Lorentzian isothermic surfaces and Bonnet pairs

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Abstract. Lorentzian surfaces in Lorentz three-space are studied using an indefinite version of the quaternions. A classification theorem for Bonnet pairs in Lorentz three-space is obtained.

1. Introduction. In recent years there has been a resurgence of interest in certain classical differential geometric objects and constructions such as isothermic surfaces, Christoffel transforms and Darboux transforms. For example one has F. Burstall's epic work [B] and the wonderful example of both mathematics and scholarship [H-J]. These use, among other tools, Clifford algebras or quaternions to explore and clarify aspects of surface theory from the late 1800's. In [KPP] the quaternions are used to study isothermic surfaces and their relationship to what the authors define as Bonnet pairs.

In [M] it is shown that for Lorentzian isothermic surfaces, one has, just as in the positive definite setting, a dual surface. However, there are two types of isothermic surfaces and the constructions of the dual surfaces are different for the two types. One can still investigate these isothermic surfaces and appropriately defined Bonnet pairs in Lorentz space using what Libermann [L1, L2] called the quaternions of the second type $\widetilde{\mathbb{H}}$. These form a real, four-dimensional associative algebra containing some non-invertible, non-zero elements and give an appropriate setting for studying Lorentzian surfaces.

2. Preliminaries

2.1. *Definitions and notations for Lorentzian surfaces in* $\mathbb{R}_1^3$. The metric in $\mathbb{R}_1^3$ is denoted by

$$\langle \vec{v}, \vec{w} \rangle = -v_1 w_1 + v_2 w_2 + v_3 w_3$$

for $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. We often use isothermal coordinates (t, s) on our surface M_1^2 defined by an immersion $X : M_1^2 \to \mathbb{R}_1^3$, by which we mean that, for the induced metric h on M_1^2 and for the associated coordinate vectors ∂t and ∂s, there is a non-zero function μ so that

$$(2.1) \qquad h(\partial t, \partial t) = -\mu^2, \quad h(\partial t, \partial s) = 0, \quad h(\partial s, \partial s) = \mu^2.$$

We can also use any set of isothermal coordinates to define null coordinates $\{x, y\}$ by

$$x = \frac{t - s}{\sqrt{2}}, \quad y = \frac{t + s}{\sqrt{2}}$$

so that $\partial x = \frac{1}{\sqrt{2}}(\partial t - \partial s)$, $\partial y = \frac{1}{\sqrt{2}}(\partial t + \partial s)$ and $h(\partial x, \partial y) = -\mu^2$.

Following Weinstein [W, p. 13], a Lorentzian surface $(M_1^2, [h])$ is defined to be M_1^2 with all the metrics conformally equivalent to h.

The fundamental equations of the immersion X are given by

$$X_{tt} = \frac{\mu_t}{\mu} X_t + \frac{\mu_s}{\mu} X_s + eN,$$

$$X_{ts} = \frac{\mu_s}{\mu} X_t + \frac{\mu_t}{\mu} X_s + fN,$$

$$(2.2) \qquad X_{ss} = \frac{\mu_t}{\mu} X_t + \frac{\mu_s}{\mu} X_s + gN,$$

$$N_t = \frac{e}{\mu^2} X_t - \frac{f}{\mu^2} X_s,$$

$$N_s = \frac{f}{\mu^2} X_t - \frac{g}{\mu^2} X_s.$$

Thus, the shape operator corresponding to N is

$$(2.3) \qquad A = \begin{bmatrix} -e/\mu^2 & -f/\mu^2 \\ f/\mu^2 & g/\mu^2 \end{bmatrix}.$$

Recall that the shape operator at each point on a Lorentzian surface falls into one of three distinct classes which we call its *algebraic type*. The shape operator is either (a) diagonalizable over $\mathbb{R}$, (b) diagonalizable over $\mathbb{C}$ but not $\mathbb{R}$ or (c) not diagonalizable over $\mathbb{C}$, and has a single real eigenvalue ([O'N]).

A Riemannian surface is called *isothermic* if there is an isothermal coordinate system for which every shape operator is diagonalized (over $\mathbb{R}$). For a Lorentzian surface we need to adjust the definition. Essentially we will call a surface isothermic if there is an isothermal coordinate system for which each shape operator is diagonalized over $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 2.1. A Lorentzian immersion is called *isothermic* if there is some isothermal coordinate system (t, s) such that each A with respect to

the basis $\{\partial t, \partial s\}$ has one of the following forms:

$$\text{(a)} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

We call the first case *real isothermic* and the second *complex isothermic*. In this paper we are working locally and define an *isothermic immersion* to be an immersion for which each point has a neighborhood which is either real isothermic or complex isothermic. Note that, in the complex isothermic case, the shape operator satisfies $e + g = 0$.

Each Lorentzian surface $(M_1^2, [h])$ has a time orientation and an integrable product structure J'. A *time orientation* is a choice of time cone in each tangent space. An *integrable product structure* is a tensor of type $(1,1)$ such that $J'^2 = \mathrm{Id}$, $J' \neq \mathrm{Id}$ and $\nabla J' = 0$, where ∇ is the Levi-Civita connection on M_1^2 defined by h. We define J' by choosing the two future directed null vectors $\{\partial x, \partial y\}$ in the chosen time cone so that $\{\partial x, \partial y\}$ is positively oriented, and set

$$J'(\partial x) = -\partial x, \quad J'(\partial y) = \partial y.$$

In the isothermal coordinates we have

(2.4) $$J'(\partial t) = \partial s, \quad J'(\partial s) = \partial t.$$

The time orientation on M_1^2 is induced from the standard time orientation on $\mathbb{R}_1^3$; see [O'N, pp. 145 and 194]. We note that J' depends only on the Lorentzian surface $(M_1^2, [h])$, its time orientation, and orientation, which we include in the definition of $(M_1^2, [h])$. This structure is analogous to the complex structure defined on a Riemann surface.

2.2. *Surfaces in* $\widetilde{\mathbb{H}}$. Our $\mathbb{R}_1^3$ will sit in $\widetilde{\mathbb{H}}$, a four-dimensional Clifford algebra given by

$$\widetilde{\mathbb{H}} = \{x_0 + x_1 i + x_2 j' + x_3 k' \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

with

$$i^2 = -1, \quad j'^2 = 1 = k'^2,$$
$$ij' = k', \ j'i = -k', \ j'k' = -i, \ k'j' = i, \ ik' = -j', \ k'i = j'.$$

This was defined by Libermann in [L1], and called a quaternionic algebra of the second type. We will call them *indefinite quaternions*. For $x, y \in \widetilde{\mathbb{H}}$ we define our inner product $x \cdot y$ by

$$x \cdot y = -\mathrm{Re}(\overline{x}y) = -x_0 y_0 - x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $\overline{x_0 + x_1 i + x_2 j' + x_3 k'} = x_0 - x_1 i - x_2 j' - x_3 k'$. We call those indefinite quaternions with $x_0 = 0$ *imaginary*, and denote them by $\mathrm{Im}\,\widetilde{\mathbb{H}}$. They inherit a metric of signature $(1,2)$ from $\widetilde{\mathbb{H}}$ and are identified in all that follows with Lorentz three-space $\mathbb{R}_1^3$. The metric chosen here in $\widetilde{\mathbb{H}}$ has signature $(2,2)$

and induces the standard metric on the imaginary quaternions, but it is the negative of the one found in [Po]. We denote $\widetilde{\mathbb{H}}$ by $\mathbb{R}_2^4$. Also note that if $x \in \operatorname{Im} \widetilde{\mathbb{H}}$ then $\overline{x} = -x$.

For imaginary x and y an important formula for calculations is

(2.5) $$xy = x \cdot y + x \times y,$$

Here $x \times y$ is the cross product in $\mathbb{R}_1^3$ given by

$$(x \times y) \cdot z = \det[z, x, y].$$

We can define, exactly as in [KPP], a wedge product on $\widetilde{\mathbb{H}}$-valued one-forms on M_1^2 by

$$\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

which satisfies the identities

$$\overline{\alpha \wedge \beta} = -\overline{\beta} \wedge \overline{\alpha}, \quad d(h\alpha) = dh \wedge \alpha + h\,d\alpha,$$
$$\alpha \wedge h\beta = \alpha h \wedge \beta, \quad d(\alpha h) = d\alpha h - \alpha \wedge dh,$$

where $h : M_1^2 \to \widetilde{\mathbb{H}}$. We also identify two-forms on M_1^2 with their quadratic forms by

$$\omega(X) = \omega(X, J'X),$$

for ω an $\widetilde{\mathbb{H}}$-valued two-form on M_1^2. If we define $*\alpha = \alpha \circ J'$ then we have

$$\alpha \wedge \beta = \alpha(*\beta) - (*\alpha)\beta,$$

since $(\alpha \wedge \beta)(U, J'U) = \alpha(U)\beta(J'U) - \alpha(J'U)\beta(U)$. Note that in the Lorentzian setting we have $*^2 = \operatorname{Id}$. There is an algebraic lemma whose statement and proof is quite similar to one in [BFLPP].

LEMMA 2.1. *For* $x, y \in \widetilde{\mathbb{H}} = \mathbb{R}_2^4$,

(1) $xy = yx$ *iff* $\operatorname{Im}(x)$ *and* $\operatorname{Im}(y)$ *are linearly dependent over* $\mathbb{R}$.
(2) $\operatorname{Im}(x^2) = 0$ *iff* x *is either real or purely imaginary.*
(3) $x^2 = 0$ *iff* $x \cdot x = 0$, *i.e.,* x *is in the null cone in* $\mathbb{R}_2^4$.
(4) $x^2 = 1$ *iff* $x = \pm 1$ *or* x *is purely imaginary and* $x^2 = x \cdot x = 1$.
(5) $x^2 = -1$ *iff* x *is purely imaginary and* $x^2 = x \cdot x = -1$.

We also need the following version of Lemma 2.1 found in [KPP].

LEMMA 2.2. $X : M_1^2 \to \mathbb{R}_1^3$ *is a conformal immersion iff there exists* $N : M_1^2 \to \widetilde{\mathbb{H}}$ *such that*

(2.6) $$(*dX)U = dX(J'U) = N\,dX(U).$$

If (2.6) *holds then* $N : M_1^2 \to S_1^2 \subset \mathbb{R}_1^3$, *where* S_1^2 *is the Lorentzian sphere in* $\mathbb{R}_1^3$ *and* N *is the oriented unit normal field to* X.

Proof. Assume first that N exists. Since dX is pointwise injective,

$$dX(U) = (*dX)(J'U) = (N\,dX)(J'U) = NN\,dX(U),$$

so that $N^2 = 1$. We see that N cannot be ± 1, so that N is purely imaginary and a map into the sphere. Next we would like to see that N is normal to the surface. Note that $-\overline{N} = N$, so that $\overline{dX(J'U)} = \overline{dX(U)}\,\overline{N}$ gives $-dX(J'U) = dX(U)N$, hence $-NdX(U) = dX(U)N$ and

$$NdX(U) + dX(U)N = 0 = 2N \cdot dX(U),$$

so that N is a normal field.

Conversely, suppose $X : M_1^2 \to \mathbb{R}_1^3$ is a conformal immersion of a Lorentzian surface and assume that $N : M_1^2 \to S_1^2$ is the unit normal field to X. We need to show that

$$dX(J'U) = NdX(U) = N \times dX(U).$$

Equivalently we must show that for the null coordinates $\{x, y\}$,

$$dX(-\partial x) = N \times X_x, \quad dX(\partial y) = N \times X_y.$$

We can see that this is so because $\det[X_t/\mu, X_s/\mu, N] = 1$. ∎

At this point we decompose our $\widetilde{\mathbb{H}}$-valued one-forms $\alpha = \alpha_+ + \alpha_-$ by defining

(2.7) $\qquad \alpha_+ = \frac{1}{2}(\alpha - N(*\alpha)), \quad \alpha_- = \frac{1}{2}(\alpha + N(*\alpha)).$

These satisfy

$$*\alpha_+ = -N\alpha_+, \quad *\alpha_- = N\alpha_-.$$

Now we can find

$$\begin{aligned} d(*dX) = d(NdX) = dN \wedge dX &= dN(*dX) - (*dN)dX \\ &= (dNN - *dN)dX = 2(dN_-)NdX. \end{aligned}$$

We can see by a calculation involving the shape operator that $(dNN - *dN)dX$ is $\operatorname{Im}\widetilde{\mathbb{H}}$-valued. This says that $dN_- \cdot NdX = 0$, which in turn implies that dN_- is a real multiple of dX. In fact we can see that $dN_- = HdX$, where H is the negative of the mean curvature of the immersion X.

Thus we have

(2.8) $\qquad\qquad\qquad d*dX = -2HNdX \cdot dX$

and we can write

(2.9) $\qquad dN = dN_- + dN_+ = HdX + \theta \quad \text{with} \quad \theta = dN_+.$

We also get the Codazzi equation by differentiating (2.9):

(2.10) $\qquad\qquad\qquad d\theta = (*dH - dHN)dX.$

2.3. *Spin equivalent immersions*

DEFINITION 2.2. Two conformal immersions $X, \widetilde{X} : (M_1^2, [h]) \to \mathbb{R}_1^3$ are called *spin equivalent* if there exists $\lambda : M_1^2 \to \widetilde{\mathbb{H}}_*$ so that

$$(2.11) \qquad d\widetilde{X} = -\left(\frac{|\lambda \cdot \lambda|}{\lambda \cdot \lambda}\right)\overline{\lambda}dX\lambda = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda}dX\lambda,$$

where $\widetilde{\mathbb{H}}_*$ are the invertible elements.

The factor $\operatorname{sgn}(\lambda \cdot \lambda)$ is not needed in the positive definite case because it is always 1 there.

We note that, up to a positive scaling factor, this is an orthogonal transformation of $\mathbb{R}_1^3$ and the mapping

$$\varrho(v) = -\frac{1}{\lambda \cdot \lambda}\,\overline{\lambda}v\lambda$$

covers the special orthogonal group of $\mathbb{R}_1^3$. Note finally that the identity component of the special orthogonal group, which consists of orientation and time orientation preserving transformations, is generated by λ with $\lambda \cdot \lambda < 0$.

Thus we can see that, locally, any two conformal immersions are spin equivalent.

If X and $\widetilde{X}$ are spin equivalent then we must have

$$0 = d(\overline{\lambda}dX\lambda) = d\overline{\lambda} \wedge dX\lambda - \overline{\lambda}dX \wedge d\lambda.$$

This shows that $\overline{\lambda}dX \wedge d\lambda$ is real, so that $dX \wedge d\lambda = -\varrho dX \cdot dX\lambda = -\varrho dXdX\lambda$ for some real-valued function ϱ on M_1^2. Then we have

$$-\varrho dXdX\lambda = dX \wedge d\lambda = dX(*d\lambda) - (*dX)d\lambda = dX(*d\lambda + Nd\lambda),$$

so that we get the following integrability condition:

$$-\varrho dX\lambda = *d\lambda + Nd\lambda.$$

We write this as a lemma:

LEMMA 2.3. *If X and $\widetilde{X}$ are spin equivalent via*

$$d\widetilde{X} = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda}dX\lambda$$

then $\lambda : M_1^2 \to \widetilde{\mathbb{H}}_*$ satisfies

$$(2.12) \qquad\qquad -\varrho dX\lambda = *d\lambda + Nd\lambda$$

for some real-valued function ϱ on M_1^2. Conversely, if $\pi_1(M_1^2) = 0$ then for a given conformal immersion $X : M_1^2 \to \mathbb{R}_1^3$, nowhere vanishing solutions $\lambda : M_1^2 \to \widetilde{\mathbb{H}}_*$ to (2.12) give all conformal immersions $\widetilde{X} : M_1^2 \to \mathbb{R}_1^3$ via the transformations $d\widetilde{X} = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda}dX\lambda$.

The next lemma calculates the shape operator of a spin transform; it gives more information than the positive definite version.

LEMMA 2.4. *Let* $X, \widetilde{X} : M_1^2 \rightarrow \mathbb{R}_1^3$ *be spin equivalent via* $d\widetilde{X} = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda}dX\lambda$. *Then*

(1) $\widetilde{N} = \lambda^{-1}N\lambda$.
(2) $d\widetilde{X} \cdot d\widetilde{X} = (\lambda \cdot \lambda)^2 dX \cdot dX$.
(3)
$$\widetilde{A} = \frac{\operatorname{sgn}(\lambda \cdot \lambda)}{\lambda \cdot \lambda} \begin{bmatrix} -2a_2 - e/\mu^2 & -f/\mu^2 + 2a_1 \\ -2a_1 + f/\mu^2 & 2a_2 + 2\varrho + g/\mu^2 \end{bmatrix},$$

where a_1, a_2 *are defined below.*

Proof. The first two statements are clear so we will calculate the shape operator of $\widetilde{X}$. Using the fact that $Nd\lambda = -*d\lambda - \varrho dX\lambda$ we find that

$$N\lambda_t = -\lambda_s - \varrho X_t\lambda, \quad N\lambda_s = -\lambda_t - \varrho X_s\lambda,$$

or

$$\lambda_t\lambda^{-1} = -N\lambda_s\lambda^{-1} - \varrho X_s, \quad \lambda_s\lambda^{-1} = -N\lambda_t\lambda^{-1} - \varrho X_t.$$

If we set

$$\lambda_t\lambda^{-1} = a_0 + a_1X_t + a_2X_s + a_3N, \quad \lambda_s\lambda^{-1} = b_0 + b_1X_t + b_2X_s + b_3N,$$

and use the equations above we see that

$$b_0 = -a_3, \quad b_1 = -a_2 - \varrho, \quad b_2 = -a_1, \quad b_3 = -a_0.$$

We now calculate

$$\begin{aligned} d\widetilde{N} &= -\lambda^{-1}d\lambda\lambda^{-1}N\lambda + \lambda^{-1}dN\lambda + \lambda^{-1}Nd\lambda \\ &= -\lambda^{-1}(d\lambda\lambda^{-1}N - dN - Nd\lambda\lambda^{-1})\lambda \\ &= \frac{1}{\lambda \cdot \lambda}\overline{\lambda}(d\lambda\lambda^{-1}N - dN - Nd\lambda\lambda^{-1})\lambda. \end{aligned}$$

Using the expression for N_t and N_s from (2.2) we can find

$$(2.13) \qquad \widetilde{A} = \frac{\operatorname{sgn}(\lambda \cdot \lambda)}{\lambda \cdot \lambda} \begin{bmatrix} -2a_2 - e/\mu^2 & -f/\mu^2 + 2a_1 \\ -2a_1 + f/\mu^2 & 2a_2 + 2\varrho + g/\mu^2 \end{bmatrix}.$$

Thus the negative of the mean curvature of $\widetilde{X}$ is

$$(2.14) \qquad \widetilde{H} = \frac{\operatorname{sgn}(\lambda \cdot \lambda)}{\lambda \cdot \lambda}(-\varrho + H). \quad \blacksquare$$

COROLLARY. *Assume that* $X, \widetilde{X} : M_1^2 \rightarrow \mathbb{R}_1^3$ *are spin equivalent with*

$$d\widetilde{X} = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda}dX\lambda.$$

Then the following are equivalent:

(1) $dX \wedge d\lambda = 0$, *which is the same as* $*d\lambda + Nd\lambda = 0$.
(2) $\operatorname{sgn}(\lambda \cdot \lambda)\widetilde{H}d\widetilde{X} \cdot d\widetilde{X} = H(dX \cdot dX)(\lambda \cdot \lambda)$.

3. Isothermic surfaces and Bonnet pairs. We begin with an alternative characterization of an isothermic surface.

PROPOSITION 3.1. *A surface* $X : M_1^2 \to \mathbb{R}_1^3$ *is isothermic iff there is a two-dimensional closed one-form* $\tau \in \Omega^1(M_1^2, \mathrm{Im}\,\widetilde{\mathbb{H}})$ *so that* $*\tau + N\tau = 0$, *or, equivalently,* $dX \wedge \tau = 0$.

Proof. If a surface X is real isothermic, then

$$\tau = -\frac{X_t}{\mu^2}\,dt + \frac{X_s}{\mu^2}\,ds,$$

while if X is complex isothermic,

$$\tau = -\frac{X_s}{\mu^2}\,dt + \frac{X_t}{\mu^2}\,ds.$$

On the other hand, if such a locally integrable one-form τ is given, then there is a surface X^c so that $dX \wedge dX^c = 0$. Setting

$$X_t^c = a_1 X_t + b_1 X_s + c_1 N, \quad X_s^c = a_2 X_t + b_2 X_s + c_2 N,$$

and calculating $X_t X_s^c - X_s X_t^c = 0$ gives

$$X_t^c = a_1 X_t + b_1 X_s, \quad X_s^c = -b_1 X_t - a_1 X_s.$$

We see that the tangent planes of the two surfaces agree. If $a_1^2 - b_1^2 > 0$ the surface is real isothermic, while the other sign gives a complex isothermic surface [M]. The only other possibility, $a_1^2 - b_1^2 = 0$, does not yield a two-dimensional one-form. ∎

DEFINITION 3.1. The Lorentzian isothermic surface X^c in Proposition 3.1 is called the *dual surface* to X.

DEFINITION 3.2. Two conformal immersions $X, \widetilde{X} : (M_1^2, [h]) \to \mathbb{R}_1^3$ form a *Bonnet pair* if $dX \cdot dX = d\widetilde{X} \cdot d\widetilde{X}$, the principal curvatures agree, and the shape operators have the same algebraic type but they are not congruent. (If one avoids umbilic points then one only needs the mean curvatures to agree.)

THEOREM 3.1. *Let* $X : M_1^2 \to \mathbb{R}_1^3$ *be an isothermic surface with dual* $X^c : M_1^2 \to \mathbb{R}_1^3$. *Choose* $\varepsilon \in \mathbb{R}$, $a \in \mathrm{Im}\,\widetilde{\mathbb{H}}$ *and set* $\lambda_\pm = \pm\varepsilon + X^c + a$, *where* ε *and* a *must be chosen so that* $\lambda_\pm$ *is invertible. Then the spin transforms* $X_\pm : M_1^2 \to \mathbb{R}_1^3$ *given by* $dX_\pm = -\mathrm{sgn}(\lambda_\pm \cdot \lambda_\pm)\overline{\lambda}_\pm dX \lambda_\pm$ *form a Bonnet pair.*

Conversely, up to a rigid transformation, every pair of Bonnet surfaces arises from a three-parameter family (determined up to scalings) of isothermic surfaces where the three parameters account for the orientation and time orientation preserving rotations of the Bonnet pairs with respect to one another.

Note that
$$\lambda_\pm \cdot \lambda_\pm = -(\pm\varepsilon + X^c + a)(\pm\varepsilon - X^c - a)$$
$$= -(\varepsilon^2 - a^2 - aX^c - X^c a - X^c X^c) = (X^c + a) \cdot (X^c + a) - \varepsilon^2.$$

Thus the condition that $\lambda_\pm$ is invertible is equivalent to requiring that X^c does not intersect the indefinite sphere centered at $-a$ with radius ε.

Proof. We first assume that we have X and X^c so that $dX \wedge dX^c = 0$. This implies that $dX \wedge d\lambda_\pm = 0$ or $*d\lambda_\pm + N d\lambda_\pm = 0$. This is (2.10) with $\varrho_\pm = 0$. We have seen that $\lambda_+ \cdot \lambda_+ = \lambda_- \cdot \lambda_-$, so that $dX_+ \cdot dX_+ = dX_- \cdot dX_-$

To see that $X_\pm$ form a Bonnet pair, we need to look at their shape operators. As we saw in the proof of Lemma 2.4 the shape operators depend on the shape operator of X and the tangential components of $(\lambda_+)_t(\lambda_+)^{-1}$ and $(\lambda_-)_t(\lambda_-)^{-1}$. In fact these are

$$(\lambda_+)_t(\lambda_+)^{-1} = \frac{1}{\lambda_+ \cdot \lambda_+} X_t^c(-\varepsilon + X^c + a),$$

$$(\lambda_-)_t(\lambda_-)^{-1} = \frac{1}{\lambda_+ \cdot \lambda_+} X_t^c(\varepsilon + X^c + a).$$

We express the tangential component of these expressions respectively as

$$((\lambda_+)_t(\lambda_+)^{-1})^T = \frac{1}{\lambda_+ \cdot \lambda_+} \left(-\varepsilon X_t^c + c_1 X_t + c_2 X_s\right),$$

$$((\lambda_-)_t(\lambda_-)^{-1})^T = \frac{1}{\lambda_+ \cdot \lambda_+} \left(\varepsilon X_t^c + c_1 X_t + c_2 X_s\right).$$

We will now make a separate calculation for the two types of isothermic surfaces, real and complex. In the case of a real isothermic surface, $X_t^c = -X_t/\mu^2$ and c_1 is zero. Indeed,

$$c_1 = \frac{1}{\mu^2} X_t \cdot (X_t/\mu^2)(X^c + a)$$

$$= -\operatorname{Re}\frac{1}{\mu^2} \overline{X}_t(X_t/\mu^2)(X^c + a) = \frac{1}{\mu^2} \operatorname{Re}(X^c + a) = 0.$$

Recalling that, for a real isothermic surface, $f = 0$, we see that the shape operators of X_+ and X_- are

$$A^+ = \frac{\operatorname{sgn}(\lambda_+ \cdot \lambda_+)}{\lambda_+ \cdot \lambda_+} \begin{bmatrix} \dfrac{-2c_2}{\lambda_+ \cdot \lambda_+} - \dfrac{e}{\mu^2} & \dfrac{2\varepsilon}{(\lambda_+ \cdot \lambda_+)\mu^2} \\[3mm] \dfrac{-2\varepsilon}{(\lambda_+ \cdot \lambda_+)\mu^2} & \dfrac{2c_2}{\lambda_+ \cdot \lambda_+} + \dfrac{g}{\mu^2} \end{bmatrix},$$

$$A^- = \frac{\operatorname{sgn}(\lambda_+ \cdot \lambda_+)}{\lambda_+ \cdot \lambda_+} \begin{bmatrix} \dfrac{-2c_2}{\lambda_+ \cdot \lambda_+} - \dfrac{e}{\mu^2} & \dfrac{-2\varepsilon}{(\lambda_+ \cdot \lambda_+)\mu^2} \\[3mm] \dfrac{2\varepsilon}{(\lambda_+ \cdot \lambda_+)\mu^2} & \dfrac{2c_2}{\lambda_+ \cdot \lambda_+} + \dfrac{g}{\mu^2} \end{bmatrix}.$$

Thus it is clear that, in this case, the shape operators have the same eigenvalues and the same algebraic type. Indeed, one can explicitly calculate the eigenvalues and eigenvectors to determine the algebraic type of the Bonnet pairs.

In the case of a complex isothermic surface, $X_t^c = -X_s/\mu^2$ and c_2 is zero as above. Recalling that, for a complex isothermic surface, $g = -e$ we see that the shape operators of X_+ and X_- are

$$A^+ = \frac{\operatorname{sgn}(\lambda_+ \cdot \lambda_+)}{\lambda_+ \cdot \lambda_+} \begin{bmatrix} \dfrac{-2\varepsilon}{\mu^2(\lambda_+ \cdot \lambda_+)} - \dfrac{e}{\mu^2} & \dfrac{2c_1}{\lambda_+ \cdot \lambda_+} - \dfrac{f}{\mu^2} \\[2ex] -\dfrac{2c_1}{\lambda_+ \cdot \lambda_+} + \dfrac{f}{\mu^2} & \dfrac{2\varepsilon}{\mu^2(\lambda_+ \cdot \lambda_+)} - \dfrac{e}{\mu^2} \end{bmatrix},$$

$$A^- = \frac{\operatorname{sgn}(\lambda_+ \cdot \lambda_+)}{\lambda_+ \cdot \lambda_+} \begin{bmatrix} \dfrac{2\varepsilon}{\mu^2(\lambda_+ \cdot \lambda_+)} - \dfrac{e}{\mu^2} & \dfrac{2c_1}{\lambda_+ \cdot \lambda_+} - \dfrac{f}{\mu^2}, \\[2ex] -\dfrac{2c_1}{\lambda_+ \cdot \lambda_+} + \dfrac{f}{\mu^2} & \dfrac{-2\varepsilon}{\mu^2(\lambda_+ \cdot \lambda_+)} - \dfrac{e}{\mu^2} \end{bmatrix}.$$

Again, the shape operators have the same eigenvalues and the same algebraic type at each corresponding point.

For the converse suppose that X_- and X_+ form a Bonnet pair and, initially, that dX_- and dX_+ are never equal. We have

$$(3.1) \qquad dX_+ = -\operatorname{sgn}(\lambda \cdot \lambda)\overline{\lambda} dX_- \lambda$$

for some $\lambda : M_1^2 \to \widetilde{\mathbb{H}}$. Since X_- and X_+ induce the same metric, we have $dX_+ \cdot dX_+ = \overline{\lambda} dX_- \lambda \cdot \overline{\lambda} dX_- \lambda = (\overline{\lambda}\lambda)^2 dX_- \cdot dX_- = dX_- \cdot dX_-$, so $\lambda \cdot \lambda = \pm 1$. We are assuming that $X_\pm$ induce the same time orientation, so that $\lambda \cdot \lambda = -1$. From equation (2.14) we see that $dX_- \wedge d\lambda = 0$. We are looking for $\lambda_\pm$ so that

$$dX_\pm = -\operatorname{sgn}(\lambda_\pm \cdot \lambda_\pm)\overline{\lambda}_\pm dX \lambda_\pm$$

with $\lambda_+ = X^c + \varepsilon$ and $\lambda_- = X^c - \varepsilon$. In this case

$$-\operatorname{sgn}(\lambda_+ \cdot \lambda_+)\overline{\lambda}_+ dX \lambda_+ = -\operatorname{sgn}(\lambda_- \cdot \lambda_-)\overline{\lambda\lambda}_- dX \lambda_- \lambda.$$

In order for this to hold we must have

$$(3.2) \qquad \lambda_+ = \lambda_- \lambda.$$

Thus, the signs of λ_- and λ_+ are equal, yielding $\lambda_- + 2\varepsilon = \lambda_+$ and $\lambda_- = 2\varepsilon(\lambda-1)^{-1}$. We set $X^c = \varepsilon + 2\varepsilon(\lambda-1)^{-1}$. One can calculate, using $\lambda \cdot \lambda = -1$, that X^c is imaginary.

Thus we can define X by

$$(3.3) \qquad dX = -\operatorname{sgn}(\lambda_- \cdot \lambda_-)\frac{\overline{\lambda}-1}{2\varepsilon} dX_- \frac{\lambda-1}{2\varepsilon}$$

since the forms on the right-hand side are closed. We can also see that $dX^c \wedge dX = 0$.

We can now follow the proof of Theorem 2.3.2 in [KNPP]. Let $X_{\pm}$ be a Bonnet pair and $\lambda : M_1^2 \to \mathbb{H}_2^3$ the function such that $dX_+ = \overline{\lambda} dX_- \lambda$, where $\mathbb{H}_2^3$ is the set of vectors in $\widetilde{\mathbb{H}}$ of length -1. Because M_1^2 is a surface, there must be an $a \in \mathbb{H}_2^3$ such that a is not in the image of λ. We see that $-\operatorname{sgn}(a \cdot a) a X_+ \overline{a} = a X_+ \overline{a}$ and X_- form a Bonnet pair satisfying $dX_- \neq a dX_+ \overline{a}$. ■

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