r)}{\psi(r)}\,dr + \tau\log r_0 \tag{2.5}$$
$$\leq \int_{r_0}^{R} \frac{\psi'(r)}{\psi(r)}\,dr + \tau\log r_0 = \log\frac{\psi(R)}{\psi(r_0)} + \tau\log r_0.$$

But $\psi(R) = R\sigma'_-(R) + \tau\sigma(R)$. It follows from the definition of $\sigma(r)$ that

$$\sigma(r) = \int_0^\alpha T^*(r,\theta,u_\phi)\cos\frac{\tau(\theta+\psi)}{p_\phi(\infty,f)}\,d\theta \leq \int_0^\alpha T^*(r,\theta,u_\phi)\,d\theta$$
$$\leq \int_0^\pi (T(r,f) + o(T(r,f))\,d\theta = \pi T(r,f) + o(T(r,f)) \qquad (r\to\infty).$$

From the monotonicity of $r\sigma'_-(r)$ we get

$$r\sigma'_-(r) \leq \int_r^{2r} \sigma'_-(t)dt \leq \sigma(2r) \leq \pi T(2r,f).$$

Thus from (2.5) we obtain

$$\tau \int\limits_{A(\phi,\tau)\cap[1,R]} \frac{dr}{r} \leq \log\frac{\psi(R)}{\psi(r_0)} + O(1) \leq \log\psi(R) + O(1)$$
$$= \log[R\sigma'_-(R) + \tau\sigma(R)] + O(1) \leq \log T(2R,f) + O(1),$$

and the proof of Lemma 2.3 is complete. ∎

Notice that it follows from Lemma 2.3 that the logarithmic density of the set $E(\phi,\tau) := \{r : h_\phi(r,\tau) \ge 0\}$ satisfies the inequalities

$$\overline{\operatorname{logdens}}\, E(\phi,\tau) \ge 1 - \lambda/\tau \quad \text{and} \quad \underline{\operatorname{logdens}}\, E(\phi,\tau) \ge 1 - \rho/\tau. \tag{2.6}$$

3. Proof of Theorem 1.1. We shall conduct the proof for $a = \infty$. Then, for $a \ne \infty$, we may apply the same considerations to the function $F(z) = 1/(f(z) - a)$.

We concentrate on the upper logarithmic density of E. If $\beta(\infty, f) = 0$ or $\gamma \le \lambda$ the theorem is straightforward. Assume that $\beta(\infty, f) > 0$. This means that $p(\infty, f) \ge 1$. We start with the case $1 \le p(\infty, f) < \infty$.

Let us recall that

$$\Delta(\infty, f) := \limsup_{r\to\infty} \frac{m(r,\infty,f)}{T(r,f)} = 1 - \liminf_{r\to\infty} \frac{N(r,\infty,f)}{T(r,f)}.$$

Thus for a fixed $\varepsilon > 0$, for $r \ge r_0(\varepsilon)$ we have

$$N(r,\infty,f) > (1 - \Delta(\infty,f) - \varepsilon)T(r,f). \tag{3.1}$$

Notice that when $\Delta(\infty, f) = 1$ we have

$$B(\gamma/p(\infty,f), \Delta(\infty,f)) = B(\gamma/p(\infty,f))$$

and the statement follows easily from Theorem C. If, on the other hand, $\Delta(\infty, f) = 0$ then also $\beta(\infty, f) = 0$. Therefore we consider $0 < \Delta(\infty, f) < 1$ and select $0 < \varepsilon < 1 - \Delta(\infty, f)$.

As $p(\infty, f) < \infty$, and $p_\phi(\infty, f)$ and $p(\infty, f)$ take only integral values, we can find $\phi(r)$ such that $p_\phi(\infty, f) = p(\infty, f)$. Let now $\phi(r)$ be a positive nondecreasing convex function of $\log r$ for $r > 0$ such that $\phi(r) = o(T(r, f))$ and $p_\phi(\infty, f) = p(\infty, f)$. Let us also take a number τ such that $\lambda < \tau < \gamma$, and $\psi = \pi p_\phi(\infty, f)/(2\tau) - \alpha$. Then

$$\begin{aligned} h_\phi(r,\tau) = {} & \frac{p_\phi^2(\infty,f)}{\pi} \tilde{u}_\phi(r) \sin\frac{\tau\alpha}{p_\phi(\infty,f)} \\ & + \tau p_\phi(\infty,f)\left\{\cos\frac{\tau\alpha}{p_\phi(\infty,f)} N(r,\infty,f) - T^*(re^{i\alpha}, f)\right\}. \end{aligned}$$

Then, as $\tilde{u}_\phi(r) \ge \mathcal{L}(r,\infty,f)$, for $r \ge r_0(\varepsilon)$ we have

$$\begin{aligned} h_\phi(r,\tau) \ge {} & \frac{p_\phi^2(\infty,f)}{\pi}\left\{\mathcal{L}(r,\infty,f)\sin\frac{\tau\alpha}{p_\phi(\infty,f)} \right. \\ & \left. - \frac{\pi\tau}{p_\phi(\infty,f)}\left[T(r,f) + \phi(r) - (1 - \Delta(\infty,f) - \varepsilon)\cos\frac{\tau\alpha}{p_\phi(\infty,f)} T(r,f)\right]\right\}. \end{aligned}$$

Let $E(\phi,\tau)$ be the set from (2.6). We get

$$\mathcal{L}(r,\infty,f) \le \frac{\frac{\pi\tau}{p_\phi(\infty,f)}}{\sin\frac{\tau\alpha}{p_\phi(\infty,f)}}\left(1-(1-\Delta(\infty,f))\cos\frac{\tau\alpha}{p_\phi(\infty,f)}\right)T(r,f) + o(T(r,f))$$

when $r\to\infty$, $r\in E(\phi,\tau)$. We now consider two cases.

First, if $\arccos(1-\Delta(\infty,f)) < \pi\tau/p_\phi(\infty,f)$ we can take

$$\alpha = \arccos(1-\Delta(\infty,f)),$$

as $\arccos(1-\Delta(\infty,f)) < \min(\pi, \pi p_\phi(\infty,f)/2\tau)$. Then $\sqrt{\Delta(\infty,f)/2} < \sin\frac{\pi\tau}{2p_\phi(\infty,f)}$, so

$$\begin{aligned} B\left(\frac{\tau}{p_\phi(\infty,f)},\Delta(\infty,f)\right) &= \frac{\pi\tau}{p_\phi(\infty,f)}\sqrt{\Delta(\infty,f)(2-\Delta(\infty,f))} \\ &= \frac{\pi\tau}{p_\phi(\infty,f)}\sin\alpha \end{aligned}$$

and we get

$$\mathcal{L}(r,\infty,f) \le \frac{\pi\tau}{p_\phi(\infty,f)}\,\frac{1-\cos\alpha\cos\frac{\tau\alpha}{p_\phi(\infty,f)}}{\sin\frac{\tau\alpha}{p_\phi(\infty,f)}}T(r,f)+o(T(r,f))$$

$$\le \frac{\pi\tau\sin\alpha}{p_\phi(\infty,f)}T(r,f)+o(T(r,f)) = B\left(\frac{\tau}{p_\phi(\infty,f)},\Delta(\infty,f)\right)T(r,f)+o(T(r,f))$$

for $r\to\infty$, $r\in E(\phi,\tau)$.

Second, let $\arccos(1-\Delta(\infty,f))\ge\pi\tau/p_\phi(\infty,f)$ and $\min\left(\pi,\frac{\pi p_\phi(\infty,f)}{2\tau}\right)=\pi$. In this case $\sqrt{\Delta(\infty,f)/2}\ge\sin\frac{\pi\tau}{2p_\phi(\infty,f)}$ and $\tau/p_\phi(\infty,f)\le 1/2$, so

$$B\left(\frac{\tau}{p_\phi(\infty,f)},\Delta(\infty,f)\right) = \frac{\pi\tau\left(1-(1-\Delta(\infty,f))\cos\frac{\pi\tau}{p_\phi(\infty,f)}\right)}{p_\phi(\infty,f)\sin\frac{\pi\tau}{p_\phi(\infty,f)}}.$$

We put $\alpha=\pi$ and directly obtain

$$\mathcal{L}(r,\infty,f)\le B(\tau/p_\phi(\infty,f),\Delta(\infty,f))T(r,f)+o(T(r,f))$$

for $r\to\infty$, $r\in E(\phi,\tau)$.

As $p_\phi(\infty,f)=p(\infty,f)$ and $\tau<\gamma$, in both cases we get

$$\begin{aligned}\mathcal{L}(r,\infty,f) &\le B(\tau/p(\infty,f),\Delta(\infty,f))T(r,f)+o(T(r,f)) \\ &< B(\gamma/p(\infty,f),\Delta(\infty,f))T(r,f)\end{aligned}$$

for $r\to\infty$, $r\in E(\phi,\tau)$. Thus $E(\phi,\tau)\subset E(\gamma)$ and it follows that

$$\overline{\operatorname{logdens}}\,E(\gamma)\ge 1-\lambda/\tau.$$

Letting $\tau\to\gamma$ leads us to the desired statement.

Finally, we consider the case when $p(\infty, f) = \infty$ (it should be stressed here that we are not sure if this case is at all possible for a meromorphic function of finite lower order). Let $p > 0$ be a fixed number. By definition, there exists a function ϕ such that $p_\phi(\infty, f) > p$. If $p_\phi(\infty, f) < \infty$ we may repeat all the previous considerations with respect to $p_\phi(\infty, f)$ and the statement follows from the fact that $B(\gamma/p_\phi(\infty, f), \Delta(\infty, f)) < B(\gamma/p, \Delta(\infty, f))$. If, on the other hand, $p_\phi(\infty, f) = \infty$ we put p instead of $p_\phi(\infty, f)$ in (2.1) and (2.2). Notice that the conclusion of Lemma 2.3 holds. Also Lemma 2.2 holds for $\phi(r)$ as $p_\phi(r, \infty, f) \neq \infty$. Since $p_\phi(\infty, f) = \infty$, we get $p_\phi(r, \infty, f) > p$ for $r \geq r_0$. This leads us directly to the conclusion in this case.

We have conducted the proof for the upper logarithmic density. The proof for the lower logarithmic density can be done in a similar way.

4. Exactness of the estimate

4.1. Case $0 < \lambda < 1$. For $0 < \lambda < 1$ and $0 < u \leq 1$ let $f_\lambda(z, u) = \prod_{n=1}^{\infty}(1 - z/a_n)$ be a canonical product of genus zero with positive zeros $\{a_n\}$ such that $n(r, 0) \sim ur^\lambda$ $(r \to \infty)$. Ryshkov [21] considered the meromorphic function

$$F_\lambda(z) = \frac{f_\lambda(z, u)}{f_\lambda(-z, v)},$$

where $0 < \lambda < 1$ and $0 \leq v \leq u \leq 1$, $u > 0$ and, by definition, $f(z, 0) \equiv 1$. This is a special case of an example given by Gol'dberg and Ostrovskiĭ [10]. Notice that F_λ has only positive zeros and negative poles, and

$$N(r, \infty, F_\lambda) = N(r, 0, f_\lambda(-z, v)) = \frac{v}{\lambda} r^\lambda + o(r^\lambda) \quad (r \to \infty).$$

It is shown in [10] that

(4.1)
$$\log |F_\lambda(re^{i\varphi})| = \frac{\pi}{\sin \pi\lambda}\{u \cos \lambda(\varphi - \pi) - v \cos \lambda\varphi\} r^\lambda + o(r^\lambda) \quad (r \to \infty)$$

uniformly in φ in every interval $[\eta, \pi - \eta]$ where $0 < \eta < \pi/2$. It follows that

$$\begin{aligned} m(r, \infty, F_\lambda) &= \frac{1}{\lambda} I(u, v, \lambda) r^\lambda + o(r^\lambda) & (r \to \infty), \\ T(r, F_\lambda) &= \frac{1}{\lambda}\{I(u, v, \lambda) + v\} r^\lambda + o(r^\lambda) & (r \to \infty), \end{aligned}$$

where

$$I(u, v, \lambda) = \begin{cases} u - v & \text{if } u \cos \pi\lambda \geq v, \\ \sqrt{u^2 + \left(\dfrac{v - u \cos \pi\lambda}{\sin \pi\lambda}\right)^2} - v & \text{if } u \geq v \geq u \cos \pi\lambda. \end{cases}$$

Thus F_λ is a meromorphic function of both lower order and order λ with $p(\infty, F_\lambda) = 1$. Moreover, as $F_\lambda(z) = \overline{F_\lambda(\overline{z})}$ it follows from (4.1) that

$$\mathcal{L}(r, \infty, F_\lambda) = \frac{\pi}{\sin \pi\lambda}\{u - v\cos \pi\lambda\}r^\lambda + o(r^\lambda) \quad (r \to \infty),$$

for r such that $F_\lambda(-r) \neq \infty$. Thus

$$\beta(\infty, F_\lambda) = \frac{\pi\lambda(u - v\cos\pi\lambda)}{\sin\pi\lambda\,(I(u, v, \lambda) + v)} \quad \text{and} \quad \Delta(\infty, F_\lambda) = \frac{I(u, v, \lambda)}{I(u, v, \lambda) + v}.$$

We also get

$$B(\lambda, \Delta(\infty, F_\lambda)) = \begin{cases} \dfrac{\pi\lambda\sqrt{2I^2 + 2vI} - I}{I + v} & \text{if } \lambda > 1/2 \text{ or } \sin(\pi\lambda/2) > \sqrt{\Delta(\infty, F_\lambda)/2}, \\ \dfrac{\pi\lambda(1 - \frac{v}{I+v}\cos\pi\lambda)}{\sin\pi\lambda} & \text{if } 0 \leq \lambda \leq 1/2 \text{ and } \sin(\pi\lambda/2) \leq \sqrt{\Delta(\infty, F_\lambda)/2}, \end{cases}$$

where $I := I(u, v, \lambda)$. By elementary computations we obtain the equality $\beta(\infty, F_\lambda) = B(\lambda, \Delta(\infty, F_\lambda))$.

Let n be a fixed positive integer. We consider the function $F(z) := F_{\lambda/n}(z^n)$. It is easy to see that $p(\infty, F) = n$. Also

$$N(r, \infty, F) = \frac{nv}{\lambda}r^\lambda + o(r^\lambda) \quad (r \to \infty).$$

It follows from (4.1) that

$$\log|F(re^{i\varphi})| = \frac{\pi}{\sin(\pi\lambda/n)}\left\{u\cos\lambda\left(\varphi - \frac{\pi}{n}\right) - v\cos\lambda\varphi\right\}r^\lambda + o(r^\lambda) \quad (r \to \infty) \tag{4.2}$$

uniformly in φ in every interval $[\eta, \pi/n - \eta]$, where $0 < \eta < \pi/(2n)$.

By a similar argument to one in [10] it can be shown that

$$m(r, \infty, F) = \frac{n}{\lambda}I\left(u, v, \frac{\lambda}{n}\right)r^\lambda + o(r^\lambda) \qquad (r \to \infty),$$

$$T(r, F) = \frac{n}{\lambda}\left\{I\left(u, v, \frac{\lambda}{n}\right) + v\right\}r^\lambda + o(r^\lambda) \quad (r \to \infty).$$

Thus F is a meromorphic function of both lower order and order λ with

$$\mathcal{L}(r, \infty, F) = \frac{\pi}{\sin(\pi\lambda/n)}\left\{u - v\cos\frac{\pi\lambda}{n}\right\}r^\lambda + o(r^\lambda) \quad (r \to \infty),$$

for r such that $F_{\lambda/n}(-r^n) \neq \infty$. Thus

$$\beta(\infty, F) = \frac{(\pi\lambda/n)(u - v\cos(\pi\lambda/n))}{\sin(\pi\lambda/n)\,(I(u, v, \lambda/n) + v)}, \qquad \Delta(\infty, F) = \frac{I(u, v, \lambda/n)}{I(u, v, \lambda/n) + v}.$$

By similar considerations to the case of F_λ it follows that

$$\beta(\infty, F) = B(\lambda/n, \Delta(\infty, F)),$$

which shows that the estimate (1.1) is sharp for meromorphic functions of lower order $0 < \lambda < 1$.

4.2. Case $\lambda = 0$. Let $r > 1$ and $\varrho(r) := \log(\log r)/\log r$. Then $\varrho(r) \to 0$ $(r \to \infty)$ and $r^{\varrho(r)} = \log r \uparrow \infty$ $(r \to \infty)$.

For $0 < u \le 1$ let $f_0(z, u) = \prod_{n=1}^{\infty}(1 - z/a_n)$ be a canonical product of genus zero with positive zeros $\{a_n\}$ such that $n(r, 0) \sim ur^{\varrho(r)}$ $(r \to \infty)$. Thus $\varrho(r)$ is a proximate order. Gol'dberg and Ostrovskiĭ [10] considered the meromorphic function

$$F_0(z) = \frac{f_0(z, u)}{f_0(-z, v)},$$

where $0 \le v \le u \le 1$, $u > 0$ and, by definition, $f_0(z, 0) \equiv 1$. Notice that

$$\begin{aligned} N(r, \infty, F_0) = N(r, 0, f_0(-z, v)) &= (v + o(1)) \int_1^r t^{\varrho(t)-1}\, dt \\ &= (v + o(1)) \frac{\log^2 r}{2} \qquad (r \to \infty). \end{aligned}$$

It is shown in [10] that

$$\log |F_0(re^{i\varphi})| = (u - v + o(1)) \frac{\log^2 r}{2} \qquad (r \to \infty) \tag{4.3}$$

uniformly in φ in every interval $[\eta, \pi - \eta]$ and $[\pi + \eta, 2\pi - \eta]$ where $0 < \eta < \pi/2$. It follows that

$$\begin{aligned} m(r, \infty, F_0) &= (u - v + o(1)) \frac{\log^2 r}{2} \qquad (r \to \infty), \\ T(r, F_0) &= (u + o(1)) \frac{\log^2 r}{2} \qquad (r \to \infty). \end{aligned}$$

Thus F_0 is a meromorphic function of both lower order and order 0 with $p(\infty, F_0) = 1$. Moreover, it follows from (4.3) that

$$\beta(\infty, F_0) = \frac{u - v}{u} = \Delta(\infty, F_0) = B(0, \Delta(\infty, F_0)).$$

Let n be a fixed positive integer. Consider the function $F(z) := F_0(z^n)$. It is easy to notice that

$$\begin{aligned} N(r, \infty, F) &= (v + o(1)) \frac{n \log^2 r}{2} \qquad (r \to \infty), \\ m(r, \infty, F) &= (u - v + o(1)) \frac{n \log^2 r}{2} \qquad (r \to \infty), \\ T(r, F) &= (u + o(1)) \frac{n \log^2 r}{2} \qquad (r \to \infty). \end{aligned}$$

Thus F is a meromorphic function of both lower order and order 0 with $p(\infty, F) = n$. It follows that

$$\beta(\infty, F) = \frac{u - v}{u} = B(0, \Delta(\infty, F)),$$

which shows that (1.1) is sharp for meromorphic functions of lower order 0.

4.3. Case $\lambda \geq 1$. Let $\lambda \geq 1$ be fixed and define

$$p = [\lambda], \qquad 0 < \beta < \frac{\pi}{2\lambda}, \qquad \alpha = (2\cos\beta\lambda)^{-1/\lambda},$$

where $[\lambda]$ is the integral part of λ. Following Edrei and Fuchs [6] and Ryshkov [21], we consider the canonical products of genus p

$$g(z) = \prod_{n=1}^{\infty} E\Big(-\frac{z}{n^{1/\lambda}}, p\Big).$$

If λ is not an integer we put

$$F_\lambda(z) := \frac{g(\alpha e^{i\beta} z) g(\alpha e^{-i\beta} z)}{g(z)}.$$

If λ is an integer we put

$$F_\lambda(z) := \frac{g(\alpha e^{i\beta} z) g(\alpha e^{-i\beta} z)}{g(z)} \exp\{((-1)^p \beta \tan\beta p + (-1)^{p-1} \log\alpha) z^p\}.$$

Thus F_λ has got zeros $a_n = \big(\frac{n}{2\cos\beta\lambda}\big) e^{i(\pi-\beta)}$ and $\overline{a}_n = \big(\frac{n}{2\cos\beta\lambda}\big) e^{i(\pi+\beta)}$ $(n = 1, 2, \ldots)$ and poles $b_n = -n^{1/\lambda}$. It follows that $n(r, F_\lambda) \sim r^\lambda$ $(r \to \infty)$ and

$$N(r, \infty, F_\lambda) = r^\lambda/\lambda + o(r^\lambda) \qquad (r \to \infty).$$

Let $0 < \eta < \min\{\beta/2, \pi - \beta\}$. It was shown in [6] that the relations

$$\log|F_\lambda(re^{i\varphi})| = o(r^\lambda) \qquad (r \to \infty) \tag{4.4}$$

if $\varphi \in [-\pi + \beta + \eta, \pi - \beta - \eta]$, and

$$\log|F_\lambda(re^{i\varphi})| = \frac{\pi \sin\lambda(\varphi + \beta - \pi)}{\cos\beta\lambda} r^\lambda + o(r^\lambda) \qquad (r \to \infty) \tag{4.5}$$

if $\varphi \in [\pi - \beta + \eta, \pi - \eta]$, hold uniformly in φ. It follows that

$$m(r, F_\lambda) = \frac{1 - \cos\beta\lambda}{\lambda\cos\beta\lambda} r^\lambda + o(r^\lambda) \qquad (r \to \infty),$$

$$T(r, F_\lambda) = \frac{r^\lambda}{\lambda\cos\beta\lambda} + o(r^\lambda) \qquad (r \to \infty),$$

so

$$\Delta(\infty, F_\lambda) = 1 - \cos\beta\lambda.$$

It is clear that F_λ is a meromorphic function of both lower order and order λ with $p(\infty, F_\lambda) = 1$. Also, for $r \neq n^{1/\lambda}$ $(n = 1, 2, \dots)$ we have

$$\mathcal{L}(r, \infty, F_\lambda) = \frac{\pi \sin \beta\lambda}{\cos \beta\lambda} r^\lambda + o(r^\lambda) \quad (r \to \infty),$$

so

$$\beta(\infty, F_\lambda) = \pi\lambda \sin \beta\lambda.$$

We get

$$B(\lambda, \Delta(\infty, F_\lambda)) = \pi\lambda\sqrt{\Delta(\infty, F_\lambda)(2 - \Delta(\infty, F_\lambda))} = \beta(\infty, F_\lambda).$$

Consider now the function $F(z) = F_{\lambda/n}(z^n)$, where n is a fixed positive integer. As in the previous cases we observe that F is a meromorphic function with both lower order and order λ. Also $p(\infty, F) = n$ and

$$B(\lambda, \Delta(\infty, F)) = \pi\frac{\lambda}{n}\sqrt{\Delta(\infty, F)(2 - \Delta(\infty, F))} = \pi\frac{\lambda}{n} \sin \beta\frac{\lambda}{n} = \beta(\infty, F).$$

Finally, let us mention that for $\Delta(\infty, f) = 1$ the equality (1.1) holds for the function $f(z) = F_{\lambda/n}(z^n)$, where F_λ is a Mittag-Leffler function of order λ, $F_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+n/\lambda)}$ [10].

References

[1] A. Baernstein, *Integral means, univalent functions and circular symmetrization*, Acta Math. 133 (1974), 139–169.

[2] P. Barry, *On a theorem of Besicovitch*, Quart. J. Math. Oxford 14 (1963), 293–302.

[3] P. Barry, *On a theorem of Kjellberg*, Quart. J. Math. Oxford 15 (1964), 179–191.

[4] E. Ciechanowicz and I. I. Marchenko, *Maximum modulus points, deviations and spreads of meromorphic functions*, in: Value Distribution Theory and Related Topics, Kluwer, 2004, 117–129.

[5] E. Ciechanowicz and I. I. Marchenko, *On deviations and strong asymptotic functions of meromorphic functions of finite lower order*, J. Math. Anal. Appl. 382 (2011), 383–398.

[6] A. Edrei and W. H. J. Fuchs, *Asymptotic behavior of meromorphic functions with extremal spread, I*, Ann. Acad. Sci. Fenn. Ser. A I 2 (1976), 67–111.

[7] M. Essen and D. F. Shea, *Applications of Denjoy integral inequalities and differential inequalities to growth problems for subharmonic and meromorphic functions*, Proc. Roy. Irish Acad. Sect. A 82 (1982), 201–216.

[8] W. H. J. Fuchs, *Topics in Nevanlinna theory*, in: Proc. NRL Conf. on Classical Function Theory (Washington, DC, 1970), Naval Res. Lab., Washington, DC, 1970, 1–32.

[9] R. Gariepy and J. L. Lewis, *Space analogues of some theorems for subharmonic and meromorphic functions*, Ark. Mat. 13 (1975), 91–105.

[10] A. A. Gol'dberg and I. V. Ostrovskiĭ, *Value Distribution of Meromorphic Functions*, Nauka, Moscow, 1970 (in Russian); English transl.: Transl. Math. Monogr. 236, Amer. Math. Soc., Providence, 2008.

[11] N. V. Govorov, *On Paley's problem*, Funktsional. Anal. i Prilozhen. 3 (1969), no. 2, 38–43 (in Russian).

[12] W. K. Hayman, *Multivalent Functions*, Cambridge Univ. Press, Cambridge, 1958.
[13] I. I. Marchenko, *On the magnitudes of deviations and spreads of meromorphic functions of finite lower order*, Mat. Sb. 186 (1995), 391–408 (in Russian), English transl.: Sb. Math. 186 (1995), 91–408.
[14] I. I. Marchenko, *An analogue of the second main theorem for the uniform metric*, Mat. Fiz. Anal. Geom. 5 (1998), 212–227 (in Russian).
[15] I. I. Marchenko, *On the Shea estimate of the magnitude of deviation of a meromorphic function*, Izv. Vyssh. Ucheb. Zaved. Mat. 2000, no. 8 (457), 46–51 (in Russian); English transl.: Russian Math. (Iz. VUZ) 44 (2000), no. 8, 44–49.
[16] I. I. Marchenko and A. I. Shcherba, *On the magnitudes of deviations of meromorphic functions*, Mat. Sb. 181 (1990), 3–24 (in Russian). English transl.: Math. USSR-Sb. 69 (1991), 1–24.
[17] R. Nevanlinna, *Single-Valued Analytic Functions*, OGIZ, Moscow, 1941 (in Russian).
[18] V. P. Petrenko, *The growth of meromorphic functions of finite lower order*, Izv. Akad. Nauk SSSR 33 (1969), 414–454 (in Russian).
[19] V. P. Petrenko, *Growth of Meromorphic Functions*, Kharkov, Vyshcha Shkola, 1978 (in Russian).
[20] L. I. Ronkin, *Introduction to the Theory of Entire Functions of Several Variables*, Nauka, Moscow, 1971 (in Russian).
[21] M. A. Ryshkov, *On the exactness of the estimate of the value of deviation for a meromorphic function*, Teor. Funktsiĭ Funktsional. Anal. Prilozhen. 37 (1982), 114–115 (in Russian).

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Received 28.12.2012
and in final form 13.4.2013 (2985)