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Differentiable solutions for a class of functional equations

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Abstract. We give a set of sufficient conditions for the existence of differentiable solutions for a functional equation involving a series of iterates, using a method different from that of Baker and Zhang [Ann. Polon. Math. 73 (2000)].

1. Introduction. Iterative functional equations are functional equations involving the iterates of unknown functions. Many authors, including Abel, Babbage, Kuczma and Schröder, have contributed to this topic with a long history (see [2]). The iterative root problem is the problem of finding a function f which equals a given function F after n iterations and it arises in the theory of dynamical systems. In this context, Zhang [9] solved the problem of finding a function f such that certain convex combinations of the iterates of f equal F, i.e.,

$$\sum_{i=1}^{n} \lambda_i f^i(x) = F(x). \tag{1.1}$$

This equation has been studied in a variety of ways and theorems on the existence and uniqueness of solutions have been established by many authors (see [3–9]). Baker and Zhang [1] studied (1.1) assuming λ_i's are functions of x and obtained continuous solutions. In this paper, we provide sufficient conditions for the existence of differentiable solutions in the interval $I = [a, b] \subset \mathbb{R}$ for the functional equation of the form

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(f^i(x)) = F(x), \qquad x \in I, \tag{1.2}$$

where $\lambda_i(x)$ is a sequence of nonnegative functions on I such that $\sum_{i=1}^{\infty} \lambda_i(x) = 1$, H_i's and F being given functions.

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2. Preliminaries. Consider the Banach space $C^1(I,\mathbb{R})$ of all continuously differentiable functions from I into $\mathbb{R}$ with the norm $\|\cdot\|_1$, where $\|\phi\|_1 = \|\phi\| + \|\phi'\|$ and $\|\phi\| = \sup_{x\in I}|\phi(x)|$. For $M, M' \geq 0$, we define

$$\mathcal{Q}^1(M,M') = \{f \in C^1(I,\mathbb{R}) : |f'(x)| \leq M,\ |f'(x) - f'(y)| \leq M'|x-y|\}$$

and for $\delta \geq 0$,

$$\mathcal{F}^1_\delta(M,M') = \{f \in \mathcal{Q}^1(M,M') : f(a) = a,\ f(b) = b,\ \delta \leq f'(x)\}.$$

The proofs of the following propositions are easy and hence omitted.

PROPOSITION 2.1 (see [4]). *If $M < 1$ or $\delta > 1$ then $\mathcal{F}^1_\delta(M,M')$ is empty, and if $M = 1$ or $\delta = 1$ then $\mathcal{F}^1_\delta(M,M')$ is either empty or contains only the identity map.*

PROPOSITION 2.2. *The set $\mathcal{F}^1_\delta(M,M')$ is a convex compact subset of $C^1(I,\mathbb{R})$.*

In view of Proposition 2.1, one cannot seek solutions of equations such as (1.2) in $\mathcal{F}^1_\delta(M,M')$ without imposing conditions on M. The following lemma is essentially due to Zhang [9] and with the assumption that $M > 1$.

LEMMA 2.3. *Let $\phi,\psi \in \mathcal{Q}^1(M,M')$. Then for $i = 1,2,\ldots,$*

(1) $|(\phi^i)'(x)| \leq M^i, \quad \forall x \in I,$

(2) $|(\phi^i)'(x_1) - (\phi^i)'(x_2)| \leq M' \dfrac{M^{i-1}}{M-1}(M^i - 1)|x_1 - x_2|, \quad \forall x_1, x_2 \in I,$

(3) $\|\phi^i - \psi^i\| \leq \dfrac{M^i - 1}{M-1}\|\phi - \psi\|,$

(4) $\|(\phi^i)' - (\psi^i)'\| \leq iM^{i-1}\|\phi' - \psi'\| + M'M^{i-2}[M^i - iM + (i-1)]\|\phi - \psi\|.$

LEMMA 2.4. *For $\delta > 0$, $M > 0$ and $M' \geq 0$, if $f \in \mathcal{F}^1_\delta(M,M')$, then f is a diffeomorphism on I onto I and $f^{-1} \in \mathcal{F}^1_{1/M}(1/\delta, M'/\delta^3)$.*

Proof. As $0 < \delta \leq f'(x) \leq M$, and $f(a) = a$, $f(b) = b$, f is a strictly increasing function from I onto itself and hence f is invertible on I and $f^{-1}(a) = a$ and $f^{-1}(b) = b$. As $(f^{-1})'(x) = 1/f'(f^{-1}(x))$, for $x \in I$ we have

$$1/M \leq (f^{-1})'(x) \leq 1/\delta. \tag{2.3}$$

As $f'(x) \geq \delta > 0$ and $|f'(x) - f'(y)| \leq M'|x-y|$, for $x,y \in I$, we have

$$\begin{aligned}|(f^{-1})'(x) - (f^{-1})'(y)| &= \left|\frac{1}{f'(f^{-1}(x))} - \frac{1}{f'(f^{-1}(y))}\right| \\ &\leq \frac{1}{\delta^2}|f'(f^{-1}(x)) - f'(f^{-1}(y))|.\end{aligned}$$

As $|f'(x) - f'(y)| \leq M'|x-y|$, for $x,y \in I$ and by (2.3), we get

$$|(f^{-1})'(x) - (f^{-1})'(y)| \leq \frac{M'}{\delta^2}|f^{-1}(x) - f^{-1}(y)| \leq \frac{M'}{\delta^3}|x-y|. \tag{2.4}$$

The result follows from (2.3) and (2.4). ∎

Lemma 2.5 (Zhang [9]). *Let ϕ_1 and ϕ_2 be two homeomorphisms from I onto itself with $|\phi_i(x_1) - \phi_i(x_2)| \le M|x_1 - x_2|$ for all $x_1, x_2 \in I$, $i = 1, 2$, for some $M > 0$. Then*

$$\|\phi_1 - \phi_2\| \le M\|\phi_1^{-1} - \phi_2^{-1}\|.$$

Lemma 2.6. *If $f, g \in \mathcal{F}^1_\delta(M, M')$ where $0 < \delta \le 1 \le M$ and $M' \ge 0$, then*

$$\|(f^{-1})' - (g^{-1})'\| \le \frac{M'}{\delta^3}\|f - g\| + \frac{1}{\delta^2}\|f' - g'\|. \tag{2.5}$$

Proof. As $f'(x) \ge 0$, for $x \in I$, we get

$$\begin{aligned} |(f^{-1})'(x) - (g^{-1})'(x)| &= \left|\frac{1}{f'(f^{-1}(x))} - \frac{1}{g'(g^{-1}(x))}\right| \\ &\le \frac{1}{\delta^2}|f'(f^{-1}(x)) - g'(g^{-1}(x))| \\ &\le \frac{1}{\delta^2}\{|f'(f^{-1}(x)) - f'(g^{-1}(x))| \\ &\quad + |f'(g^{-1}(x)) - g'(g^{-1}(x))|\} \\ &\le \frac{1}{\delta^2}\{M'\|f^{-1} - g^{-1}\| + \|f' - g'\|\}. \end{aligned}$$

Using Lemma 2.5 for f^{-1} and g^{-1}, we have

$$\|(f^{-1})' - (g^{-1})'\| \le \frac{M'}{\delta^3}\|f - g\| + \frac{1}{\delta^2}\|f' - g'\| \tag{2.6}$$

and thus the result follows. ∎

Proposition 2.7. *The function f is a solution for the functional equation*

$$\sum_{i=1}^{\infty} \lambda_i(x) H_i(f^i(x)) = F(x), \qquad \forall x \in [a, b], \tag{2.7}$$

if and only if $g(x) = h^{-1}(f(h(x)))$ is a solution for the functional equation

$$\sum_{i=1}^{\infty} \mu_i(x) R_i(g^i(x)) = G(x), \qquad \forall x \in [0, 1], \tag{2.8}$$

where $\mu_i(x) = \lambda_i(h(x))$, $R_i(x) = h^{-1}(H_i(h(x)))$, $G(x) = h^{-1}(F(h(x)))$ and $h(x) = a + x(b - a)$ for $x \in [0, 1]$ with $\sum_{i=1}^{\infty} \lambda_i(x) = 1$.

Proof. Let f be a solution for (2.7). Note that h and h^{-1} are affine and continuous maps and $\sum_{i=1}^{\infty} \mu_i(x) = 1$ for $x \in [0, 1]$. Therefore for $x \in [0, 1]$,

$$
\begin{aligned}
\sum_{i=1}^{\infty} \mu_i(x) R_i(g^i(x)) &= \lim_{n\to\infty} \frac{1}{\sum_{j=1}^n \mu_j(x)} \sum_{i=1}^{n} \mu_i(x) R_i(g^i(x)) \\
&= \lim_{n\to\infty} \frac{1}{\sum_{j=1}^n \mu_j(x)} \sum_{i=1}^{n} \mu_i(x) h^{-1}(H_i(h(h^{-1} f^i(h(x))))) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu_i(x)}{\sum_{j=1}^n \mu_j(x)} h^{-1}(H_i(f^i(h(x)))) \\
&= \lim_{n\to\infty} h^{-1}\Big[\frac{1}{\sum_{j=1}^n \mu_j(x)} \sum_{i=1}^{n} \mu_i(x) H_i(f^i(h(x)))\Big] \\
&= h^{-1}\Big[\lim_{n\to\infty} \sum_{i=1}^{n} \mu_i(x) H_i(f^i(h(x)))\Big] \\
&= h^{-1}\Big[\sum_{i=1}^{\infty} \mu_i(x) H_i(f^i(h(x)))\Big] = h^{-1}(F(h(x))).
\end{aligned}
$$

Thus

$$
\sum_{i=1}^{\infty} \mu_i(x) R_i(g^i(x)) = G(x). \tag{2.9}
$$

The converse follows similarly. ∎

In view of Proposition 2.7, it suffices to prove the existence of solution for the functional equation (1.2) on $I = [0, 1]$.

3. Existence, uniqueness and stability. We begin with a lemma concerning infinite series. Its proof is straightforward and hence omitted.

LEMMA 3.1. *Let $L_i \geq 1$, $\alpha_i, \beta_i, \Lambda_i$ and L'_i be nonnegative numbers for $i \in \mathbb{N}$ and $M > 1$ satisfying*

$$
\sum_{i=1}^{\infty} \beta_i < \infty, \quad \sum_{i=1}^{\infty} \alpha_i L_i M^{2i} < \infty, \quad \sum_{i=1}^{\infty} \Lambda_i [L_i + L'_i] M^{2i} < \infty.
$$

Then for any $\delta > 0$ and $M' \geq 0$, the following series denoted by K_1 to K_6 are convergent:

$$
K_1 = \sum_{i=1}^{\infty} \Big\{ \frac{1}{\delta} \alpha_i + \Lambda_i L_i M^{i-1} \Big\},
$$

$$
K_2 = \sum_{i=1}^{\infty} \Big\{ \frac{1}{\delta^2} \beta_i + \frac{2}{\delta} \alpha_i L_i M^{i-1} + \Lambda_i L'_i M^{2(i-1)} \Big\},
$$

$$K_3 = \frac{1}{\delta^3}\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\Lambda_i L_i M^{i-2}\,\frac{M^{i-1}-1}{M-1},$$

$$K_4 = \frac{1}{\delta}\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\Lambda_i L_i\,\frac{M^{i-1}-1}{M-1},$$

$$\begin{aligned}K_5 = {}&\frac{1}{\delta^2}\sum_{i=1}^{\infty}\beta_i + \frac{1}{\delta}\sum_{i=1}^{\infty}\alpha_i L_i\,\frac{M^{i-1}-1}{M-1} + \frac{1}{\delta}\sum_{i=1}^{\infty}\alpha_i L_i M^{i-1}\\ &+ \sum_{i=1}^{\infty}\Lambda_i L_i' M^{i-1}\frac{M^{i-1}-1}{M-1}\\ &+ M'\biggl\{\frac{1}{\delta^3}\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\Lambda_i L_i M^{i-3}[M^{i-1}-(i-1)M+(i-2)]\biggr\},\end{aligned}$$

$$K_6 = \frac{1}{\delta^2}\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\Lambda_i L_i (i-1)M^{i-2}.$$

Now we state our main result on the existence of a solution to (1.2).

THEOREM 3.2. *In addition to the hypotheses of Lemma* 3.1, *let* λ_i *be a sequence of nonnegative functions on* I *such that* $\lambda_i(x) \in \mathcal{Q}^1(\alpha_i,\beta_i)$ *and* $\gamma_i \le \lambda_i(x) \le \Lambda_i$ *for* $i = 1,2,\ldots,$ *and* $\sum_{i=1}^{\infty}\lambda_i(x) = 1$ *for* $x \in I$. *Let* $H_i \in \mathcal{F}^1_{l_i}(L_i, L_i')$ *where* l_i *are nonnegative numbers for* $i = 1,2,\ldots.$ *Suppose* $0<\delta<1$, $M^* \ge 0$, *and*

$$K_0 = \sum_{i=1}^{\infty}\gamma_i l_i \delta^{i-1} - \frac{1}{\delta}\sum_{i=1}^{\infty}\alpha_i > M^2 K_3.$$

Then for any function F *in* $\mathcal{F}^1_{K_1\delta}(K_0 M, M^*)$, *the functional equation*

$$\sum_{i=1}^{\infty}\lambda_i(x)H_i(f^i(x)) = F(x)$$

has a solution f *in* $\mathcal{F}^1_{\delta}(M,M')$ *for every* $M' \ge (M^* + M^2K_2)/(K_0 - M^2K_3)$.

The following lemmata will lead directly to the proof of Theorem 3.2. For $f \in \mathcal{F}^1_{\delta}(M,M')$, we define $L_f : I \to I$ by

$$L_f(x) = \sum_{i=1}^{\infty}\lambda_i(f^{-1}(x))H_i(f^{i-1}(x)), \qquad x \in I. \tag{3.10}$$

LEMMA 3.3. *In addition to the hypotheses of Theorem 3.2, suppose that* $f \in \mathcal{F}^1_{\delta}(M,M')$. *Then* $L_f \in \mathcal{F}^1_{K_0}(K_1, K_2 + M'K_3)$.

Proof. It is clear that $L_f(0)=0$ and $L_f(1)=1$. Because $K_1=\sum_{i=1}^{\infty}\{\delta^{-1}\alpha_i + \Lambda_i L_i M^{i-1}\} < \infty$, the function L_f is differentiable and

$$L'_f(x) = \sum_{i=1}^{\infty} \lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) + \sum_{i=1}^{\infty} \lambda_i(f^{-1}(x))H'_i(f^{i-1}(x))(f^{i-1})'(x)$$

and by the hypothesis of Theorem 3.2, we have

$$0 < K_0 \le L'_f(x) \le K_1. \tag{3.11}$$

Thus

$$1/K_1 \le (L_f^{-1})'(x) \le 1/K_0. \tag{3.12}$$

From the definition of $L_f(x)$ and the triangle inequality, for $x, y \in I$,

$$\begin{aligned}
&|L'_f(x) - L'_f(y)| \\
&\le \sum_{i=1}^{\infty}\{|\lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) + \lambda_i(f^{-1}(x))H'_i(f^{i-1}(x))(f^{i-1})'(x) \\
&\quad - \lambda'_i(f^{-1}(y))(f^{-1})'(y)H_i(f^{i-1}(y)) - \lambda_i(f^{-1}(y))H'_i(f^{i-1}(y))(f^{i-1})'(y)|\} \\
&\le \sum_{i=1}^{\infty}\{|\lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) - \lambda'_i(f^{-1}(y))(f^{-1})'(y)H_i(f^{i-1}(y))| \\
&\quad + |\lambda_i(f^{-1}(x))H'_i(f^{i-1}(x))(f^{i-1})'(x) - \lambda_i(f^{-1}(y))H'_i(f^{i-1}(y))(f^{i-1})'(y)|\} \\
&=: \sum_{i=1}^{\infty}(A_i + B_i).
\end{aligned}$$

By Lemma 2.4, $H_i(x) \le 1$ and by the definition of $\lambda_i(x)$ and $H_i(x)$, for all i,

$$\begin{aligned}
A_i &\le |\lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) - \lambda'_i(f^{-1}(y))(f^{-1})'(y)H_i(f^{i-1}(x))| \\
&\quad + |\lambda'_i(f^{-1}(y))(f^{-1})'(y)H_i(f^{i-1}(x)) - \lambda'_i(f^{-1}(y))(f^{-1})'(y)H_i(f^{i-1}(y))| \\
&\le |\lambda'_i(f^{-1}(x))(f^{-1})'(x) - \lambda'_i(f^{-1}(y))(f^{-1})'(y)| \\
&\quad + \frac{1}{\delta}\,\alpha_i|H_i(f^{i-1}(x)) - H_i(f^{i-1}(y))| \\
&\le |\lambda'_i(f^{-1}(x)) - \lambda'_i(f^{-1}(y))|\,|(f^{-1})'(x)| \\
&\quad + |\lambda'_i(f^{-1}(y))|\,|(f^{-1})'(x) - (f^{-1})'(y)| + \frac{1}{\delta}\,\alpha_i L_i|f^{i-1}(x) - f^{i-1}(y)|.
\end{aligned}$$

In view of Lemmas 2.3 and 2.4, we get

$$(3.13)\quad \begin{aligned} A_i &\le \frac{1}{\delta}\,\beta_i|f^{-1}(x)-f^{-1}(y)| + \frac{M'}{\delta^3}\,\alpha_i|x-y| + \frac{1}{\delta}\,\alpha_i L_i M^{i-1}|x-y| \\ &\le \left\{\frac{1}{\delta^2}\,\beta_i + \frac{M'}{\delta^3}\,\alpha_i + \frac{1}{\delta}\,\alpha_i L_i M^{i-1}\right\}|x-y|. \end{aligned}$$

Again for each i,

$$\begin{aligned} B_i \le{}& |\lambda_i(f^{-1}(x))H_i'(f^{i-1}(x)) - \lambda_i(f^{-1}(y))H_i'(f^{i-1}(y))|\,|(f^{i-1})'(x)| \\ &+ |\lambda_i(f^{-1}(y))|\,|H_i'(f^{i-1}(y))|\,|(f^{i-1})'(x) - (f^{i-1})'(y)|. \end{aligned}$$

By (1) and (2) of Lemma 2.3 and definitions of $\lambda_i(x)$ and $H_i(x)$, we get

$$\begin{aligned} B_i &\le M^{i-1}|\lambda_i(f^{-1}(x))H_i'(f^{i-1}(x)) - \lambda_i(f^{-1}(y))H_i'(f^{i-1}(y))| \\ &\quad + \Lambda_i L_i|(f^{i-1})'(x) - (f^{i-1})'(y)| \\ &\le M^{i-1}\{|\lambda_i(f^{-1}(x)) - \lambda_i(f^{-1}(y))|\,|H_i'(f^{i-1}(x))| \\ &\quad + |\lambda_i(f^{-1}(y))|\,|H_i'(f^{i-1}(x)) - H_i'(f^{i-1}(y))|\} \\ &\quad + M'\Lambda_i L_i M^{i-2}\,\frac{M^{i-1}-1}{M-1}\,|x-y|. \end{aligned}$$

Using the definitions of $\lambda_i(x)$ and $H_i(x)$, and Lemmas 2.3 and 2.4, we get

$$(3.14)\quad \begin{aligned} B_i &\le \alpha_i L_i M^{i-1}|f^{-1}(x)-f^{-1}(y)| + \Lambda_i L_i' M^{i-1}|f^{i-1}(x)-f^{i-1}(y)| \\ &\quad + M'\Lambda_i L_i M^{i-2}\,\frac{M^{i-1}-1}{M-1}\,|x-y| \\ &\le \frac{1}{\delta}\,\alpha_i L_i M^{i-1}|x-y| + \Lambda_i L_i' M^{2(i-1)}|x-y| \\ &\quad + M'\Lambda_i L_i M^{i-2}\,\frac{M^{i-1}-1}{M-1}\,|x-y|. \end{aligned}$$

From (3.13) and (3.14), for $x, y \in I$ we get

$$(3.15)\qquad |L_f'(x) - L_f'(y)| \le (K_2 + M'K_3)|x-y|.$$

Thus, by (3.11) and (3.15), L_f is in $\mathcal{F}^1_{K_0}(K_1, K_2+M'K_3)$. ∎

LEMMA 3.4. *In addition to the hypotheses of Theorem* 3.2, *suppose that* $f, g \in \mathcal{F}^1_\delta(M, M')$. *Then*

(i) $\|L_f - L_g\| \le K_4\|f-g\|$,

(ii) $\|L_f^{-1} - L_g^{-1}\| \le \dfrac{K_4}{K_0}\,\|f-g\|$,

(iii) $\|L_f' - L_g'\| \le K_5\|f-g\| + K_6\|f'-g'\|$,

(iv) $\|(L_f^{-1})' - (L_g^{-1})'\| \le \left[\dfrac{K_4}{K_0^3}\,(K_2+M'K_3) + \dfrac{K_5}{K_0^2}\right]\|f-g\| + \dfrac{K_6}{K_0^2}\,\|f'-g'\|$.

Proof. For $x \in I$ and $f, g \in \mathcal{F}^1_\delta(M, M')$,

$$|L_f(x) - L_g(x)| \le \sum_{i=1}^{\infty} |\lambda_i(f^{-1}(x))H_i(f^{i-1}(x)) - \lambda_i(g^{-1}(x))H_i(g^{i-1}(x))|$$
$$\le \sum_{i=1}^{\infty}\{|\lambda_i(f^{-1}(x)) - \lambda_i(g^{-1}(x))|\,|H_i(f^{i-1}(x))|$$
$$+ |\lambda_i(g^{-1}(x))|\,|H_i(f^{i-1}(x)) - H_i(g^{i-1}(x))|\}.$$

From Lemmas 2.3 and 2.5 and the definitions of $\lambda_i(x)$ and $H_i(x)$, we get

$$|L_f(x) - L_g(x)| \le \sum_{i=1}^{\infty}\{\alpha_i|f^{-1}(x) - g^{-1}(x)| + \Lambda_i L_i \|f^{i-1} - g^{i-1}\|\}$$
$$\le \sum_{i=1}^{\infty}\left\{\frac{1}{\delta}\,\alpha_i + \Lambda_i L_i\,\frac{M^{i-1} - 1}{M - 1}\right\}\|f - g\|.$$

Therefore,

$$\|L_f - L_g\| \le K_4\|f - g\|, \tag{3.16}$$

proving (i). By (3.12), for $f \in \mathcal{F}^1_\delta(M, M')$,

$$|L_f^{-1}(x) - L_f^{-1}(y)| \le \frac{1}{K_0}\,|x - y|. \tag{3.17}$$

So, from Lemma 2.5, and (3.16) we get

$$\|L_f^{-1} - L_g^{-1}\| \le \frac{1}{K_0}\,\|L_f - L_g\| \le \frac{K_4}{K_0}\,\|f - g\|, \tag{3.18}$$

proving (ii). Now for $x \in I$,

$$|L'_f(x) - L'_g(x)|$$
$$\le \sum_{i=1}^{\infty}\{|\lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) + \lambda_i(f^{-1}(x))H'_i(f^{i-1}(x))(f^{i-1})'(x)$$
$$- \lambda'_i(g^{-1}(x))(g^{-1})'(x)H_i(g^{i-1}(x)) - \lambda_i(g^{-1}(x))H'_i(g^{i-1}(x))(g^{i-1})'(x)|\}$$
$$\le \sum_{i=1}^{\infty}\{|\lambda'_i(f^{-1}(x))(f^{-1})'(x)H_i(f^{i-1}(x)) - \lambda'_i(g^{-1}(x))(g^{-1})'(x)H_i(g^{i-1}(x))|$$
$$+ |\lambda_i(f^{-1}(x))H'_i(f^{i-1}(x))(f^{i-1})'(x) - \lambda_i(g^{-1}(x))H'_i(g^{i-1}(x))(g^{i-1})'(x)|\}$$
$$=: \sum_{i=1}^{\infty}(C_i + D_i).$$

Using Lemmas 2.3 and 2.4, for each i we get

$$(3.19)\quad \begin{aligned} C_i &\le |\lambda_i'(f^{-1}(x))(f^{-1})'(x)-\lambda_i'(g^{-1}(x))(g^{-1})'(x)|\,|H_i(f^{i-1}(x))| \\ &\quad +|\lambda_i'(g^{-1}(x))(g^{-1})'(x)|\,|H_i(f^{i-1}(x))-H_i(g^{i-1}(x))| \\ &\le |\lambda_i'(f^{-1}(x))-\lambda_i'(g^{-1}(x))|\,|(f^{-1})'(x)| \\ &\quad +|\lambda_i'(g^{-1}(x))|\,|(f^{-1})'(x)-(g^{-1})'(x)| \\ &\quad +\frac{1}{\delta}\,\alpha_i L_i\,\frac{M^{i-1}-1}{M-1}\,\|f-g\| \\ &\le \frac{1}{\delta^2}\,\beta_i\|f-g\|+\frac{M'}{\delta^3}\,\alpha_i\|f-g\|+\frac{1}{\delta^2}\,\alpha_i\|f'-g'\| \\ &\quad +\frac{1}{\delta}\,\alpha_i L_i\,\frac{M^{i-1}-1}{M-1}\,\|f-g\|. \end{aligned}$$

Further, for each i,

$$\begin{aligned} D_i &= |\lambda_i(f^{-1}(x))H_i'(f^{i-1}(x))(f^{i-1})'(x)-\lambda_i(g^{-1}(x))H_i'(g^{i-1}(x))(g^{i-1})'(x)| \\ &\le |\lambda_i(f^{-1}(x))-\lambda_i(g^{-1}(x))|\,|H_i'(f^{i-1}(x))|\,|(f^{i-1})'(x)| \\ &\quad +|\lambda_i(g^{-1}(x))|\,|H_i'(f^{i-1}(x))(f^{i-1})'(x)-H_i'(g^{i-1}(x))(g^{i-1})'(x)| \\ &\le M^{i-1}\{|\lambda_i(f^{-1}(x))-\lambda_i(g^{-1}(x))|\,|H_i'(f^{i-1}(x))| \\ &\quad +|\lambda_i(g^{-1}(x))|\,|H_i'(f^{i-1}(x))-H_i'(g^{i-1}(x))|\}+\Lambda_i L_i\|(f^{i-1})'-(g^{i-1})'\|. \end{aligned}$$

Applying Lemma 2.3, we get

$$(3.20)\quad \begin{aligned} D_i &\le \frac{1}{\delta}\,\alpha_i L_i M^{i-1}\|f-g\|+\Lambda_i L_i' M^{i-1}\|f^{i-1}-g^{i-1}\| \\ &\quad +\Lambda_i L_i(i-1)M^{i-2}\|f'-g'\| \\ &\quad +M'\Lambda_i L_i M^{i-3}[M^{i-1}-(i-1)M+(i-2)]\|f-g\| \\ &\le \frac{1}{\delta}\,\alpha_i L_i M^{i-1}\|f-g\|+\Lambda_i L_i' M^{i-1}\,\frac{M^{i-1}-1}{M-1}\,\|f-g\| \\ &\quad +\Lambda_i L_i(i-1)M^{i-2}\|f'-g'\| \\ &\quad +M'\Lambda_i L_i M^{i-3}[M^{i-1}-(i-1)M+(i-2)]\|f-g\|. \end{aligned}$$

Using (3.19) and (3.20), we get

$$(3.21)\quad |L_f'(x)-L_g'(x)|\le K_5\|f-g\|+K_6\|f'-g'\|.$$

Thus (iii) is established.

For $f,g\in\mathcal{F}_\delta^1(M,M')$ and $x\in I$,

$$|(L_f^{-1})'(x)-(L_g^{-1})'(x)|=\left|\frac{1}{L_f'(L_f^{-1}(x))}-\frac{1}{(L_g'(L_g^{-1}(x))}\right|.$$

Using (3.11) we have

$$\begin{aligned}|(L_f^{-1})'(x)-(L_g^{-1})'(x)| &\le \frac{1}{K_0^2}\,|L_f'(L_f^{-1}(x))-L_g'(L_g^{-1}(x))|\\ &\le \frac{1}{K_0^2}\,\{|L_f'(L_f^{-1}(x))-L_g'(L_f^{-1}(x))|\\ &\quad +|L_g'(L_f^{-1}(x))-L_g'(L_g^{-1}(x))|\}.\end{aligned}$$

From (3.15) and (3.21) we have

$$\begin{aligned}|(L_f^{-1})'(x)-(L_g^{-1})'(x)| \le \frac{1}{K_0^2}\,\{&K_5\|f-g\|+K_6\|f'-g'\|\\ &+(K_2+M'K_3)\|L_f^{-1}-L_g^{-1}\|\}.\end{aligned}$$

Further using (3.18) to estimate $\|L_f^{-1}-L_g^{-1}\|$, we get

$$\begin{aligned}\|(L_f^{-1})'-(L_g^{-1})'\| \le \frac{1}{K_0^2}\,\{&K_5\|f-g\|+K_6\|f'-g'\|\\ &+\frac{K_4}{K_0}\,(K_2+M'K_3)\|f-g\|\},\end{aligned}$$

thus proving (iv) of Lemma 3.4 as well. ∎

Proof of Theorem 3.2. Define $T:\mathcal{F}_\delta^1(M,M')\to C^1(I,\mathbb{R})$ by

$$Tf(x)=(L_f^{-1})(F(x)) \quad \text{for } f\in\mathcal{F}_\delta^1(M,M'). \tag{3.22}$$

By the definitions of L_f and F we get $Tf(0)=0$ and $Tf(1)=1$. Now

$$Tf'(x)=(L_f^{-1})'(F(x))F'(x) \tag{3.23}$$

and so $\delta=\frac{1}{K_1}K_1\delta\le Tf'(x)\le\frac{1}{K_0}K_0M=M$. Hence

$$\delta\le Tf'(x)\le M. \tag{3.24}$$

For x,y in I,

$$\begin{aligned}|Tf'(x)-Tf'(y)| &\le |(L_f^{-1})'(F(x))F'(x)-(L_f^{-1})'(F(y))F'(y)|\\ &\le |(L_f^{-1})'(F(x))-(L_f^{-1})'(F(y))|\,|F'(x)|\\ &\quad +|(L_f^{-1})'(F(y))|\,|F'(x)-F'(y)|.\end{aligned}$$

Applying Lemma 2.4 for $L_f\in\mathcal{F}_{K_0}^1(K_1,K_2+M'K_3)$ and using the definition of F, we get

$$\begin{aligned}|Tf'(x)-Tf'(y)| &\le \frac{K_2+M'K_3}{K_0^3}\,K_0M|F(x)-F(y)|+\frac{M^*}{K_0}\,|x-y|\\ &\le \frac{M^2(K_2+M'K_3)+M^*}{K_0}\,|x-y|.\end{aligned}$$

As $M' \geq (M^* + M^2K_2)/(K_0 - M^2K_3)$, we have

$$|Tf'(x) - Tf'(y)| \leq M'|x - y|. \tag{3.25}$$

Hence T is a self-map of $\mathcal{F}^1_\delta(M, M')$.

We now prove that T is continuous. For $f, g \in \mathcal{F}^1_\delta(M, M')$ and $x \in I$,

$$\begin{aligned}|Tf(x) - Tg(x)| &= |L_f^{-1}(F(x)) - L_g^{-1}(F(x))| \\ &\leq \|L_f^{-1} - L_g^{-1}\| \leq \frac{1}{K_0}\,\|L_f - L_g\|.\end{aligned}$$

So, by Lemma 3.4, we have

$$\|Tf - Tg\| \leq \frac{K_4}{K_0}\,\|f - g\| \tag{3.26}$$

and

$$\begin{aligned}|Tf'(x) - Tg'(x)| &= |(L_f^{-1})'(F(x))F'(x) - (L_g^{-1})'(F(x))F'(x)| \\ &\leq K_0M|(L_f^{-1})'(F(x)) - (L_g^{-1})'(F(x))|.\end{aligned}$$

Using Lemma 3.4(iv), we get

$$\begin{aligned}&|Tf'(x) - Tg'(x)| \\ &\quad\leq K_0M\left\{\frac{K_4(K_2 + M'K_3)}{K_0^3}\,\|f - g\| + \frac{K_5}{K_0^2}\,\|f - g\| + \frac{K_6}{K_0^2}\,\|f' - g'\|\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|Tf' - Tg'\| \\ &\quad\leq \frac{MK_4(K_2 + M'K_3)}{K_0^2}\,\|f - g\| + \frac{MK_5}{K_0}\,\|f - g\| + \frac{MK_6}{K_0}\,\|f' - g'\|.\end{aligned} \tag{3.27}$$

Consequently, from (3.26) and (3.27), we have

$$\begin{aligned}\|Tf - Tg\|_1 &= \|Tf - Tg\| + \|Tf' - Tg'\| \\ &\leq \frac{K_4}{K_0}\,\|f - g\| + \frac{MK_4}{K_0^2}\,(K_2 + M'K_3)\|f - g\| \\ &\quad + \frac{MK_5}{K_0}\,\|f - g\| + \frac{MK_6}{K_0}\,\|f' - g'\|.\end{aligned}$$

Hence

$$\|Tf - Tg\|_1 \leq \varrho\|f - g\|_1, \tag{3.28}$$

where

$$\varrho = \max\left\{\frac{K_4}{K_0} + \frac{MK_4}{K_0^2}\,(K_2 + M'K_3) + \frac{MK_5}{K_0}, \frac{MK_6}{K_0}\right\}.$$

This proves that T is continuous. By Proposition 2.2, T is a continuous self-map of the convex, compact subset $\mathcal{F}^1_\delta(M, M')$ of $C^1(I, \mathbb{R})$. So, by

Schauder's fixed point theorem, T has a fixed point, which is a solution of the functional equation (1.2). ∎

REMARK 3.5. Clearly, by the hypothesis of Theorem 3.2, γ_1 and l_1 are positive.

THEOREM 3.6. *In addition to the hypotheses of Theorem* 3.2, *let*

$$\varrho = \max\left\{ \frac{K_4}{K_0} + \frac{MK_4}{K_0^2}\,(K_2 + M'K_3) + \frac{MK_5}{K_0}, \frac{MK_6}{K_0} \right\} < 1.$$

Then for $F \in \mathcal{F}^1_{K_1\delta}(K_0M, M^*)$, *the functional equation* (1.2) *has a unique solution* f *in* $\mathcal{F}^1_\delta(M, M')$ *and the solution* f *depends continuously on the function* F.

Proof. As $\varrho < 1$, the uniqueness of the solution follows from the Banach contraction principle for T. Let $F, G \in \mathcal{F}^1_{K_1\delta}(K_0M, M^*)$, and let f and g be the solutions for the functional equations involving F and G respectively. Thus $L_f^{-1}(F(x)) = f(x)$ and $L_g^{-1}(G(x)) = g(x)$ for $x \in I$. So,

$$\begin{aligned} |f(x) - g(x)| &= |L_f^{-1}(F(x)) - L_g^{-1}(G(x))| \\ &\le |L_f^{-1}(F(x)) - L_g^{-1}(F(x))| + |L_g^{-1}(F(x)) - L_g^{-1}(G(x))| \\ &\le \|L_f^{-1} - L_g^{-1}\| + \frac{1}{K_0}\,\|F - G\|. \end{aligned}$$

Using Lemma 3.4, we get

$$\|f - g\| \le \frac{K_4}{K_0}\,\|f - g\| + \frac{1}{K_0}\,\|F - G\|. \tag{3.29}$$

From (3.12) and the definition of F, for x in I, we have

$$\begin{aligned} |f'(x) - g'(x)| &= |(L_f^{-1})'(F(x))F'(x) - (L_g^{-1})'(G(x))G'(x)| \\ &\le |(L_f^{-1})'(F(x)) - (L_g^{-1})'(G(x))||F'(x)| \\ &\quad + |(L_g^{-1})'(G(x))|\,|F'(x) - G'(x)| \\ &\le K_0M\{|(L_f^{-1})'(F(x)) - (L_g^{-1})'(F(x))| \\ &\quad + |(L_g^{-1})'(F(x)) - (L_g^{-1})'(G(x))|\} + \frac{1}{K_0}\,|F'(x) - G'(x)|. \end{aligned}$$

By Lemmas 3.3 and 3.4(iv), we get

$$\begin{aligned} \|f' - g'\| \le K_0M\left\{ \|(L_f^{-1})' - (L_g^{-1})'\| + \frac{K_2 + M'K_3}{K_0^3}\,\|F - G\| \right\} \\ + \frac{1}{K_0}\,\|F' - G'\| \end{aligned}$$

$$\leq K_0 M\left\{\frac{K_4}{K_0^3}(K_2+M'K_3)\|f-g\|+\frac{K_5}{K_0^2}\|f-g\|+\frac{K_6}{K_0^2}\|f'-g'\|\right\}$$
$$+\frac{K_0 M}{K_0^3}\,(K_2+M'K_3)\|F-G\|+\frac{1}{K_0}\,\|F'-G'\|.$$

Thus we have

$$\|f'-g'\|\leq\frac{M}{K_0}\left\{K_5\|f-g\|+K_6\|f'-g'\|+\frac{K_2+M'K_3}{K_0}\,\|F-G\|\right.$$
$$\left.+\frac{K_4(K_2+M'K_3)}{K_0}\,\|f-g\|\right\}+\frac{1}{K_0}\,\|F'-G'\|. \tag{3.30}$$

From (3.29) and (3.30) we get

$$\|f-g\|_1\leq\left\{\frac{K_4+MK_5}{K_0}+\frac{MK_4(K_2+M'K_3)}{K_0^2}\right\}\|f-g\|$$
$$+\frac{MK_6}{K_0}\,\|f'-g'\|+\left\{\frac{1}{K_0}+\frac{M(K_2+M'K_3)}{K_0^2}\right\}\|F-G\|_1$$
$$\leq\varrho\|f-g\|_1+\left\{\frac{1}{K_0}+\frac{M(K_2+M'K_3)}{K_0^2}\right\}\|F-G\|_1.$$

Thus

$$\|f-g\|_1\leq\frac{1}{1-\varrho}\left\{\frac{1}{K_0}+\frac{M(K_2+M'K_3)}{K_0^2}\right\}\|F-G\|_1, \tag{3.31}$$

which proves the stability. ∎

Now we give an example to illustrate the main theorem.

Example 1. Consider the functional equation

$$\sum_{i=1}^{\infty}\lambda_i(x)H_i(f^i(x))=\frac{1}{2}\left[1+x-\cos\frac{\pi x}{2}\right],\qquad\forall x\in[0,1], \tag{3.32}$$

where

$$\lambda_1(x)=1-\frac{1}{k}\,(e^x-1-x),\qquad\lambda_i(x)=\frac{1}{k}\,\frac{x^i}{i!}\qquad\text{for } i=2,3,\ldots,$$

with $k=27e^9$, $F(x)=\frac{1}{2}[1+x-\cos(\pi x/2)]\in\mathcal{F}^1_{1/2}((\pi+2)/4,\pi^2/8)$, $H_1(x)=x\in\mathcal{F}^1_1(1,0)$ and $H_i(x)=x^{i-1}\in\mathcal{F}^1_0(i-1,(i-1)(i-2))$ for each $i\geq 2$.

Here $l_1=1$, $L_1=1$, $L'_1=0$, $\gamma_1=1-(e-1)/k$, $\Lambda_1=1$, $\alpha_1=(e-1)/k$, $\beta_1=e/k$ and $l_i=0$, $L_i=i-1$, $L'_i=(i-1)(i-2)$, $\gamma_i=0$,

$$\Lambda_i=\frac{1}{k}\,\frac{1}{i!},\qquad\alpha_i=\frac{1}{k}\,\frac{1}{(i-1)!},\qquad\beta_i=\frac{1}{k}\,\frac{1}{(i-2)!}$$

for each $i\geq 2$.

For $M = 3$ and $\delta = 1/4$ it is easily seen that the series $\sum_{i=1}^{\infty} \beta_i$, $\sum_{i=1}^{\infty} \Lambda_i \{L_i + L_i'\} M^{2i}$ and $\sum_{i=1}^{\infty} \alpha_i L_i M^{2i}$ are convergent. Now

$$\begin{aligned} K_1 &= \sum_{i=1}^{\infty} \left\{ \frac{1}{\delta} \alpha_i + \Lambda_i L_i M^{i-1} \right\} \\ &= 4 \left\{ \frac{e-1}{k} + \sum_{i=2}^{\infty} \frac{1}{k} \frac{1}{(i-1)!} \right\} + 1 + \frac{1}{k} \sum_{i=2}^{\infty} \frac{i-1}{i!} M^{i-1} \\ &\leq 4 \left\{ \frac{e-1}{k} + \frac{e-1}{k} \right\} + \frac{1}{k} \sum_{i=2}^{\infty} \frac{1}{(i-1)!} M^{i-1} + 1 \\ &\leq \frac{8(e-1)}{k} + \frac{1}{k} (e^M - 1) + 1 = \frac{1}{27e^9} [8(e-1) + e^3 - 1] + 1 < 2 \end{aligned}$$

and hence $K_1 \delta < 1/2$. Moreover,

$$\begin{aligned} K_0 &= \sum_{i=1}^{\infty} \gamma_i l_i \delta^{i-1} - \frac{1}{\delta} \sum_{i=1}^{\infty} \alpha_i = \gamma_1 - \frac{1}{\delta} \sum_{i=1}^{\infty} \alpha_i = 1 - \frac{e-1}{k} - \frac{2(e-1)}{\delta k} \\ &= 1 - \frac{1}{k\delta} [(e-1)(2+\delta)] = 1 - \frac{4}{27e^9} [(e-1)(2+1/4)] > 2/3 \end{aligned}$$

and

$$\begin{aligned} K_3 &= \frac{1}{\delta^3} \sum_{i=1}^{\infty} \alpha_i + \frac{1}{M-1} \sum_{i=1}^{\infty} \Lambda_i L_i M^{i-2} (M^{i-1} - 1) \\ &\leq 4^3 \cdot \frac{2(e-1)}{k} + \frac{1}{k(M-1)} \sum_{i=2}^{\infty} \frac{i-1}{i!} M^{2i-3} \\ &\leq \frac{2^7(e-1)}{k} + \frac{M}{k(M-1)} \sum_{i=2}^{\infty} \frac{1}{(i-2)!} M^{2(i-2)} \leq \frac{2^7(e-1)}{k} + \frac{M e^{M^2}}{k(M-1)} \\ &= \frac{2^7(e-1)}{27e^9} + \frac{1}{27e^9} \frac{3}{2} e^9 \leq \frac{1}{2 \cdot 3^3} + \frac{1}{18} = \frac{2}{27} \end{aligned}$$

so that $M^2 K_3 \leq 2/3$. This implies that $K_0 > M^2 K_3$. As $K_0 M > 2$ and $K_1 \delta < 1/2$ we have $F \in \mathcal{F}^1_{1/2}((\pi + 2)/4, \pi^2/8) \subseteq \mathcal{F}^1_{K_1\delta}(K_0 M, M^*)$ where $M^* = \pi^2/8$. It is easy to prove that $K_2 < 1$ and $K_0 - M^2 K_3 \geq 1/4$. Thus by Theorem 3.2, the functional equation (3.32) has a solution f in $\mathcal{F}^1_{0.25}(3, M')$ for every $M' \geq 4(\pi^2/8 + 9)$, since

$$4\left(\frac{\pi^2}{8} + 9\right) > \frac{M^* + M^2 K_2}{K_0 - M^2 K_3}.$$

We also deduce a corollary which provides a set of sufficient conditions for the existence of smooth solutions to the functional equation

$$\sum_{i=1}^{n} \lambda_i(x) H_i(f^i(x)) = F(x). \tag{3.33}$$

COROLLARY 3.7. *Let $\lambda_i \in \mathcal{Q}^1(\alpha_i, \beta_i)$ be nonnegative functions on I for each $i = 1, \dots, n$ such that $\gamma_i \le \lambda_i(x) \le \Lambda_i$ where $\alpha_i, \beta_i, \gamma_i$ and Λ_i are nonnegative numbers for $i = 1, \dots, n$ and $\sum_{i=1}^{n} \lambda_i(x) = 1$ for $x \in I$. Let $H_i \in \mathcal{F}^1_{l_i}(L_i, L_i')$ where l_i, L_i, L_i' are nonnegative numbers for $i = 1, \dots, n$. Let $0 < \delta < 1$, $M > 1$ and $M^* \ge 0$. Define*

$$K_0 = \sum_{i=1}^{n} \gamma_i l_i \delta^{i-1} - \frac{1}{\delta} \sum_{i=1}^{n} \alpha_i,$$

$$K_1 = \sum_{i=1}^{n} \left\{ \frac{1}{\delta}\, \alpha_i + \Lambda_i L_i M^{i-1} \right\},$$

$$K_2 = \sum_{i=1}^{n} \left\{ \frac{1}{\delta^2}\, \beta_i + \frac{2}{\delta}\, \alpha_i L_i M^{i-1} + \Lambda_i L_i' M^{2(i-1)} \right\},$$

$$K_3 = \frac{1}{\delta^3} \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \Lambda_i L_i M^{i-2}\, \frac{M^{i-1} - 1}{M - 1}.$$

Suppose that $K_0 > M^2 K_3$. Then for any F in $\mathcal{F}^1_{K_1 \delta}(K_0 M, M^*)$, the functional equation (3.33) *has a solution f in $\mathcal{F}^1_{\delta}(M, M')$ for every $M' \ge (M^* + M^2 K_2)/(K_0 - M^2 K_3)$.*

Proof. This follows directly from Theorem 3.2, upon choosing $\lambda_i(x) = 0$ and $H_i(x) = 0$ for $i > n$. ∎

EXAMPLE 2. Consider the functional equation

$$4[(4242 - x) f(x) + (f^2(x))^2 + x (f^3(x))^5] = 4243(3x^2 + x), x \in [0, 1]. \tag{3.34}$$

In order to apply Corollary 3.7, set $\lambda_1 = (4242 - x)/4243$, $\lambda_2 = 1/4243$, $\lambda_3 = x/4243$, $H_1(x) = x$, $H_2(x) = x^2$, $H_3(x) = x^5$ and $F(x) = \frac{1}{4}(3x^2 + x)$. It can be easily seen that

$$\gamma_1 = \frac{4241}{4243}, \quad \Lambda_1 = \frac{4242}{4243}, \quad \alpha_1 = \frac{1}{4243}, \quad \beta_1 = 0, \quad l_1 = 1 = L_1, \quad L_1' = 0;$$

$$\gamma_2 = \frac{1}{4243} = \Lambda_2, \quad \alpha_2 = \beta_2 = l_2 = 0, \quad L_2 = 2 = L_2';$$

$$\gamma_3 = 0, \quad \Lambda_3 = \frac{1}{4243} = \alpha_3, \quad \beta_3 = 0 = l_3, \quad L_3 = 5, \quad L_3' = 20.$$

For $\delta = 1/8$, $M = 2$ and $M^* = 3/2$, it is readily seen that

$$K_0 = \frac{4225}{4243}, \quad K_1 = \frac{4282}{4243}, \quad K_2 = \frac{664}{4243}, \quad K_3 = \frac{1056}{4243}.$$

Hence

$$K_0 - M^2 K_3 = \frac{4225}{4243} - 4 \cdot \frac{1056}{4243} = \frac{1}{4243} \quad \text{and} \quad \frac{M^* + M^2 K_2}{K_0 - M^2 K_3} = 18041/2.$$

Clearly

$$K_1 \delta < \frac{1}{4} \le F'(x) = \frac{1}{4}(6x+1) \le \frac{7}{4} \le K_0 M.$$

Thus $F \in \mathcal{F}^1_{K_1\delta}(K_0 M, M^*)$.

By Corollary 3.7, the functional equation (3.34) has a solution f in $\mathcal{F}^1_{1/8}(2, M')$ for every $M' \ge 18041/2$.

The concluding corollary answers a question on the existence of smooth solution raised by Baker and Zhang [1].

COROLLARY 3.8. *Let $\lambda_i \in \mathcal{Q}^1(\alpha_i, \beta_i)$ be nonnegative functions on I for each $i = 1, \ldots, n$ such that $\gamma_i \le \lambda_i(x) \le \Lambda_i$ where $\alpha_i, \beta_i, \gamma_i$ and Λ_i are nonnegative numbers for $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i(x) = 1$ for $x \in I$. Let $0 < \delta < 1$, $M > 1$ and $M^* \ge 0$. Define*

$$K_0 = \sum_{i=1}^n \gamma_i \delta^{i-1} - \frac{1}{\delta} \sum_{i=1}^n \alpha_i,$$

$$K_1 = \sum_{i=1}^n \left\{ \frac{1}{\delta}\, \alpha_i + \Lambda_i M^{i-1} \right\},$$

$$K_2 = \sum_{i=1}^n \left\{ \frac{1}{\delta^2}\, \beta_i + \frac{2}{\delta}\, \alpha_i M^{i-1} \right\},$$

$$K_3 = \frac{1}{\delta^3} \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \Lambda_i M^{i-2}\, \frac{M^{i-1}-1}{M-1}.$$

Suppose further that $K_0 > M^2 K_3$. Then for any F in $\mathcal{F}^1_{K_1\delta}(K_0 M, M^*)$, the functional equation

$$\sum_{i=1}^n \lambda_i(x) f^i(x) = F(x)$$

has a solution f in $\mathcal{F}^1_\delta(M, M')$ for every $M' \ge (M^* + M^2 K_2)/(K_0 - M^2 K_3)$.

Proof. This follows directly from Corollary 3.7, upon choosing $H_i(x) = x$ for $i = 1, \ldots, n$. ∎

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