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An existence and localization theorem for the solutions of a Dirichlet problem

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Abstract. We establish an existence theorem for a Dirichlet problem with homogeneous boundary conditions by using a general variational principle of Ricceri.

1. Introduction and statement of the result. The aim of this paper is to establish the following result:

Theorem 1. *Suppose that Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 3$, with sufficiently regular boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Let $s = 2N/(N-2)$ and suppose that there exist $\gamma \in \,]2/s, 1[$, $p \in \,]1, \gamma s - 1[$, $\alpha \in L^{\gamma s/(\gamma s - 1)}(\Omega)$ and $\beta \in L^{1/(1-\gamma)}(\Omega)$ such that*

$$|f(x,t)| \leq \alpha(x) + \beta(x)|t|^p \quad \text{almost everywhere in } \Omega \times \mathbb{R} \tag{1}$$

and

$$\|\alpha\|^{p-1}_{L^{s\gamma/(s\gamma-1)}(\Omega)} \|\beta\|_{L^{1/(1-\gamma)}(\Omega)} < \frac{(p-1)^{p-1}}{p^p\, c_1(\gamma)^{p-1} c_2(\gamma)^{p+1}}, \tag{2}$$

where $c_1(\gamma)$ and $c_2(\gamma)$ are the embedding constants of $W^{1,2}(\Omega)$ in $L^{s\gamma}(\Omega)$ and $L^{(p+1)/\gamma}(\Omega)$ respectively. Then the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

has a strong solution $\overline{u}$ such that $\|\overline{u}\| \leq k$, where k is the smallest real root of the equation

$$c_2(\gamma)^{p+1} \|\beta\|_{L^{1/(1-\gamma)}(\Omega)} r^p + c_1(\gamma) \|\alpha\|_{L^{s\gamma/(s\gamma-1)}(\Omega)} - r = 0.$$

We recall that a *strong solution* of problem (P) is a function $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ such that

$$-\Delta u(x) = f(x, u(x)) \quad \text{almost everywhere in } \Omega,$$

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while a *weak solution* of problem (P) is a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla v(x)\,dx - \int_\Omega f(x,u(x))v(x)\,dx = 0$$

for all $v \in W_0^{1,2}(\Omega)$.

The proof of Theorem 1 is essentially based on the following general variational principle of B. Ricceri which has already been used in several works [6, 7, 8].

Theorem A ([5, Theorem 2.5]). *Let X a real reflexive Banach space, and $\Phi, \Psi : X \to \mathbb{R}$ two sequentially weakly lower semicontinuous and Gateaux differentiable functionals. Moreover, assume that Ψ is (strongly) continuous and satisfies*

$$\lim_{\|x\|\to\infty} \Psi(x) = \infty.$$

For every $r > \inf_X \Psi$, put

$$\varphi(r) = \inf_{x\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(x) - \inf_{\mathrm{cl}_w(\Psi^{-1}(]-\infty,r[))}\Phi}{r - \Psi(x)}$$

where $\mathrm{cl}_w(\Psi^{-1}(]-\infty,r[))$ is the closure of $\Psi^{-1}(]-\infty,r[)$ with respect to the weak topology. Then, for every $r > \inf_X \Psi$ and every $\lambda > \varphi(r)$, the functional $\Phi + \lambda\Psi$ has at least one critical point in $\Psi^{-1}(]-\infty,r[)$.

From Theorem A, we deduce that if there exists $r > \inf_X \Psi$ such that

$$\inf_{x\in\Psi^{-1}(]-\infty,r[)} \frac{\Phi(x) - \inf_{\mathrm{cl}_w(\Psi^{-1}(]-\infty,r[))}\Phi}{r - \Psi(x)} < \frac{1}{2}, \tag{3}$$

then the functional $\Phi + \frac{1}{2}\Psi$ has at least one critical point in $\Psi^{-1}(]-\infty,r[)$.

We apply Theorem A to suitable functionals Φ and Ψ satisfying condition (3) and defined on the Sobolev space $W_0^{1,2}(\Omega)$ equipped with the norm $\|\cdot\| = (\int_\Omega |\nabla(\cdot)|^2 d\,x)^{1/2}$.

There are many works in which problem (P) is studied. Nevertheless, because of the type of our assumptions, only a few results can be compared with ours. Among these, we find interesting Theorem 3.1 of [4]. In that result the strong solution of problem (P) belongs to $W_0^{1,\sigma}(\Omega) \cap W^{2,\sigma}(\Omega)$ for some $\sigma > N/2$ and the right hand side is a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ which can be discontinuous in both variables while, in our case, f must be a Carathéodory function. In [4], the following condition on f is imposed:

(4) There exist $r > 0$ and $\sigma > N/2$ such that $x \mapsto \sup_{|t|\le Br} |f(x,t)|$ belongs to $L^\sigma(\Omega)$, and its norm in this space is not greater than r, where

$$B = \sup_{u\in W_0^{1,\sigma}(\Omega)\cap W^{2,\sigma}(\Omega)} \frac{\operatorname{ess\,sup}_{x\in\Omega} |u(x)|}{\|u\|_{L^\sigma(\Omega)}}.$$

It is easily seen that condition (4) is satisfied if f is a Carathéodory function and there exist $\alpha, \beta \in L^\sigma(\Omega)$ and $p > 1$ such that

$$|f(x,t)| \le \alpha(x) + \beta(x)|t|^p \quad \text{almost everywhere in } \Omega \times \mathbb{R} \tag{5}$$

and

$$\|\alpha\|^{p-1}_{L^\sigma(\Omega)} \|\beta\|_{L^\sigma(\Omega)} < \frac{(p-1)^{p-1}}{p^p B^p}. \tag{6}$$

Conditions (5) and (6) are similar to (1) and (2) respectively but, in our case, the summability conditions on α and β are different and this allows us to treat some problems which are not covered by Theorem 3.1 of [4]. A simple example in which conditions (1) and (2) are fulfilled and condition (4) is violated, is the following.

Example 1. Let $N = 3$ and $\sigma = 2$. Then

$$\tilde{c} := c_1(\gamma)^{p-1} c_2(\gamma)^{p+1} \le m(\Omega)^{\frac{p-1-2\gamma p+6\gamma^2}{6\gamma}} c^{2p},$$

where

$$c = \sup_{u \in W_0^{1,2}(\Omega)} \frac{\|u\|_{L^6(\Omega)}}{\|u\|}.$$

The exact value of c^2 is $\frac{2}{3\pi}\sqrt[3]{\frac{2}{\pi}}$ (see [3, p. 115]).

Now, we take two functions $\alpha \in L^{4/3}(\Omega)$, $\beta \in L^3(\Omega)$ non-negative almost everywhere in Ω and such that

$$\|\alpha\|_{L^{4/3}(\Omega)} \|\beta\|_{L^3(\Omega)} < \frac{9\pi^2 \sqrt[3]{\pi^2}}{m(\Omega)^{1/4} 16 \sqrt[3]{4}}. \tag{7}$$

Put

$$f(x,t) = \alpha(x) + \beta(x)|t|^2 \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Since $s = 6$, if we choose $\gamma = 2/3$ then $1/(1-\gamma) = 3$ and $s\gamma/(s\gamma - 1) = 4/3$. Further, in this setting, we have

$$\tilde{c} \le m(\Omega)^{1/4} c^2.$$

Thus α, β, f satisfy all the assumptions of Theorem 1 with $p = 2$ and $\gamma = 2/3$.

Observe that if $\alpha \notin L^{3/2}(\Omega)$, then condition (4) is violated because $\sup_{|t| \le Br} f(\cdot, t) \notin L^\sigma(\Omega)$ for all $\sigma > 3/2$ and $r > 0$.

Another result where the growth condition on f is similar to ours is Theorem 1 of [10]. Nevertheless, if the function f is as in the previous example with $\alpha \notin L^\infty(\Omega)$ and β is not identically 0, then f does not satisfy the assumptions in [10].

2. Proof of Theorem 1. We put

$$\Psi(u) = \|u\|^2, \qquad \Phi(u) = \int_{\Omega} \Big(\int_0^{u(x)} f(x,t)\,dt \Big)\,dx$$

for all $u \in W_0^{1,2}(\Omega)$. Thanks to condition (2) the operator Φ is well defined, Gateaux differentiable and sequentially weakly lower semicontinuous. So is the operator Ψ, which is strongly continuous as well. Now, the weak solutions of problem (P) are exactly the critical points of the functional $\frac{1}{2}\Psi + \Phi$. Moreover, by Theorem 8.2′ of [1], the weak solutions of problem (P) are strong solutions. Thus, our goal is achieved if we show that condition (1) holds with this choice of Φ and Ψ. In fact, in this case, given a sequence $\{r_n\}_{n\in\mathbb{N}}$ of real numbers with $r_n > k$ and $r_n \to k$ we find a strong solution u_n of problem (P) whose norm is smaller than r_n, so $\{u_n\}_{n\in\mathbb{N}}$ admits a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, weakly converging to $\overline{u}$ in $W_0^{1,2}(\Omega)$ and, by an embedding theorem, strongly converging to $\overline{u}$ in $L^p(\Omega)$. Now, given $v \in W_0^{1,2}(\Omega)$, taking into account the growth condition on f, we deduce from well known results that the sequence $\{f(\cdot, u_n(\cdot))v(\cdot)\}_{n\in\mathbb{N}}$ converges almost everywhere to $f(\cdot.\overline{u}(\cdot))v(\cdot)$ and it is dominated by an $L^1(\Omega)$ function. Hence, $\int_\Omega f(x,u_n(x))v(x)\,dx \to \int_\Omega f(x,\overline{u}(x))v(x)\,dx$. By the previous considerations we conclude that $\overline{u}$ is a solution of problem (P) with $\|\overline{u}\| \le k$.

For all $r > 0$ put

$$\mu(r) = \sup_{\|v\|\le r} \int_\Omega \Big(\int_0^{v(x)} f(x,t)\,dt \Big)\,dx.$$

We observe that, thanks to condition (2) and the fact that $W_0^{1,2}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for all $q \in \,]2, 2N/(N-2)[$, the function μ is well defined. Obviously, μ is non-decreasing in $]0,\infty[$.

Now, as is easily seen, condition (1) is equivalent to

$$\inf_{r>0} \inf_{s<r} \frac{\mu(r)-\mu(s)}{r^2 - s^2} < \frac{1}{2}.$$

This, in turn, is equivalent to the existence of $r > 0$ and $s < r$ such that

$$\mu(s) - \frac{1}{2}s^2 > \mu(r) - \frac{1}{2}r^2.$$

This holds if there exists $r > 0$ such that

$$\limsup_{h\to 0} \frac{\mu(r+h)-\mu(r)}{h} < r.$$

We show that this is indeed case. In fact, for every $r > 0$ and every $h \in \,]-r, \infty[$ one has

$$
\begin{aligned}
&\frac{1}{h}\left(\mu(r+h)-\mu(r)\right)\\
&\le \frac{1}{|h|}\Big|\sup_{\|v\|\le r+h}\int_{\Omega}\Big(\int_0^{v(x)} f(x,t)\,dt\Big)\,dx-\sup_{\|v\|\le r}\int_{\Omega}\Big(\int_0^{v(x)} f(x,t)\,dt\Big)\,dx\Big|\\
&= \frac{1}{|h|}\Big|\sup_{\|v\|\le 1}\int_{\Omega}\Big(\int_0^{(r+h)v(x)} f(x,t)\,dt\Big)\,dx-\sup_{\|v\|\le 1}\int_{\Omega}\Big(\int_0^{rv(x)} f(x,t)\,dt\Big)\,dx\Big|\\
&\le \frac{1}{|h|}\sup_{\|v\|\le 1}\int_{\Omega}\Big|\int_{rv(x)}^{(r+h)v(x)} |f(x,t)|\,dt\Big|\,dx\\
&\le \sup_{\|v\|\le 1}\int_{\Omega}\left(\alpha(x)\left|\frac{r+h-r}{h}\right|\cdot|v(x)|+\frac{\beta(x)}{q}\left|\frac{(r+h)^q-r^q}{h}\right|\cdot|v(x)|^q\right)dx\\
&\le \|\alpha\|_{L^{s\gamma/(s\gamma-1)}(\Omega)}\|v\|_{L^{s\gamma}(\Omega)}+\frac{1}{q}\,\|\beta\|_{L^{1/(1-\gamma)}(\Omega)}\|v\|^q_{L^{q/\gamma}(\Omega)}\left|\frac{(r+h)^q-r^q}{h}\right|\\
&\le c_1(\gamma)\|\alpha\|_{L^{s\gamma/(s\gamma-1)}(\Omega)}+c_2(\gamma)^q\,\frac{1}{q}\,\|\beta\|_{L^{1/(1-\gamma)}(\Omega)}\left|\frac{(r+h)^q-r^q}{h}\right|,
\end{aligned}
$$

where $q=p+1$. Consequently, we have

$$
\limsup_{h\to 0}\frac{1}{h}\left(\mu(r+h)-\mu(r)\right)\le c_1(\gamma)\|\alpha\|_{L^{s\gamma/(s\gamma-1)}(\Omega)}+c_2(\gamma)^q\|\beta\|_{L^{1/(1-\gamma)}(\Omega)}r^p.
$$

Thus, if we put

$$
\varphi(r)=c_1(\gamma)\|\alpha\|_{L^{s\gamma/(s\gamma-1)}(\Omega)}+c_2(\gamma)^q\|\beta\|_{L^{1/(1-\gamma)}(\Omega)}r^p-r
$$

for all $r>0$, we only have to prove that $\inf_{r>0}\varphi(r)<0$. By using an elementary differential method, we see that the infimum is attained at

$$
r=\left(pc_2(\gamma)^{p+1}\|\beta\|_{L^{1/(1-\gamma)}(\Omega)}\right)^{1/(1-p)}
$$

and (2) entails that it is negative.

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