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The theorem of Forelli for holomorphic mappings into complex spaces

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Dedicated to Professor Nguyen Thanh Van
on the occasion of his 60th birthday

Abstract. Generalizations of the theorem of Forelli to holomorphic mappings into complex spaces are given.

1. Introduction. The classical Hartogs theorem states that if a complex-valued function $f(z_1, \ldots, z_n)$ defined for $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ $(n \geq 2)$ is separately holomorphic, i.e. holomorphic with respect to each variable separately when the other variables are fixed, then f is jointly holomorphic. Equivalently, if f is holomorphic on each line which is parallel to some coordinate axis, then f is jointly holomorphic. Much attention has been given to generalizing this theorem, and many Hartogs-type theorems for separately holomorphic mappings have been obtained by various authors (see [Te], [S], [NZ1], [NZ2], [Shi2], [TM], [JP]).

Modifying the point of view of the above-mentioned theorem, in 1978, F. Forelli proved the following remarkable result (see [Ru, p. 60] or [Sha, p. 49]). If f is a function defined in the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$, holomorphic on the intersection of $\mathbb{B}^n$ with every complex line l passing through the origin, and if f is of class $\mathcal{C}^\infty$ in a neighbourhood of the origin, then it is holomorphic in $\mathbb{B}^n$.

Our main aim in this article is to generalize the theorem of Forelli to holomorphic mappings into complex spaces. Namely, we are going to prove the following:

Theorem A. *Let M be a complex space and $\mathbb{B}^n$ the open unit ball of $\mathbb{C}^n$. Let $f : \mathbb{B}^n \to M$ be a mapping such that f is holomorphic on the intersection of $\mathbb{B}^n$ with every complex line l passing through the origin, and f is of*

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class $\mathcal{C}^\infty$ in a neighbourhood of the origin. Then there exists a pluripolar subset S of $\mathbb{P}^{n-1}(\mathbb{C})$ such that f is holomorphic in a neighbourhood of $\mathbb{B}^n - \bigcup\{l : l \in S\}$.

THEOREM B. *Let M be a complex space of Hartogs type. Then M has the Forelli property.*

THEOREM C. *Let M be a holomorphically convex Kähler complex space. Then M has the Hartogs extension property if and only if M has the Forelli property.*

THEOREM D. *Let M be a holomorphically convex compact Kähler manifold. Let $f : \mathbb{B}^n \to M$ be a mapping such that f is holomorphic on the intersection of $\mathbb{B}^n$ with every complex line l passing through the origin, and f is of class $\mathcal{C}^\infty$ in a neighbourhood of the origin. Then f is meromorphic in $\mathbb{B}^n$.*

2. Preliminaries

2.1. DEFINITION. For $r > 0$ put $\Delta_r = \Delta(0,r) = \{|z| < r\} \subset \mathbb{C}$ and $\Delta_1 = \Delta$.

Let X be a complex space. We say that X has the *Hartogs extension property* (briefly X has (HEP)) if every holomorphic mapping, from a Riemann domain Ω over a Stein manifold into X, can be extended holomorphically to $\widehat{\Omega}$, the envelope of holomorphy of Ω.

Let $H_2(r) = \{(z_1, z_2) \in \Delta^2 : |z_1| < r \text{ or } |z_2| > 1 - r\}$ $(0 < r < 1)$ denote the 2-dimensional Hartogs domain.

It is well known ([Shi1] or [I]) that X has (HEP) iff every holomorphic mapping $f : H_2(r) \to X$ extends holomorphically over Δ^2.

The class of complex spaces having (HEP) is large: it contains the taut complex spaces [Fu], complex Lie groups [ASY], and complete hermitian complex manifolds with non-positive holomorphic sectional curvature [Shi1]. In particular, Ivashkovich [I] showed that a holomorphically convex Kähler manifold has (HEP) iff it contains no rational curves. This was generalized to holomorphically convex Kähler spaces by Do Duc Thai [T].

2.2. DEFINITION. Let M be a complex space.

(i) An open subset A of M is said to be of *type* (H) if there exists a biholomorphic mapping from A onto an analytic subset of a complex space having (HEP).

(ii) The space M is said to be *of Hartogs type* if for each $p \in M$ there exists a neighbourhood W_p of p and $r_p > 0$ and a neighbourhood S_p of p of type (H) such that for each $f \in \mathrm{Hol}(\Delta, M)$, if $f(0) \in W_p$ then $f(\Delta_{r_p}) \subset S_p$.

The class of complex spaces of Hartogs type is rather large. It is easy to see that it contains the complex spaces having (HEP) and the hyperbolic complex spaces.

2.3. Definition. Let M be a complex space. We say that M has the *Forelli property* for the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ (briefly M has (FP)) if whenever a mapping $f : \mathbb{B}^n \to M$ is holomorphic on the intersection of $\mathbb{B}^n$ with every complex line l passing through the origin, and f is of class $\mathcal{C}^\infty$ in a neighbourhood of the origin, then f is holomorphic in $\mathbb{B}^n$.

Examples. (a) The complex plane $\mathbb{C}$ has the Forelli property (see [Ru, p. 60]).

(b) Every complex space of Stein type has the Forelli property (see [TP]).

2.4. Let l_a be a complex line passing through the origin of $\mathbb{C}^n$. Then in $\mathbb{C}^n$, the set l_a is given by $\{t(a_1, \ldots, a_n) : t \in \mathbb{C}\}$. Thus we can consider l_a as a point $a = [a_1 : \ldots : a_n]$ in $\mathbb{P}^{n-1}(\mathbb{C})$.

2.5. Let S be a subset of a complex manifold M. We say that S is *pluripolar* if for any $x_0 \in S$ there are an open neighbourhood U of x_0 in M and a plurisubharmonic function $\varphi : U \to [-\infty, \infty)$ such that $S \cap U \subset \{\varphi = -\infty\}$.

2.6. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we let $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. For each $R > 0$ put $\mathbb{B}^n_R = \mathbb{B}^n(0, R) = \{z \in \mathbb{C}^n : \|z\| < R\}$, $\mathbb{B}^n = \mathbb{B}^n_1$.

3. Proofs of the main results. In order to prove Theorem A we need the following lemma:

3.1. Lemma ([Shi2]). *Let M be a complex space. Let U, V be open sets in $\mathbb{C}^m, \mathbb{C}^n$ respectively and let K be a connected compact set in $\mathbb{C}^n$ containing V. Let $f : U \times V \to M$ be a holomorphic map. If f_z extends holomorphically to K for all $z \in U$, then there exists a closed pluripolar subset E of U and a holomorphic map $\widetilde{f} : (U - E) \times K \to M$ such that $f = \widetilde{f}$ on $(U - E) \times V$.*

3.2. *Proof of Theorem A.* By the theorem of Forelli [Ru, p. 60], there exists $r_0 > 0$ such that

$$f \text{ is holomorphic in } \mathbb{B}^n_{r_0}. \tag{1}$$

Put $\mathbb{B}^n_* = \mathbb{B}^n - \{z_n = 0\}$. Consider the holomorphic mapping $\varphi : \mathbb{B}^n_* \to \mathbb{C}^n$ given by $\varphi(z_1, \ldots, z_n) = (z_1/z_n, \ldots, z_{n-1}/z_n, z_n)$. Put $\varphi(\mathbb{B}^n_*) = T$ and define $\varphi_1 : \mathbb{B}^n_* \to T$ by $\varphi_1(z) = \varphi(z)$ for $z \in \mathbb{B}^n_*$. Then φ_1 is biholomorphic.

Put $g = f \circ \varphi_1^{-1} : T \to M$ and

$$T_{R,h} = \{t = (t', z_n) \in T : \|t'\| < R \text{ and } 0 < |z_n|^2 < h/(1 + R^2)\}$$

for $R > 0$ and $0 < h \leq 1$. It is easy to see that $\{T_{R,h}\}$ is a family of open sets which is increasing when h is increasing and $T = \bigcup\{T_{R,1} : R > 0\} = \bigcup\{T_{R,1} : R \in \mathbb{Q}_+^*\}$. From (1), it follows that g is holomorphic in T_{R,r_0^2} for all $R > 0$.

Define

$$\widetilde{\Delta}_R = \overline{\Delta}_{\sqrt{1/(1+R^2)}} = \{z \in \mathbb{C} : |z| \leq \sqrt{1/(1+R^2)}\},$$
$$S_R = \{w' \in \mathbb{B}_R^{n-1} : g \text{ does not extend to any neighbourhood of } (w' \times \widetilde{\Delta}_R) \cap \varphi_1(\mathbb{B}_*^n)\}.$$

It is easy to see that S_R is closed. We now prove that S_R is pluripolar. Indeed, by the hypothesis and since

$$\frac{1}{1 + \|w'\|^2} > \frac{1}{1 + R^2} \qquad \text{for each } w' \in \mathbb{B}_R^{n-1},$$

it follows that the mapping $g_{w'}(w_n) = g(w', w_n) = f(w_n w', w_n)$ is holomorphic on some neighbourhood of $\widetilde{\Delta}_R$. From Lemma 3.1, there exists a closed pluripolar subset S'_R of $\mathbb{B}_R^{n-1}$ such that g extends to a holomorphic mapping $\widetilde{g} : (\mathbb{B}_R^{n-1} - S'_R) \times \widetilde{\Delta}_R) \to M$. Clearly, $S_R \subset S'_R$, and hence S_R is pluripolar.

Put $\widetilde{S}_R = S_R \times \widetilde{\Delta}_R$ and $\widetilde{S} = \bigcup_{R \in \mathbb{Q}_+^*} \widetilde{S}_R$. Clearly, $\widetilde{S}$ is a pluripolar subset of T.

Take any point $z = (z', z_n) \in T - \widetilde{S}$. Since $T = \bigcup_{R \in \mathbb{Q}_+^*} T_{R,1}$, there exists $R \in \mathbb{Q}_+^*$ such that $z \in T_{R,1}$. On the other hand, by the definition of S_R and $\widetilde{S}_R$, we get $z' \notin S_R$. Thus g extends holomorphically over a neighbourhood of $(z' \times \widetilde{\Delta}_R) \cap T$. This means that g is holomorphic on an open neighbourhood of z. This also implies that g is holomorphic on an open neighbourhood of $T - \widetilde{S}$.

Consider the mapping $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ given by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$ and put $T^* = \{z : z \in T \text{ and } p(z) \notin \bigcup_{R \in \mathbb{Q}_+^*} S_R\}$. Since $T^* \subset T - \widetilde{S}$, it follows that g is holomorphic on an open neighbourhood of T^*.

Since $\mathbb{B}^n = \bigcup_{j=1}^n (\mathbb{B}^n - \{z_j = 0\}) \cup \mathbb{B}_{r_0}^n$, we conclude that f is holomorphic on an open neighbourhood of $\mathbb{B}^n - \bigcup_{a \in S} l_a$, where S is pluripolar in $\mathbb{P}^{n-1}(\mathbb{C})$. ∎

In order to prove Theorem B we need the following lemma:

3.3. Lemma ([Shi2]). *Let M be a complex space having* (HEP). *Let U, V be domains in $\mathbb{C}^m, \mathbb{C}^n$ respectively and let V_0 be an open subset of V. If $f : U \times V_0 \to M$ is a holomorphic map such that f_z extends holomorphically to V for almost all $z \in U$, then f extends holomorphically to $U \times V$.*

3.4. *Proof of Theorem B.* By the theorem of Forelli [Ru, p. 60], there exists $r_0 > 0$ such that f is holomorphic in $\mathbb{B}_{r_0}^n$. Put $r^* = \sup\{r \in (0, 1) : f \text{ is holomorphic in } \mathbb{B}_r^n\}$. Then f is holomorphic in $\mathbb{B}_{r^*}^n$. Suppose $r^* < 1$.

STEP 1. Take $p_0 \in \partial\mathbb{B}^n_{r^*}$. For the point $f(p_0) \in M$ take $W_0 = W_{f(p_0)}$, $r_0 = r_{f(p_0)}$, $S_0 = S_{f(p_0)}$ as in Definition 2.2(ii) of Hartogs type, i.e. for each $\varphi \in \mathrm{Hol}(\Delta, M)$, if $\varphi(0) \in W_0$ then $\varphi(\Delta_{r_0}) \subset S_0$.

Since

$$\lim_{\alpha\to 1^-} \frac{r^*(1-\alpha)}{1-\alpha(r^*)^2} = 0 < r_0,$$

there exists $\alpha_0 \in (0,1)$ such that

$$\frac{r^*(1-\alpha_0)}{1-\alpha_0(r^*)^2} < r_0, \qquad f(\alpha_0 p_0) \in W_0.$$

Since

$$\lim_{p\to p_0} \frac{\|p\|(1-\alpha_0)}{1-\alpha_0\|p\|^2} = \frac{r^*(1-\alpha_0)}{1-\alpha_0(r^*)^2} < r_0,$$

there exists $\mathbb{B}(p_0,\delta) \subset \mathbb{B}^n$ such that $\|p\|(1-\alpha_0)/(1-\alpha_0\|p\|^2) < r_0$ for each $p \in \mathbb{B}(p_0,\delta)$ and

$$f(\alpha_0\mathbb{B}(p_0,\delta)) = f(\mathbb{B}(\alpha_0 p_0, \alpha_0\delta)) \subset W_0.$$

We now prove that $f(\mathbb{B}(p_0,\delta)) \subset S_0$. Indeed, take $p \in \mathbb{B}(p_0,\delta)$. Consider the Möbius map $\psi : \Delta \to \Delta$ given by

$$\psi(z) = \frac{z - \|\alpha_0 p\|}{1 - \|\alpha_0 p\| z}.$$

Put $\psi(\|p\|) = p'$. Consider the map $\varphi : \Delta \to \mathbb{B}^n$ given by $\varphi(z) = z.p/\|p\|$ and the composite map $\phi := f \circ \varphi \circ \psi^{-1} : \Delta \to M$. Then $\phi(0) = f(\alpha_0 p) \in W_0$, $\phi(p') = f(p)$. On the other hand, since

$$|p'| = \frac{\|p\|(1-\alpha_0)}{1-\alpha_0\|p\|^2} < r_0,$$

we have $p' \in \Delta_{r_0}$, and hence $\phi(p') = f(p) \in S_0$.

STEP 2. We now prove that, for each $p_0 \in \partial\mathbb{B}^n_{r^*}$, there exists $\delta_{p_0} > 0$ such that the restriction of f to $\mathbb{B}(p_0, \delta_{p_0})$ is holomorphic. Without loss of generality we may assume that $p_0 = (0,\ldots,0,r^*)$.

By using again the mappings φ_1, g and the definitions of T, $T_{R,h}$, we find that g is holomorphic in $T_{R,(r^*)^2}$ for all $R > 0$. By Step 1 and since φ_1 is biholomorphic, there exists $\delta > 0$ such that $g(\mathbb{B}(p_0,\delta))$ is contained in a subset S_0 of Hartogs type. Note that $\varphi_1(p_0) = p_0$.

Take a sufficiently small $\delta_1 > 0$ such that $\Delta^{n-1}_{\delta_1} \times \Delta(p_0,\delta_1) \subset \mathbb{B}(p_0,\delta)$. Since

$$\lim_{\delta\to 0^+} \frac{(r^*)^2}{1+(n-1)\delta^2} = (r^*)^2 > \left(r^* - \frac{\delta_1}{4}\right)^2,$$

there exists $\delta_2 > 0$ such that

$$(*) \qquad \frac{(r^*)^2}{1+(n-1)\delta_2^2} > \left(r^* - \frac{\delta_1}{4}\right)^2, \quad 0 < \delta_2 < \delta_1.$$

This implies $\Delta_{\delta_2}^{n-1} \times \Delta(r^* - \delta_1/2, \delta_1/4) \subset \Delta_{\delta_2}^{n-1} \times \Delta(r^*, \delta_1) \subset \mathbb{B}(p_0, \delta)$ and $\Delta_{\delta_2}^{n-1} \times \Delta(r^* - \delta_1/2, \delta_1/4) \subset T_{\delta_2,(r^*)^2}$.

By Lemma 3.3, g is holomorphic in $\Delta_{\delta_2}^{n-1} \times \Delta(r^*, \delta_1)$. Thus the assertion of Step 2 follows from the fact that φ_1 is a biholomorphic mapping.

STEP 3. For each $p \in \overline{\mathbb{B}^n_{r^*}}$ put

$$\delta_p = \sup\{\delta : f \text{ is holomorphic in } \mathbb{B}(p, \delta)\}.$$

By Step 2, we know that δ_p is positive.

On the other hand, it is easy to see that

$$|\delta_{p_0} - \delta_{p_1}| \leq \|p_0 - p_1\|, \quad \forall p_0, p_1 \in \overline{\mathbb{B}^n_{r^*}}.$$

This implies that the function $\delta : \overline{\mathbb{B}^n_{r^*}} \to \mathbb{R}^+_*$ is continuous. Hence $\min_{p\in\overline{\mathbb{B}^n_{r^*}}} \delta(p) = \delta_{r^*} > 0$. Then f is holomorphic in $\mathbb{B}^n_{r^*+\delta_{r^*}} \supsetneq \mathbb{B}^n_{r^*}$. This is a contradiction. ∎

The following lemma plays an essential role in proving Theorem C:

3.5. LEMMA ([T]). *Let M be a holomorphically convex Kähler complex space. Then M has* (HEP) *if and only if M contains no rational curves.*

3.6. *Proof of Theorem C. Sufficiency.* This follows immediately from Theorem B.

Necessity. By Lemma 3.5, it suffices to prove that M contains no rational curve.

Suppose that

(1) there exists a rational curve $\varphi : \mathbb{P}^1(\mathbb{C}) \to M$ and $\varphi \neq$ const.

Consider the mapping $f : \mathbb{B}^2 \to \mathbb{P}^1(\mathbb{C})$ given by $(z, w) \mapsto [(z + w - 1)^2 : (z - w)^2]$ for each $(z, w) \neq (1/2, 1/2)$ and $f(1/2, 1/2) = [1 : 1]$. Then it is easy to check that f is $\mathcal{C}^\infty$ in an open neighbourhood of the origin and the restriction of f to each complex line through the origin is holomorphic. Since M has (FP), $\varphi \circ f$ is holomorphic. In particular, it is continuous, and hence the following limit exists:

$$(2) \qquad \lim_{(z,w)\to(1/2,1/2)} (\varphi \circ f)(z, w) = a \in M.$$

From (1), it follows that

$$(3) \qquad \varphi^{-1}(a) \text{ is a finite set in } \mathbb{P}^1(\mathbb{C}).$$

Put $w = 1/2 + \lambda(z - 1/2)$, $\lambda \in \mathbb{C}$. Then

$$\lim_{(z,w)\to(1/2,1/2)} f(z, w) = [(1 + \lambda)^2 : (1 - \lambda)^2].$$

By (2), we have $\{[(1+\lambda)^2 : (1-\lambda)^2] : \lambda \in \mathbb{C}\} \subset \varphi^{-1}(a)$. This contradicts (3). The proof is complete. ∎

In order to prove Theorem D we need the following lemma:

3.7. Lemma ([Shi 3]). *Let M be a complex space having the meromorphic extension property. Let U, V be open sets in $\mathbb{C}^m, \mathbb{C}^n$ respectively, and let V_0 be an open subset of V. Let $f : U \times V_0 \to M$ be a meromorphic mapping. If f_z has a meromorphic extension to V for almost all $z \in U$, then f has a meromorphic extension to $U \times V$.*

3.8. *Proof of Theorem D.* We use the argument of the first part of Theorem A. By the theorem of Forelli [Ru, p. 60], there exists $r_0 > 0$ such that g is holomorphic in T_{R,r_0^2} for all $R > 0$.

From Lemma 3.7, we deduce that g is meromorphic in $T_{R,1}$. Since $T = \bigcup_{R>0} T_{R,1}$, g is meromorphic in T. On the other hand, since $\mathbb{B}^n = \bigcup_{i=1}^n (\mathbb{B}^n - \{z_i = 0\}) \cup \mathbb{B}^n_{r_0}$, we conclude that f is meromorphic in $\mathbb{B}^n$.

3.9. Remark. The Kähler property in Theorem D cannot be omitted. Consider the Hopf surface $S = \mathbb{C}^2 - \{0\}/z \sim 2z$ and the canonical projection $\varphi : \mathbb{C}^2 - \{0\} \to S$. Then φ is holomorphic on any complex curve through 0 but does not extend meromorphically to $\mathbb{C}^2$. Let $f : \mathbb{B}^2 \to S$ be the holomorphic mapping given by $f(z,w) = \varphi((z+w-1)^2, (z-w)^2)$. It is easy to see that the limit $\lim_{t\to 0} \varphi(t,t)$ exists and equals $f(1/2, 1/2)$. But f is not meromorphic at 0.

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