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On a problem concerning quasianalytic local rings

by Hassan Sfouli (Kénitra)

Abstract. Let $(\mathcal{C}_n)_n$ be a quasianalytic differentiable system. Let $m \in \mathbb{N}$. We consider the following problem: let $f \in \mathcal{C}_m$ and $\widehat{f}$ be its Taylor series at $0 \in \mathbb{R}^m$. Split the set $\mathbb{N}^m$ of exponents into two disjoint subsets A and B, $\mathbb{N}^m = A \cup B$, and decompose the formal series $\widehat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Do there exist $g, h \in \mathcal{C}_m$ with Taylor series at zero G and H, respectively? The main result of this paper is the following: if we have a positive answer to the above problem for some $m \geq 2$, then the system $(\mathcal{C}_n)_n$ is contained in the system of analytic germs. As an application of this result, we give a simple proof of Carleman's theorem (on the non-surjectivity of the Borel map in the quasianalytic case), under the condition that the quasianalytic classes considered are closed under differentiation, for $n \geq 2$.

1. Introduction. In this paper we consider the following problem concerning quasianalytic classes, posed in [N1, N2] for a quasianalytic Denjoy–Carleman class.

Problem. Let $(\mathcal{C}_n)_n$ be a quasianalytic differentiable system. Let $f \in \mathcal{C}_m$ and $\widehat{f}$ be its Taylor series at $0 \in \mathbb{R}^m$. Split the set $\mathbb{N}^m$ of exponents into two disjoint subsets A and B, $\mathbb{N}^m = A \cup B$, and decompose the formal series $\widehat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Do there exist $g, h \in \mathcal{C}_m$ with Taylor series at zero G and H, respectively?

This problem is related to the question whether polynomials are dense in a certain Hilbert space associated with a quasianalytic Denjoy–Carleman class, investigated by Thilliez [T] in connection with his proof of Carleman's theorem on the failure of surjectivity for the Borel mapping. Clearly, the above problem is trivial for the analytic system. In this paper, we solve this problem for quasianalytic differentiable systems: we prove that if the answer to this problem is affirmative for a given quasianalytic differentiable system and some dimension $m \geq 2$, then the system is contained in the

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analytic system. Furthermore, we present a simple proof of Carleman's theorem (on the non-surjectivity of the Borel map in the quasianalytic case) for quasianalytic differentiable Denjoy–Carleman classes. Finally, we give a negative answer to the above problem for a differentiable analytic system of an o-minimal structure. The work of this paper has been motivated by a question (concerning the above problem) asked by K. J. Nowak [N2], who was interested mainly in the diagonal splitting of exponents.

2. Quasianalytic differentiable systems. Let $(X_1,\dots,X_n)$ be an n-tuple of distinct indeterminates with $n\in\mathbb{N}$. The ring of formal series in $X_1,\dots,X_n$ over the field $\mathbb{R}$ of real numbers will be denoted by $\mathbb{R}[[X_1,\dots,X_n]]$, and the subring of $\mathbb{R}[[X_1,\dots,X_n]]$ of formal series which converge in some neighborhood of the origin in $\mathbb{R}^n$ will be denoted by $\mathbb{R}\langle X_1,\dots,X_n\rangle$.

Denote by $\mathcal{A}_n$ (resp. $\mathcal{E}_n$) the ring of real-analytic (resp. smooth) function germs at the origin of $\mathbb{R}^n$, and by $\mathcal{P}_n$ the ring of germs, at the origin in $\mathbb{R}^n$, of polynomial functions. Clearly, $\mathcal{P}_n\subseteq\mathcal{A}_n\subseteq\mathcal{E}_n$ for all $n\in\mathbb{N}$, and $\mathcal{A}_n$ is isomorphic to $\mathbb{R}\langle X_1,\dots,X_n\rangle$.

Definition 2.1. A *differentiable system* is a sequence

$$\mathcal{C}=\{\mathcal{C}_n;\, n\in\mathbb{N}\}$$

such that, for each $n\in\mathbb{N}$, $\mathcal{C}_n$ is a local subring of $\mathcal{E}_n$ and the following hold:

(C1) $\mathcal{P}_n\subseteq\mathcal{C}_n\subseteq\mathcal{E}_n$;
(C2) if $\varphi_1,\dots,\varphi_n\in\mathcal{C}_p$ are such that $\varphi_1(0)=\cdots=\varphi_n(0)=0$, then for every $f\in\mathcal{C}_n$ the composition $f(\varphi_1,\dots,\varphi_n)$ belongs to $\mathcal{C}_p$;
(C3) $\partial f/\partial x_i\in\mathcal{C}_n$ for every $f\in\mathcal{C}_n$ and each $i=1,\dots,n$.

Let

$$\widehat{\cdot}:\mathcal{C}_n\to\mathbb{R}[[X_1,\dots,X_n]]$$

be the map which associates to each $f\in\mathcal{C}_n$ its Taylor expansion. We consider the following condition:

(C4) $\widehat{\cdot}$ is an injective homomorphism.

Definition 2.2. A differentiable system is called *quasianalytic* if the condition (C4) holds.

Now, we consider the following two conditions:

(C5) For each $n\ge 2$ and each $f\in\mathcal{C}_n$ there is a neighborhood, U, of the origin in $\mathbb{R}^n$ such that the functions $x\mapsto f(x+a)$, $a\in U$, belong to $\mathcal{C}_n$.

(S_m) Let $f\in\mathcal{C}_m$ and $\widehat{f}$ be its Taylor series at $0\in\mathbb{R}^m$. Split the set $\mathbb{N}^m$ of exponents into two disjoint subsets A and B, $\mathbb{N}^m=A\cup B$, and decompose the formal series $\widehat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Then there exist $g, h \in \mathcal{C}_m$ with Taylor series at zero G and H, respectively.

We can now state the main result of this paper, proved in Section 3.

THEOREM 2.3. *Let $\mathcal{C} = (\mathcal{C}_n)_n$ be a quasianalytic differentiable system such that there is an integer $m \geq 2$ for which the condition* (S_m) *holds. Then:*

(1) $\mathcal{C}_1 \subseteq \mathcal{A}_1$.
(2) *If $\mathcal{A}_n \subseteq \mathcal{C}_n$ for all $n \geq 2$, then $\mathcal{C}_n = \mathcal{A}_n$ for every $n \in \mathbb{N}$.*
(3) *If the condition* (C5) *holds, then $\mathcal{C}_n \subseteq \mathcal{A}_n$ for every $n \in \mathbb{N}$.*

3. Proof of Theorem 2.3. For the proof of Theorem 2.3 we need a few lemmas.

LEMMA 3.1. *Let $\mathcal{C} = (\mathcal{C}_n)_n$ be a quasianalytic differentiable system. Assume that there is an integer $m \geq 2$ such that the condition* (S_m) *holds. Then the conditions* (S_1) *and* (S_2) *hold.*

Proof. Let $f \in \mathcal{C}_2$. Put $g(x_1, \ldots, x_m) := f(x_1, x_2)$. By (C2), $g \in \mathcal{C}_m$. Split $\mathbb{N}^2$ into disjoint subsets A and B, $\mathbb{N}^2 = A \cup B$, and decompose the formal series $\widehat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Put $A' = A \times \mathbb{N}^{m-2}$ and $B' = B \times \mathbb{N}^{m-2}$. Clearly, $A' \cup B' = \mathbb{N}^m$ and $A' \cap B' = \emptyset$. Now, we decompose $\widehat{g}$ into the sum of two formal series G' and H', supported by A' and B', respectively. Then there exist $g_0, g_1 \in \mathcal{C}_m$ with Taylor series at zero G' and H', respectively. By (C2), the germs f_0 and f_1 given by $f_0(x_1, x_2) = g_0(x_1, x_2, 0, \ldots, 0)$ and $f_1(x_1, x_2) = g_1(x_1, x_2, 0, \ldots, 0)$ belong to $\mathcal{C}_2$. Clearly, $\widehat{f_0} = \widehat{g_0} = G' = G$ and $\widehat{f_1} = \widehat{g_1} = H' = H$. Therefore, the condition (S_2) holds. Using the same discussion (with obvious changes), we can see that (S_1) holds. ∎

Proof of Theorem 2.3(1). Only in this proof $X = X_1$, $Y = X_2$, $x = x_1$ and $y = x_2$. Let $f \in \mathcal{C}_1$. Put $g(x, y) = f(x + y)$. We have $g \in \mathcal{C}_2$. Put

$$\widehat{g} = \sum_{(\alpha,\beta) \in \mathbb{N}^2} a_{\alpha,\beta} X^\alpha Y^\beta.$$

For each $l \in \{0, 1, 2, 3\}$, put $A_l = \{(k, 4p + l);\ k, p \in \mathbb{N}\}$. Clearly the sets A_l, $l = 0, 1, 2, 3$, are disjoint and

$$\bigcup_{l=0}^{3} A_l = \mathbb{N}^2.$$

Decompose the formal series $\widehat{g}$ into the sum of formal series H_0, H_1, H_2 and H_3, supported by A_0, A_1, A_2 and A_3, respectively. By Lemma 3.1, for each $l \in \{0,1,2,3\}$ there exists $g_l \in \mathcal{C}_2$ such that $\widehat{g}_l = H_l$. We have

$$\widehat{f}(X+iY) = \widehat{g}(X,iY) = \sum_{(\alpha,\beta)\in\mathbb{N}^2} a_{\alpha,\beta} X^\alpha (i)^\beta Y^\beta = \widehat{g}_0 + i\widehat{g}_1 - \widehat{g}_2 - i\widehat{g}_3.$$

Put $u = g_0 - g_2$ and $v = g_1 - g_3$. Hence

$$\widehat{f}(X+iY) = \widehat{u} + i\widehat{v}. \tag{3.1}$$

From (3.1), we have the Cauchy–Riemann equalities

$$\frac{\partial\widehat{u}}{\partial X} = \frac{\partial\widehat{v}}{\partial Y} \quad\text{and}\quad \frac{\partial\widehat{v}}{\partial X} = -\frac{\partial\widehat{u}}{\partial Y}.$$

Then, by quasianalyticity, these equalities are satisfied by the functions u and v themselves.

The function given by $F(x+iy) = u(x,y) + iv(x,y)$ is then holomorphic in a neighborhood of the origin in $\mathbb{C}$. Thus u and v belong to $\mathcal{A}_2$. Since $\widehat{f} \in \mathbb{R}[[X]]$, $\widehat{f}(X) = \widehat{u}(X,0) + i\widehat{v}(X,0) = \widehat{u}(X,0)$. Clearly the function u_0 given by $u_0(x) = u(x,0)$ belongs to $\mathcal{A}_1 \cap \mathcal{C}_1$ and $\widehat{u}_0 = \widehat{f}$. Therefore, by quasianalyticity, $f = u_0 \in \mathcal{A}_1$. ∎

Let $F \in \mathbb{R}[[X_1,\dots,X_n]]$ and $\mathbb{S}^{n-1}$ be the unit sphere of $\mathbb{R}^n$. If $\xi \in \mathbb{S}^{n-1}$, write $F_\xi(t) = f(\xi t) \in \mathbb{R}[[t]]$.

Lemma 3.2 ([AM]). *Let $F \in \mathbb{R}[[X_1,\dots,X_n]]$. Assume that $F_\xi(t) \in \mathbb{R}\langle t\rangle$ for each $\xi \in \mathbb{S}^{n-1}$. Then $F \in \mathbb{R}\langle X_1,\dots,X_n\rangle$.*

Corollary 3.3. *Let $\mathcal{C} = (\mathcal{C}_n)_n$ be a quasianalytic differentiable system. Assume that there is an integer $m \ge 2$ for which the condition* (S_m) *holds. Then for all n and all $f \in \mathcal{C}_n$ we have $\widehat{f} \in \mathbb{R}\langle X_1,\dots,X_n\rangle$.*

Proof. Let $f \in \mathcal{C}_n$. By Theorem 2.3(1), for each $\xi \in \mathbb{S}^{n-1}$, we have $\widehat{f_\xi} \in \mathbb{R}\langle t\rangle$. Hence by Lemma 3.2, $\widehat{f} \in \mathbb{R}\langle X_1,\dots,X_n\rangle$. ∎

Proof of Theorem 2.3(2). Let $f \in \mathcal{C}_n$. By Corollary 3.3, we have $\widehat{f} \in \mathbb{R}\langle X_1,\dots,X_n\rangle$. Let g be the germ of $\widehat{f}$ at the origin of $\mathbb{R}^n$. By the hypothesis, $g \in \mathcal{C}_n$. Since $\widehat{g} = \widehat{f}$, $f = g \in \mathcal{A}_n$ by quasianalyticity. ∎

Proof of Theorem 2.3(3). Let $f \in \mathcal{C}_n$. As before, $\widehat{f} \in \mathbb{R}\langle X_1,\dots,X_n\rangle$. Let g be the germ of $\widehat{f}$ at the origin of $\mathbb{R}^n$. By the hypothesis, there exists $\varepsilon > 0$ such that the functions $x \mapsto f(x)$ and $x \mapsto g(x)$ are well defined on

$$\mathbb{B} := \{x;\ \|x\| < \varepsilon\}$$

and for all $a \in \mathbb{B}$ the function $x \mapsto f(x+a)$ belongs to $\mathcal{C}_n$. Using again Theorem 2.3(1), it is easy to see that for all $\xi \in \mathbb{S}^{n-1}$ and $t \in (-\varepsilon,\varepsilon)$ we have $f(t\xi) = g(t\xi)$. Therefore, $f = g \in \mathcal{A}_n$. ∎

The following corollary is an extension of [ES, Theorem 2].

COROLLARY 3.4. *Let $\mathcal{C} = (\mathcal{C}_n)$ be a quasianalytic differentiable system. Then for $n \geq 2$ the Taylor map $T : \mathcal{C}_n \to \mathbb{R}[[X_1, \ldots, X_n]]$ is not surjective.*

Proof. Clearly, if $T : \mathcal{C}_n \to \mathbb{R}[[X_1, \ldots, X_n]]$ is surjective for some $n \geq 2$, then $T : \mathcal{C}_2 \to \mathbb{R}[[X_1, X_2]]$ is surjective. It suffices to prove that $T : \mathcal{C}_2 \to \mathbb{R}[[X_1, X_2]]$ is not surjective. Suppose it is. Clearly, $T : \mathcal{C}_1 \to \mathbb{R}[[X_1]]$ is surjective. It is easy to see that the condition (S_2) holds for $\mathcal{C}_2$. Hence, by Theorem 2.3(1), $\mathcal{C}_1 \subseteq \mathcal{A}_1$. This is a contradiction because $T : \mathcal{C}_1 \to \mathbb{R}[[X_1]]$ must be surjective. ∎

4. On Carleman's theorem. We use the following notation: for any multi-index $J = (j_1, \ldots, j_n)$ in $\mathbb{N}^n$, we denote the length $j_1 + \cdots + j_n$ of J by the corresponding lower case letter j. We put $D^J = \partial^j / \partial x_1^{j_1} \ldots \partial x_n^{j_n}$, $J! = j_1! \ldots j_n!$ and $X^J = X_1^{j_1} \ldots X_n^{j_n}$, where $X = (X_1, \ldots, X_n)$.

The map $T_0 : \mathcal{E}_n \to \mathbb{R}[[X]]$ defined by

$$T_0 f = \sum_{J \in \mathbb{N}^n} \frac{D^J f(0)}{J!} X^J$$

will be called the *Borel map*.

Now, let $M = (M_j)_{j \geq 0}$ be an increasing sequence of real numbers with $M_0 = 1$. Denote by $\mathcal{E}_n(M)$ the set of elements f of $\mathcal{E}_n$ for which there exist a neighborhood U of 0 and positive constants C and A such that

$$|D^J f(x)| \leq C A^j j! M_j \quad \text{for all } J \in \mathbb{N}^n \text{ and } x \in U.$$

We clearly have

$$\mathcal{A}_n \subseteq \mathcal{E}_n(M) \subseteq \mathcal{E}_n.$$

In the same spirit, denote by $\mathcal{F}_n(M)$ the set of elements

$$F = \sum_{J \in \mathbb{N}^n} F_J X^J$$

of $\mathbb{R}[[X]]$ for which there exist positive constants C and A such that

$$|F_J| \leq C A^j M_j \quad \text{for all } J \in \mathbb{N}^n. \tag{4.1}$$

The Borel map then obviously satisfies

$$T_0 \mathcal{E}_n(M) \subseteq \mathcal{F}_n(M).$$

One cannot hope to get much more information on the sets $\mathcal{E}_n(M)$ and $\mathcal{F}_n(M)$ without additional assumptions on the sequence M. From now on, we shall always make the following assumption:

the sequence M is logarithmically convex,

that is, M_{j+1}/M_j increases. Under this assumption, $\mathcal{E}_n(M)$ is a local ring and is closed under composition in the sense of the condition (C2).

The ring $\mathcal{E}_n(M)$ is stable under derivation if and only if

$$\sup_{j\geq 1}\,(M_{j+1}/M_j)^{1/j} < \infty.$$

The local ring $\mathcal{E}_n(M)$ is quasianalytic if and only if

$$\sum_{j=0}^{\infty} M_j/((j+1)M_{j+1}) = \infty.$$

EXAMPLE 4.1. Let α be a real number with $1 \geq \alpha > 0$. Put $M_j = (\log(j+e))^{\alpha j}$. Then $(\mathcal{E}_n(M))_n$ is a quasianalytic differentiable system (see [T]).

In the following, we present a simple proof of Carleman's theorem (see [T]) for the quasianalytic Denjoy–Carleman classes that are closed under differentiation.

THEOREM 4.2. *Assume that $(\mathcal{E}_n(M))_n$ is a quasianalytic differentiable system. Let $n \geq 2$ be an integer. If $\mathcal{A}_n \neq \mathcal{E}_n(M)$, then the map $T_0 : \mathcal{E}_n(M) \to \mathcal{F}_n(M)$ is not surjective.*

Proof. Suppose otherwise. Let

$$F = \sum_{J\in\mathbb{N}^n} F_J X^J \in \mathcal{F}_n(M),$$

and let

$$G = \sum_{J\in\mathbb{N}^n} G_J X^J$$

be such that $G_J \in \{0, F_J\}$. By (4.1),

$$|F_J| \leq CA^j M_j \quad \text{for all } J \in \mathbb{N}^n.$$

Then

$$|G_J| \leq CA^j M_j \quad \text{for all } J \in \mathbb{N}^n.$$

Thus $G \in \mathcal{F}_n(M)$. Hence the condition (S_n) holds for the local ring $\mathcal{F}_n(M)$. Since the map T_0 is an isomorphism, (S_n) holds for the local ring $\mathcal{E}_n(M)$. Now, by Theorem 2.3(2), $\mathcal{A}_p = \mathcal{E}_p(M)$ for all $p \in \mathbb{N}$. Therefore, T_0 is not surjective. ■

5. The problem and some o-minimal structures. Let $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \times, <, 0, 1)$ be the ordered field of real numbers. In this section we will give a negative answer to the above problem for the rings of definable C^∞ germs in the following o-minimal structures: $\mathcal{R}_1 := (\overline{\mathbb{R}}, \exp|_{[0,1]})$, $\mathcal{R}_2 := (\overline{\mathbb{R}}, \sin|_{[0,1]})$ and the expansion $\mathbb{R}_{\mathcal{G}}$ of the real field $\overline{\mathbb{R}}$ generated by multisummable real series [DS]. The o-minimality of $\mathbb{R}_{\mathcal{G}}$ was proved in [DS].

Among the basic operations of $\mathbb{R}_{\mathcal{G}}$ we have the C^∞ functions $f : [0,1] \to \mathbb{R}$ whose restriction to $(0,1]$ extends to holomorphic functions on a sector

$$S(R,\phi) := \{z \in \mathbb{C};\ |z| < R,\ |\arg z| < \phi\}$$

for some $R > 1$ and $\phi > \pi/2$, such that there exist positive constants A, B with $|f^{(n)}(z)| \le AB^n (n!)^2$ for all $z \in S(R,\phi)$, and

$$\lim_{S(R,\phi)\ni z\to 0} f^{(n)}(z) = f^{(n)}(0).$$

An example of such a function is

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \qquad \text{for } 0 \le x \le 1.$$

Its Taylor expansion at 0 is the *divergent* series $\sum_{n=0}^\infty (-1)^n n! x^n$.

Let $g : (-1,1) \to \mathbb{R}$ be the function given by $g(x) = f(x^2)$. The Taylor expansion of g at 0 is the divergent series $\widehat{g} := \sum_{n=0}^\infty (-1)^n n! x^{2n}$. Split the set $\mathbb{N}$ of exponents into $A = \{4p;\ p \in \mathbb{N}\}$ and $B = \mathbb{N} \setminus A$, and decompose $\widehat{g}$ into the sum of formal series G and H, supported by A and B, respectively. Clearly the series $G = \sum_{p=0}^\infty (2p)! x^{4p}$ is divergent. By [DS, Corollary 8.6], there is no C^∞ function $h : (-\varepsilon,\varepsilon) \to \mathbb{R}$, for some $\varepsilon > 0$, that is definable in $\mathbb{R}_{\mathcal{G}}$ and whose Taylor series at 0 is G. Therefore, we have a negative answer to the problem for the ring of germs, at the origin of $\mathbb{R}^n$, of $\mathbb{R}_{\mathcal{G}}$-definable C^∞ functions at the origin of $\mathbb{R}^n$ for all $n \ge 1$.

We have

$$\sin(x) := \sum_{p=0}^\infty (-1)^p \frac{x^{2p+1}}{(2p+1)!},\ \ \cos(x) := \sum_{p=0}^\infty (-1)^p \frac{x^{2p}}{(2p)!},\ \ \exp(x) := \sum_{p=0}^\infty \frac{x^p}{p!}.$$

Clearly, if we have a positive answer to the problem for the ring of germs (at the origin of $\mathbb{R}$) of C^∞ $\mathcal{R}_2$-definable functions at $0 \in \mathbb{R}$, then the germ, at 0, of

$$\exp(x) = \sum_{p=0}^\infty \frac{x^{2p+1}}{(2p+1)!} + \sum_{p=0}^\infty \frac{x^{2p}}{(2p)!}$$

is definable in $\mathcal{R}_2$. This contradicts Bianconi's theorem [B]. On the other hand if we have a positive answer to the problem for the ring of germs (at the origin of $\mathbb{R}$) of C^∞ $\mathcal{R}_1$-definable functions at $0 \in \mathbb{R}$, then the germ, at 0, of

$$\sin(x) := \sum_{p=0}^\infty (-1)^p \frac{x^{2p+1}}{(2p+1)!}$$

is definable in $\mathcal{R}_2$, also contrary to Bianconi's theorem [B].

It is not clear to us what is the answer to the problem for the ring of germs of Nash functions at the origin of $\mathbb{R}$. In particular, we do not know if the germ at 0 of the function

$$x \mapsto \sum_{p \in \mathbb{P}} x^p,$$

where $\mathbb{P}$ is the set of prime numbers, is a Nash function or not.

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References

[AM] S. Abhyankar and T. Moh, *Reduction theorem for divergent power series*, J. Reine Angew. Math. 241 (1970), 27–33.

[B] R. Bianconi, *Nondefinability results for expansions of the field of real numbers by the exponential function and by the restricted sine function*, J. Symbolic Logic 62 (1997), 1173–1178.

[DS] L. van den Dries and P. Speissegger, *The field of reals with multisummable series and the exponential function*, Proc. London Math. Soc. 81 (2000), 513–565.

[ES] A. Elkhadiri and H. Sfouli, *Weierstrass division theorem in quasianalytic local rings*, Studia Math. 185 (2008), 83–86.

[N1] K. Nowak, *On two problems concerning quasianalytic Denjoy–Carleman classes*, RAAG Preprint 241, 2007, and IMUJ Preprint 19, 2007.

[N2] K. Nowak, *Quantifier elimination, valuation property and preparation theorem in quasianalytic geometry via transformation to normal crossings*, Ann. Polon. Math. 96 (2009), 247–282.

[T] V. Thilliez, *On quasianalytic local rings*, Expo. Math. 26 (2008), 1–23.

Hassan Sfouli
Département de Mathématiques
Faculté des Sciences
Université Ibn Tofail
BP 133 Kénitra, Maroc
E-mail: hassansfouli@hotmail.com

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