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Asymptotic stability in L^1 of a transport equation

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Abstract. We study the asymptotic behaviour of solutions of a transport equation. We give some sufficient conditions for the complete mixing property of the Markov semigroup generated by this equation.

1. Introduction. In this paper we study the equation

$$\frac{\partial u}{\partial t} + \lambda u = Au + \lambda Pu(t,\cdot), \tag{1}$$

where

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2(a_{ij}(x)u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial(b_i(x)u)}{\partial x_i}, \tag{2}$$

$\lambda \ge 0$ and $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a Markov operator. Equation (1) can be interpreted as a diffusion process with jumps (see [12]). Our aim is to give some sufficient conditions for complete mixing (see [7]) of the Markov semigroup generated by (1). Asymptotic properties of similar equations were investigated in [12] and [5]. The complete mixing property of diffusion processes (without jumps) was studied in [11] and [3]. In this paper we generalize these results to equation (1). Our proofs are based on results from [2] (see also [1] and [8]), where some spectral techniques of studying asymptotics of semigroups of linear operators are developed.

The paper is organized as follows. In Section 2 we rewrite (1) as an evolution equation in L^1 and give some basic definitions. Section 3 starts with Theorem 2 which is an answer to an open problem posed in [13]. This is the problem of characterisation of asymptotics of the parabolic equation (1) by solutions of a proper elliptic equation. We underline that Theorem 2 in case $\lambda = 0$ is stated in [2]. Theorem 2 allows us to formulate in Theorem 3 some

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sufficient conditions for asymptotic stability of (1) in the one-dimensional case.

2. Preliminaries. We denote by D the set of all nonnegative elements of $L^1(\mathbb{R}^d)$ with norm one. The elements of D will be called *densities*. A linear operator $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is called a *Markov operator* if $P(D) \subset D$. Every Markov operator is bounded. Equation (1) can be rewritten as an evolution equation

$$u'(t) = (A - \lambda I + \lambda P)u(t), \qquad u(0) = u_0, \tag{3}$$

where $P : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a Markov operator. We assume that $a_{ij} \in C^3_{\mathrm{b}}(\mathbb{R}^d)$ and $b_i \in C^2_{\mathrm{b}}(\mathbb{R}^d)$ for $i, j = 1, \ldots, d$, where $C^k_{\mathrm{b}}(\mathbb{R}^d)$ is the space of k times differentiable bounded functions on $\mathbb{R}^d$ whose derivatives of order $\le k$ are continuous and bounded. We also assume that A is an elliptic operator, i.e.

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \mu|\xi|^2 \tag{4}$$

for some $\mu > 0$ and all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

The domain of the operator A is given by

$$D(A) = \{f \in L^1(\mathbb{R}^d) \cap C^{1+a}(\mathbb{R}^d) : Af \in L^1(\mathbb{R}^d)\}.$$

Here $C^{1+a}(\mathbb{R}^d)$ is the space of all differentiable functions having absolutely continuous derivatives. It is well known that under the above assumptions the operator A generates a continuous semigroup $(T(t))_{t\ge 0}$ of Markov operators on $L^1(\mathbb{R}^d)$.

The semigroup $(T(t))_{t\ge 0}$ is an integral semigroup with strictly positive and continuous kernel, i.e. there exists a continuous function $p : (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ such that

$$T(t)f(x) = \int_{\mathbb{R}^d} p(t,x,y)f(y)\,dy. \tag{5}$$

From the Phillips perturbation theorem equation (3) generates a continuous semigroup $(S(t))_{t\ge 0}$ of Markov operators on $L^1(\mathbb{R}^d)$.

We say that a semigroup $(P(t))_{t\ge 0}$ of Markov operators is *completely mixing* if for any two densities f and g we have

$$\|P(t)f - P(t)g\| \to 0 \quad \text{as } t \to \infty, \tag{6}$$

where $\|\cdot\|$ denotes the norm in $L^1(\mathbb{R}^d)$.

Let

$$L^1_0(\mathbb{R}^d) = \Big\{f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(x)\,dx = 0\Big\}.$$

Then condition (6) is equivalent to

$$\lim_{t\to\infty} \|P(t)f\| = 0 \qquad \text{for every } f \in L^1_0(\mathbb{R}^d). \tag{7}$$

If B is a linear operator then by $\sigma(B)$, $P\sigma(B)$ and $R\sigma(B)$ we denote respectively the spectrum, point spectrum and residual spectrum of B.

The following theorem will be useful in studying the complete mixing property of Markov semigroups.

THEOREM 1. *Let $(P(t))_{t\ge 0}$ be a continuous semigroup of contractions on a Banach space $(X, \|\cdot\|_X)$. If Z is the generator of $(P(t))_{t\ge0}$ with adjoint Z^* then denote by $\overline{N}$ the weak* closure of the linear span of unitary eigenvectors of Z^*, where a unitary eigenvector is an eigenvector corresponding to the eigenvalue $\mu \in P\sigma(Z^*)\cap i\mathbb{R}$. If the set $\sigma(Z)\cap i\mathbb{R}$ is countable then for every $x \in X$,*

$$\lim_{t\to\infty} \|P(t)x\|_X = \sup\{|\varphi(x)| : \varphi \in \overline{N},\ \|\varphi\|_{X^*} \le 1\}. \tag{8}$$

REMARK 1. Theorem 1 is a special case of a result by Batty, Brzeźniak and Greenfield [2], who consider contractive representations of abstract semigroups in Banach spaces.

Let $\mathcal{A}f = Af - \lambda f + \lambda Pf$. Then (3) can be rewritten as

$$u'(t) = \mathcal{A}u(t) \tag{9}$$

and u satisfies the initial condition $u(0) = u_0$, $u_0 \in L^1(\mathbb{R}^d)$. Let A^* be the linear operator given by

$$A^*g(x) = \sum_{i,j=1}^d a_{ij}(x)\,\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x)\,\frac{\partial g}{\partial x_i}(x), \tag{10}$$

and denote by $P^* : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ the adjoint operator of P. Then the adjoint operator of $\mathcal{A}$ is of the form

$$\mathcal{A}^*g = A^*g - \lambda g + \lambda P^*g. \tag{11}$$

3. Results.

Theorem 1 allows us to formulate the following result.

THEOREM 2. *The semigroup $(S(t))_{t\ge0}$ generated by equation* (1) *is completely mixing iff the only bounded solutions of*

$$\mathcal{A}^*g = 0 \tag{12}$$

are constant functions.

In the case when $\lambda = 0$, i.e. $\mathcal{A} = A$, the above theorem can be found in [2] (see Examples 6.1 and 6.4) but it is formulated only for doubly stochastic and one-dimensional equations. For the convenience of the reader we give its proof.

Proof of Theorem 2. Let $(S(t))_{t\geq 0}$ be completely mixing. We denote by $(S^*(t))_{t\geq 0}$ the conjugate semigroup to $(S(t))_{t\geq 0}$, i.e. $S^*(t): L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ and for all $f \in L^1(\mathbb{R}^d)$, $g \in L^\infty(\mathbb{R}^d)$ and $t > 0$ we have

$$\int_{\mathbb{R}^d} (S(t)f)g = \int_{\mathbb{R}^d} f(S^*(t)g).$$

If $\mathcal{A}^*g = 0$ for some $g \in L^\infty(\mathbb{R}^d)$ then g is a fixed point of $(S^*(t))_{t\geq 0}$ and for all $f \in L^1_0(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} fg = \int_{\mathbb{R}^d} (S(t)f)g \to 0 \quad \text{as } t \to \infty.$$

This implies that $g = \text{const}$.

Suppose now that every bounded solution of (12) is constant. From [10] the semigroup $(T(t))_{t\geq 0}$ generated by A is analytic, which implies that $(S(t))_{t\geq 0}$ generated by $\mathcal{A}$ is also analytic. For such semigroups we always have $\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \{0\}$ (see [4]), so the assumptions of Theorem 1 are satisfied.

Since $P\sigma(\mathcal{A}^*) \subset R\sigma(\mathcal{A}) \subset \sigma(\mathcal{A})$, the only unitary eigenvectors of $\mathcal{A}^*$ correspond to the eigenvalue $\mu = 0$, and are constant by our assumption. From (8) we get

$$\lim_{t\to\infty} \|S(t)f\| = \Big|\int_{\mathbb{R}^d} f(x)\,dx\Big| = 0 \quad \text{for every } f \in L^1_0(\mathbb{R}^d)$$

and the semigroup $(S(t))_{t\geq 0}$ is completely mixing.

In the rest of this paper we investigate the one-dimensional case of equation (1). If $d = 1$ then the formulas for A and A^* are as follows:

$$Af = \frac{d^2}{dx^2}(a(x)f) - \frac{d}{dx}(b(x)f), \tag{13}$$

$$A^*g = a(x)\,\frac{d^2g}{dx^2} + b(x)\,\frac{dg}{dx}, \tag{14}$$

where $a \in C^3_{\mathrm{b}}(\mathbb{R})$, $b \in C^2_{\mathrm{b}}(\mathbb{R})$ and $\inf_{x\in\mathbb{R}} a(x) > 0$.

A mapping $\mathcal{P}: \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$, where $\mathcal{B}(\mathbb{R})$ is the family of all Borel measurable subsets of $\mathbb{R}$, is called a *transition probability function* if $\mathcal{P}(x,\cdot)$ is a probability measure for all $x \in \mathbb{R}$ and $\mathcal{P}(\cdot, B)$ is a Borel measurable function for every $B \in \mathcal{B}(\mathbb{R})$. From now on we assume that the operator P in (3) is induced by a transition probability function $\mathcal{P}$, i.e. for every $f \in D$ the measure

$$\mu_f(B) = \int_{\mathbb{R}} f(x)\mathcal{P}(x,B)\,dx$$

is absolutely continuous with respect to the Lebesgue measure and

$$Pf = \frac{d\mu_f}{dx}. \tag{15}$$

The formula for the dual operator is as follows:

$$P^* g(x) = \int_{\mathbb{R}} g(y)\, \mathcal{P}(x, dy). \tag{16}$$

The following result gives a sufficient condition for complete mixing of the semigroup generated by equation (1).

THEOREM 3. *Let P be a Markov operator induced by a transition probability function $\mathcal{P}$ satisfying the following condition.*

(T) *There exist nonnegative constants M and L such that*

$$\mathcal{P}(x, [-M, x]) = 1 \quad \textit{for } x \ge L, \qquad \mathcal{P}(x, [x, M]) = 1 \quad \textit{for } x \le -L.$$

If

$$\int_0^{+\infty} \exp\Big(-\int_0^x \frac{b(r)}{a(r)}\, dr\Big) dx = \int_{-\infty}^0 \exp\Big(-\int_0^x \frac{b(r)}{a(r)}\, dr\Big) dx = +\infty \tag{17}$$

then the semigroup $(S(t))_{t\ge 0}$ generated by (1) *is completely mixing.*

We need the following two lemmas.

LEMMA 1. *Let $\alpha < x_0 < \beta$. Assume that a function $u : [\alpha, \beta] \to [0, 1)$ has absolutely continuous derivative on $[x_0, \beta]$ and satisfies*

$$a(x)u''(x) + b(x)u'(x) \ge 0 \quad \textit{a.e. in } [x_0, \beta].$$

Also assume that $u'(x_0) > 0$. Then there exists a function $v : [\alpha, \beta] \to [0, 1)$ with absolutely continuous derivative and constants $\alpha < x_1 < x_0 < x_2 < \beta$ such that

$$v(x) = u(x_0) \quad \textit{for } x \in [\alpha, x_1],$$

$$v(x) = u(x) \quad \textit{for } x \in [x_2, \beta],$$

$$a(x)v''(x) + b(x)v'(x) \ge 0 \quad \textit{a.e. in } [\alpha, \beta].$$

Moreover, $v'(x) \ge 0$, $v''(x) \ge 0$ *and* $v'''(x) \ge 0$ *for* $x \in (x_1, x_2)$.

Proof. Fix positive constants c, d satisfying

$$\frac{1}{3} < \frac{d}{c+d} < \frac{1}{2}.$$

Since

$$\frac{u(x_0 + dh) - u(x_0)}{(c+d)h} \to \frac{d}{c+d}\, u'(x_0) \quad \text{as } h \to 0$$

and u' is continuous, there exists $\delta > 0$ such that for $0 < h < \delta$ we have

$$\frac{u'(x_0 + dh)}{3} < \frac{u(x_0 + dh) - u(x_0)}{(c + d)h} < \frac{u'(x_0 + dh)}{2}. \tag{18}$$

There exist positive constants μ, B such that

$$\mu \le a(x), \qquad |b(x)| \le B \qquad \text{for } x \in \mathbb{R}. \tag{19}$$

We take $h > 0$ such that

$$h < \min\left(\delta, \frac{\mu}{2Bd}, \frac{\mu}{2Bc}\right)$$

and put $x_1 = x_0 - ch$, $x_2 = x_0 + dh$. There exists a polynomial $f(x) = \gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0$ such that $f(x_1) = u(x_0)$, $f'(x_1) = 0$, $f(x_2) = u(x_2)$ and $f'(x_2) = u'(x_2)$. Such an f satisfies

$$\begin{aligned} f(x_1) &= u(x_0), \\ f'(x_1) &= 0, \\ f''(x_1) &= \frac{6}{(x_2 - x_1)^2}\left[u(x_2) - u(x_0) - \frac{x_2 - x_1}{3}\, u'(x_2)\right], \\ f'''(x_1) &= \frac{12}{(x_2 - x_1)^3}\left[\frac{x_2 - x_1}{2}\, u'(x_2) - (u(x_2) - u(x_0))\right]. \end{aligned} \tag{20}$$

From (18) and (20) we have

$$f''(x_1) > 0, \qquad f'''(x_1) > 0. \tag{21}$$

Consequently,

$$f'(x) > 0, \qquad f''(x) > 0 \qquad \text{for } x \in (x_1, x_2). \tag{22}$$

By the definition of x_1 and x_2 and from (21) it follows that for $x \in (x_1, x_2)$ we have

$$\begin{aligned} (x - x_1) f''(x_1) &\le \frac{\mu}{B}\, f''(x_1), \\ (x - x_1)^2\, \frac{f'''(x_1)}{2} &\le \frac{\mu}{B}\, f'''(x_1)(x - x_1). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\mu}{B}\,[f''(x_1) + f'''(x_1)(x - x_1)] \\ &\qquad \ge f''(x_1)(x - x_1) + \frac{f'''(x_1)}{2}\,(x - x_1)^2 \qquad \text{for } x \in (x_1, x_2). \end{aligned}$$

Since $f'(x_1) = 0$ it follows that

$$\mu f''(x) \ge B f'(x) \quad \text{for } x \in (x_1, x_2).$$

From (19) and (22) we have

$$a(x) f''(x) + b(x) f'(x) \ge 0 \quad \text{for } x \in (x_1, x_2).$$

If we define

$$v(x)=\begin{cases} u(x_0) & \text{for } x\in[\alpha,x_1],\\ f(x) & \text{for } x\in(x_1,x_2),\\ u(x) & \text{for } x\in[x_2,\beta],\end{cases}$$

then v has all the required properties.

LEMMA 2. *For fixed* $x_0\in\mathbb{R}$ *and* $\eta>0$ *let* $u:(x_0-\eta,\infty)\to[0,1)$ *be a function with absolutely continuous derivative such that* $\sup_{x\in(x_0,\infty)}u(x)=1$ *and* $u'(x_0)>0$. *Define*

$$U=\{x\in(x_0,\infty): u(y)\le u(x) \text{ for } y\in(x_0,x)\}.$$

If $a(x)u''(x)+b(x)u'(x)\ge 0$ *a.e. in* U, *then there exists a nondecreasing function* $v:(x_0,\infty)\to[0,1)$ *such that* $\sup_{x\in(x_0,\infty)}v(x)=1$, v' *is absolutely continuous*, *and*

$$a(x)v''(x)+b(x)v'(x)\ge 0 \quad \text{a.e. in } (x_0,\infty).$$

Proof. For $x>x_0$ define

$$c(x)=\sup\{y\in U: y\le x\},\qquad d(x)=\inf\{y\in U: x\le y\}.$$

Set $K=\{x\ge x_0: c(x)<x<d(x)\}$. Then $K=\bigcup_{n=1}^{\infty}(c_n,d_n)$, where the intervals in the union are pairwise disjoint. We have $u(x)\le u(c_n)=u(d_n)$ for $x\in(c_n,d_n)$, $n\in\mathbb{N}$. Define $\widetilde{v}_1:(x_0,\infty)\to[0,1)$ by

$$\widetilde{v}_1(x)=\begin{cases} u(d_1) & \text{for } x\in(c_1,d_1),\\ u(x) & \text{for } x\in(x_0,\infty)\setminus(c_1,d_1).\end{cases}$$

If $u'(d_1)=0$ then set $v_1=\widetilde{v}_1$. If $u'(d_1)>0$ then there exists $\varepsilon_1>0$ such that $u'(x)>0$ for all $x\in[d_1,d_1+\varepsilon_1]$. According to Lemma 1 we modify $\widetilde{v}_1$ on $[c_1,d_1+\varepsilon_1]$, obtaining a function $v:(x_0,\infty)\to[0,1)$ such that v_1 has absolutely continuous derivative,

$$v_1(x)=u(x)\quad \text{for } x\in(x_0,\infty)\setminus(c_1,d_1+\varepsilon_1),$$
$$a(x)v_1''(x)+b(x)v_1'(x)\ge 0\quad \text{a.e. in } (c_1,d_1+\varepsilon_1).$$

By induction we define a sequence of functions $v_n:(x_0,\infty)\to[0,1)$ with absolutely continuous derivatives such that

$$v_n(x)=u(x)\qquad \text{for } x\in(x_0,\infty)\setminus\bigcup_{k=1}^{n}(c_k,d_k+\varepsilon_k),$$

$$v_m(x)=v_n(x)\quad \text{for } x\in\bigcup_{k=1}^{n}(c_k,d_k+\varepsilon_k),\ m\ge n,$$

$$a(x)v_n''(x)+b(x)v_n'(x)\ge 0\quad \text{for } x\in\bigcup_{k=1}^{n}(c_k,d_k+\varepsilon_k),$$

where $\varepsilon_k\ge 0$ and $n\in\mathbb{N}$.

The sequence $(v_n)_{n\in\mathbb{N}}$ is pointwise convergent to a nondecreasing function $f:(x_0,\infty)\to[0,1)$. There exists $C\subset(x_0,\infty)$ with Lebesgue measure zero such that f is differentiable on $(x_0,\infty)\setminus C$. Set $D=\overline{\{c_n:n\in\mathbb{N}\}}$, where $\overline{K}$ stands for the closure of K. Define $g:(x_0,\infty)\to\mathbb{R}$ by

$$g(x)=\begin{cases} f'(x) & \text{for } x\notin D,\\ u'(x) & \text{for } x\in D.\end{cases}$$

Since $C\subset D$, it follows that g is continuous. Moreover, the sequence $(v_n)_{n\in\mathbb{N}}$ is pointwise convergent to g. Since the variation is an additive function of the interval, the definition of v_n implies that

$$\bigvee_\alpha^\beta v_n' \le \bigvee_\alpha^\beta u' < \infty \quad \text{for } n\in\mathbb{N} \text{ and } [\alpha,\beta]\subset(x_0,\infty).$$

This implies that g is of bounded variation on compact intervals. Thus g is a.e. differentiable. Since g is absolutely continuous on $(c_n,d_n+\varepsilon_n)$ we have

$$\int_{c_n}^{d_n+\varepsilon_n} g'(x)\,dx = g(d_n+\varepsilon_n)-g(c_n).$$

Moreover, outside $\bigcup_{n=1}^\infty(c_n,d_n+\varepsilon_n)$ we have $g'(x)=u''(x)$ for a.e. x, u' is absolutely continuous and $g(c_n)=u'(c_n)$, $g(d_n+\varepsilon_n)=u'(d_n+\varepsilon_n)$. This implies that $\int_x^y g'(t)\,dt=g(y)-g(x)$ for every $x<y$, which means that g is absolutely continuous.

Finally, define $v(x)=u(x_0)+\int_{x_0}^x g(t)\,dt$ for $x\in(x_0,\infty)$. The function v has all the required properties.

Proof of Theorem 3. By Theorem 2 it suffices to show that the only bounded solutions of

$$\mathcal{A}^*g = A^*g-\lambda g+\lambda P^*g=0 \tag{23}$$

are constants.

Let $u\in L^\infty(\mathbb{R})$ be a nonconstant function. We show that u cannot satisfy (23). If $g\in L^\infty(\mathbb{R}^d)$ satisfies (23) then $g+c$, where $c\in\mathbb{R}$, also satisfies (23). Hence, by the linearity of (23), we can assume that $u\ge 0$ and $\|u\|_\infty=1$.

Let us consider two cases.

Case 1: *There exists $x_0\in\mathbb{R}$ such that $u(x_0)\ge u(x)$ for $x\in\mathbb{R}$.* A function g satisfies (23) if and only if

$$\lambda R(\lambda,A)^*P^*g=g, \tag{24}$$

where $R(\lambda,A)^*$ stands for the dual of the resolvent operator $R(\lambda,A)$. Since

$$R(\lambda,A)f=\int_0^\infty e^{-\lambda t}T(t)f\,dt,$$

from (5) it follows that $R(\lambda, A)$ is an integral operator, i.e.

$$R(\lambda, A)f(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy, \tag{25}$$

where

$$K(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt > 0 \tag{26}$$

is a continuous and strictly positive kernel. From the fact that $(T(t))_{t\geq 0}$ is a Markov semigroup it follows that

$$\lambda \int_{\mathbb{R}} K(x,y)\,dx = 1 \quad \text{for } y \in \mathbb{R}. \tag{27}$$

For the dual operator $R(\lambda, A)^*$ we have

$$R(\lambda, A)^* g(x) = \int_{\mathbb{R}} K(y,x)g(y)\,dy. \tag{28}$$

If $P^*u \equiv \text{const}$ then since A^* is a differential operator it follows that $\lambda R(\lambda, A)^* P^* u \equiv \text{const}$ so equation (24) is not satisfied. If $P^*u \not\equiv \text{const}$ then $P^*u(x) \leq u(x_0)$ for $x \in \mathbb{R}$, because the dual operator of a Markov operator is a nonnegative contraction on L^∞. Moreover, there exists a set $B \subset \mathbb{R}$ with positive Lebesgue measure such that $P^*u(x) < u(x_0)$ for $x \in B$. It follows from (27) and (28) that

$$\lambda R(\lambda, A)^* P^* u(x_0) < \lambda \int_B K(y,x_0)u(x_0)\,dy + \lambda \int_{\mathbb{R}\setminus B} K(y,x_0)u(x_0)\,dy = u(x_0).$$

Since $R(\lambda, A)^* P^* u$ is continuous, u does not satisfy (24).

Case 2: *We have* $u(x) < 1$ *for all* $x \in \mathbb{R}$. From the symmetry of the conditions in (T) and (17) we can assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = +\infty$ such that $\lim_{n\to\infty} u(x_n) = 1$. We take $x_0 > L$ such that $u(x) \leq u(x_0)$ for $x \in [-M, x_0]$. Define a set U as in the statement of Lemma 2, i.e.

$$U = \{x \in (x_0, \infty) : u(y) \leq u(x) \text{ for } y \in (x_0, x)\}.$$

There exists a set $V \subset U$ with positive Lebesgue measure such that $A^*u(x) < 0$ for $x \in V$. Otherwise, if

$$A^*u(x) = a(x)u''(x) + b(x)u'(x) \geq 0 \quad \text{a.e. in } U,$$

then it follows from Lemma 2 that there exists a nondecreasing function $v : (x_0, \infty) \to [0,1)$ with absolutely continuous derivative such that $\sup_{x\in(x_0,\infty)} v(x) = 1$ and

$$a(x)v''(x) + b(x)v'(x) \geq 0 \quad \text{a.e. in } (x_0, \infty). \tag{29}$$

Take $x_1 > x_0$ such that $v'(x_1) > 0$. By standard arguments from the theory of differential inequalities it follows from (29) that

$$v(x) \geq v(x_1) + v'(x_1) \int_{x_1}^{x} \exp\left(-\int_{x_1}^{s} \frac{b(r)}{a(r)}\, dr\right) ds. \tag{30}$$

By (17) the right side of (30) tends to ∞ as $x \to \infty$, which contradicts the boundedness of v.

It follows from (16), (T) and (23) that for $x \in V$ we have

$$0 > A^* u(x) = u(x) - P^* u(x) = \int_{-M}^{x} [u(x) - u(y)]\, \mathcal{P}(x, dy) \geq 0,$$

which implies that u cannot satisfy (23). This completes the proof.

Example 1. Suppose that $S : \mathbb{R} \to \mathbb{R}$ is a measurable transformation satisfying

$$l(S^{-1}(B)) = 0 \quad \text{for all } B \in \mathcal{B}(\mathbb{R}) \text{ with } l(B) = 0,$$

where $l : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ is the Lebesgue measure restricted to the Borel subsets of $\mathbb{R}$. If in the definition of the operator P we put

$$\mathcal{P}(x, B) = \delta_{S(x)}(B) \quad \text{for } x \in \mathbb{R} \text{ and } B \in \mathcal{B}(\mathbb{R}),$$

where δ_x is the Dirac measure at x, then P becomes the Frobenius–Perron operator corresponding to S, i.e. the unique Markov operator satisfying

$$\int_{S^{-1}(A)} f(x)\, dx = \int_A Pf(x)\, dx \quad \text{for all } f \in L^1(\mathbb{R}) \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

Then condition (T) is of the form

(T′) There exist nonnegative constants M and L such that

$$-M \leq S(x) \leq x \quad \text{for } x \geq L, \qquad x \leq S(x) \leq M \quad \text{for } x \leq -L.$$

In this case equation (1) describes the evolution of densities under the diffusion process perturbed by deterministic jumps induced by the mapping S.

Remark 2. Condition (17) is strictly connected with the complete mixing property of the semigroup $(T(t))_{t\geq 0}$ generated by A. Namely, in [11] it is shown that $(T(t))_{t\geq 0}$ is completely mixing if and only if

$$\int_{\mathbb{R}} \exp\left(-\int_0^x \frac{b(r)}{a(r)}\, dr\right) dx = \infty.$$

References

[1] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. 306 (1988), 837–852.

[2] C. J. K. Batty, Z. Brzeźniak and D. A. Greenfield, *A quantitative asymptotic theorem for contraction semigroups with countable unitary spectrum*, Studia Math. 121 (1996), 167–183.

[3] Z. Brzeźniak and B. Szafirski, *Asymptotic stability of L^1 norm of solutions to parabolic equations*, Bull. Polish Acad. Sci. Math. 39 (1991), 1–10.

[4] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer, New York, 2000.

[5] D. Jama, *Asymptotic stability of an integro-differential equation of parabolic type*, Ann. Polon. Math. 47 (1986), 65–78.

[6] A. Lasota and M. C. Mackey, *Chaos, Fractals and Noise. Stochastic Aspects of Dynamics*, Applied Math. Sci. 97, Springer, New York, 1994.

[7] M. Lin, *Mixing for Markov operators*, Z. Wahrsch. Verw. Gebiete 19 (1971), 231–242.

[8] Yu. I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. 88 (1989), 37–42.

[9] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lecture Notes in Math. 1184, Springer, Berlin, 1986.

[10] D. W. Robinson, *Elliptic Operators and Lie Groups*, Oxford Univ. Press, Oxford, 1991.

[11] R. Rudnicki, *Asymptotical stability in L^1 of parabolic equations*, J. Differential Equations 102 (1993), 391–401.

[12] —, *Asymptotic behaviour of an integro-parabolic equation*, Bull. Polish Acad. Sci. Math. 40 (1992), 111–128.

[13] —, *Asymptotic properties of the Fokker–Planck equation*, in: Chaos—The Interplay between Stochastic and Deterministic Behaviour (Karpacz, 1995), P. Gorbaczewski, M. Wolf and A. Weron (eds.), Lecture Notes in Phys. 457, Springer, Berlin, 1995, 517–521.

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