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Canonical Poisson–Nijenhuis structures on higher order tangent bundles

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Abstract. Let M be a smooth manifold of dimension $m > 0$, and denote by S_{can} the canonical Nijenhuis tensor on TM. Let Π be a Poisson bivector on M and Π^T the complete lift of Π on TM. In a previous paper, we have shown that $(TM, \Pi^T, S_{\mathrm{can}})$ is a Poisson–Nijenhuis manifold. Recently, the higher order tangent lifts of Poisson manifolds from M to T^rM have been studied and some properties were given. Furthermore, the canonical Nijenhuis tensors on T^AM are described by A. Cabras and I. Kolář [Arch. Math. (Brno) 38 (2002), 243–257], where A is a Weil algebra. In the particular case where $A = J_0^r(\mathbb{R},\mathbb{R}) \simeq \mathbb{R}^{r+1}$ with the canonical basis (e_α), we obtain for each $0 \le \alpha \le r$ the canonical Nijenhuis tensor S_α on T^rM defined by the vector e_α. The tensor S_α is called the canonical Nijenhuis tensor on T^rM of degree α. In this paper, we show that if (M, Π) is a Poisson manifold, then for each α with $1 \le \alpha \le r$, $(T^rM, \Pi^{(c)}, S_\alpha)$ is a Poisson–Nijenhuis manifold. In particular, we describe other prolongations of Poisson manifolds from M to T^rM and we give some of their properties.

1. Introduction. Let M be a smooth manifold of dimension $m > 0$. We denote by $\pi_M : TM \to M$ the tangent vector bundle and by $\pi_M^* : T^*M \to M$ the cotangent vector bundle. We also denote by $\langle\cdot,\cdot\rangle_M : TM \times_M T^*M \to \mathbb{R}$ the usual canonical pairing. Let S be a $(1,1)$-tensor field on M. The *Nijenhuis torsion* of S is defined by, for any $X, Y \in \mathfrak{X}(M)$,

$$T_S(X,Y) = [SX, SY] - S([SX,Y] + [X,SY] - S[X,Y]).$$

If $T_S = 0$, then S is said to be a *Nijenhuis tensor* and the pair (M,S) is called a *Nijenhuis manifold*. Let Π be a Poisson bivector on M. We denote by $S\Pi$ the $(2,0)$-tensor field associated with the vector bundle morphism $S \circ \sharp_\Pi$ from T^*M to TM defined for any 1-forms ω, ϖ by

$$S\Pi(\omega,\varpi) = \langle \omega, S\circ\sharp_\Pi(\varpi)\rangle_M = \langle S^*\omega, \sharp_\Pi(\varpi)\rangle_M = \Pi(S^*\omega, \varpi),$$

where S^* denotes the dual map of S. Let (M,S) be a Nijenhuis manifold and Π a Poisson bivector on M. The Poisson structure Π and the Nijenhuis

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tensor S are called *compatible* (see [KO], [KM2], [V1] or [V2]) if

$$\sharp_\Pi \circ S^* = S \circ \sharp_\Pi \quad \text{and} \quad \nabla_{\Pi S}(\omega, \varpi) = 0$$

for any $\omega, \varpi \in \Omega^1(M)$, where

$$\nabla_{\Pi S}(\omega, \varpi) = [\omega, \varpi]_{S\Pi} - ([S^*\omega, \varpi]_\Pi + [\omega, S^*\varpi]_\Pi - S^*[\omega, \varpi]_\Pi)$$

and $[\cdot,\cdot]_\Pi$, $[\cdot,\cdot]_{S\Pi}$ are the Koszul brackets induced by the bivectors Π and $S\Pi$. In particular, the bivector $S\Pi$ defined by the vector bundle morphism $S \circ \sharp_\Pi : T^*M \to TM$ over id_M is a Poisson bivector on M.

Let $(x^1, \ldots, x^m)$ be a local coordinate system of M such that

$$S = S^i_j dx^j \otimes \frac{\partial}{\partial x^i} \quad \text{and} \quad \Pi = \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Writing

$$\nabla_{\Pi S} = \Gamma^{ij}_k dx^k \otimes \left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\right)$$

we have

$$\Gamma^{ij}_k = \Pi^{\ell j} \frac{\partial S^i_k}{\partial x^\ell} + \Pi^{i\ell} \frac{\partial S^j_k}{\partial x^\ell} - S^\ell_k \frac{\partial \Pi^{ij}}{\partial x^\ell} + S^j_\ell \frac{\partial \Pi^{i\ell}}{\partial x^k} - \Pi^{\ell j} \frac{\partial S^i_\ell}{\partial x^k}.$$

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps, and $\mathcal{FM}$ the category of all smooth fibred manifolds and fibred morphisms. For any integer $r \geq 1$ and any manifold M, we put $T^rM = J^r_0(\mathbb{R}, M)$. The elements of T^rM are said to be 1-*dimensional velocities* of order r on M. The smooth map $\pi^r_M : T^rM \to M$ defined by $\pi^r_M(j^r_0\varphi) = \varphi(0)$ for $j^r_0\varphi \in T^rM$ defines the structure of a smooth fiber bundle. Usually, the manifold T^rM with the projection π^r_M is called the *tangent bundle of* M *of order* r. On the other hand, every smooth map $f : M \to N$ extends to an $\mathcal{FM}$-morphism $T^rf : T^rM \to T^rN$ defined by $T^rf(j^r_0\varphi) = j^r_0(f \circ \varphi)$. Hence, T^r is a functor $\mathcal{M}f \to \mathcal{FM}$ and it preserves products.

Let (U, x^i) be a local coordinate system of M. The local coordinate system of T^rM over T^rU is such that the coordinate functions (x^i_β) with $i = 1, \ldots, m$ and $\beta = 0, \ldots, r$ are given by

$$\begin{cases} x^i_0(j^r_0 g) = x^i(g(0)), \\ x^i_\beta(j^r_0 g) = \dfrac{1}{\beta!} \dfrac{d^\beta(x^i \circ g)}{dt^\beta}(t)\Big|_{t=0}. \end{cases}$$

In the following, the coordinate function x^i_0 is denoted by x^i. For $r = 1$, we obtain the usual tangent functor denoted by T.

In this paper, we generalize the work of [KW]. The main results are Theorems 4.1, 4.2 and 5.2: given a Poisson manifold (M, Π) and the canonical Nijenhuis tensor field S_α of degree α on T^rM defined below, we prove that $(T^rM, \Pi^{(c)}, S_\alpha)$ is a Poisson–Nijenhuis manifold, where the Poisson bivector

$\Pi^{(c)}$ on T^rM is the complete lift of Π to T^rM defined in [KWN]; moreover, we study some properties of the Poisson bivector Π^α defined by $S_\alpha \circ \sharp_{\Pi^{(c)}}$.

In this paper, all manifolds and mappings are assumed to be of class C^∞. We shall fix a natural number $r \geq 1$.

2. Preliminaries

2.1. The canonical isomorphism $\kappa_M^r : T^rTM \to TT^rM$. For each $\beta \in \{0, \dots, r\}$, we denote by τ_β the canonical linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by

$$\tau_\beta(j_0^r g) = \frac{1}{\beta!} \frac{d^\beta}{dt^\beta}(g(t))\bigg|_{t=0}, \qquad \text{for } g \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Let M be a smooth manifold of dimension $m > 0$. For $f \in C^\infty(M)$, we set $f^{(\beta)} = \tau_\beta \circ T^r f$. The smooth map $f^{(\beta)}$ is called the β-*prolongation* of f; it is defined for any $j_0^r\varphi \in T^rM$ by

$$f^{(\beta)}(j_0^r\varphi) = \frac{1}{\beta!} \frac{d^\beta (f \circ \varphi)}{dt^\beta}(t)\bigg|_{t=0}.$$

It follows that $x_\beta^i = (x^i)^{(\beta)}$ on T^rU with coordinate system $(x^1, \dots, x^m)$.

For each manifold M, there is a canonical diffeomorphism (see [GMP], [KMS])

$$\kappa_M^r : T^rTM \to TT^rM,$$

which is an isomorphism of vector bundles from

$$T^r(\pi_M) : T^rTM \to T^rM \quad \text{to} \quad \pi_{T^rM} : TT^rM \to T^rM$$

such that $T(\pi_M^r) \circ \kappa_M^r = \pi_{TM}^r$ and for any smooth map $f : M \to N$ we have the equality

$$\kappa_N^r \circ T^rTf = TT^rf \circ \kappa_M^r.$$

Let $(x^1, \dots, x^m)$ be a local coordinate system of M. We introduce the coordinates $(x^i, \dot{x}^i)$ in TM, $(x^i, \dot{x}^i, x_\beta^i, \dot{x}_\beta^i)$ in T^rTM and $(x^i, x_\beta^i, \dot{x}^i, \dot{x_\beta^i})$ in TT^rM. We have

$$\kappa_M^r(x^i, \dot{x}^i, x_\beta^i, \dot{x}_\beta^i) = (x^i, x_\beta^i, \dot{x}^i, \dot{x_\beta^i})$$

with $\dot{x_\beta^i} = \dot{x}_\beta^i$.

2.2. The canonical isomorphism $\alpha_M^r : T^*T^rM \to T^rT^*M$. For any manifold M, there is a canonical diffeomorphism

$$\alpha_M^r : T^*T^rM \to T^rT^*M$$

which is an isomorphism of the vector bundles

$$\pi_{T^rM}^* : T^*T^rM \to T^rM \quad \text{and} \quad T^r(\pi_M^*) : T^rT^*M \to T^rM$$

dual to κ_M^r with respect to the pairings $\langle \cdot,\cdot\rangle'_{T^rM} = \tau_r \circ T^r(\langle \cdot,\cdot\rangle_M)$ and $\langle \cdot,\cdot\rangle_{T^rM}$, i.e. for any $(u,u^*) \in T^rTM \oplus T^*T^rM$,

$$\langle \kappa_M^r(u), u^*\rangle_{T^rM} = \langle u, \alpha_M^r(u^*)\rangle'_{T^rM}.$$

Let $(x^1,\dots,x^m)$ be a local coordinate system of M. We introduce the coordinates (x^i,p_j) in T^*M, $(x^i,p_j,x^i_\beta,p^\beta_j)$ in T^rT^*M and $(x^i,x^i_\beta,\pi_j,\pi^\beta_j)$ in T^*T^rM. We have

$$\alpha_M^r(x^i,\pi_j,x^i_\beta,\pi^\beta_j) = (x^i,x^i_\beta,p_j,p^\beta_j) \quad \text{with} \quad \begin{cases} p_j = \pi^r_j, \\ p^\beta_j = \pi^{r-\beta}_j. \end{cases}$$

We denote $(\alpha_M^r)^{-1}$ by ε_M^r.

3. Canonical Nijenhuis tensor on higher order tangent bundles

3.1. Higher order lifting of vector fields. Let (E,M,π) be a vector bundle, and consider the vector bundle morphism $\chi_E^{(\alpha)} : T^rE \to T^rE$ defined by

$$\chi_E^{(\alpha)}(j_0^r\Psi) = j_0^r(t^\alpha\Psi)$$

where $\Psi \in C^\infty(\mathbb{R},E)$ and $t^\alpha\Psi$ is the smooth map defined for any $t\in\mathbb{R}$ by

$$(t^\alpha\Psi)(t) = t^\alpha\Psi(t).$$

Let X be a vector field on the manifold M. We define the *α-prolongation* of X, denoted $X^{(\alpha)}$, by

$$X^{(\alpha)} = \kappa_M^r \circ \chi_{TM}^{(\alpha)} \circ T^rX.$$

When $\alpha = 0$, it is called the *complete lift* of X to T^rM, and it is denoted by $X^{(c)}$. We put $X^{(\alpha)} = 0$ for $\alpha > r$ or $\alpha < 0$.

If (U,x^i) is a local coordinate system of M such that $X = X^i\frac{\partial}{\partial x^i}$, then

$$X^{(\alpha)} = (X^i)^{(\beta-\alpha)}\frac{\partial}{\partial x^i_\beta}.$$

Proposition 3.1.

(i) *For $X\in\mathfrak{X}(M)$, $f\in C^\infty(M)$ and $\alpha,\beta\in\{0,\dots,r\}$, we have*

$$X^{(\alpha)}(f^{(\beta)}) = (X(f))^{(\beta-\alpha)}.$$

(ii) *For $X,Y\in\mathfrak{X}(M)$ and $\alpha,\beta\in\{0,\dots,r\}$, we have:*

$$[X^{(\alpha)},Y^{(\beta)}] = [X,Y]^{(\alpha+\beta)}. \tag{3.1}$$

(iii) *The set $\{X^{(\beta)} \mid X\in\mathfrak{X}(M),\ \beta=0,\dots,r\}$ generates the $C^\infty(T^rM)$-module $\mathfrak{X}(T^rM)$.*

3.2. Higher order tangent lifts of 1-forms. Let $\omega \in \Omega^1(M)$. We define the *α-lift* of ω, denoted $\omega^{(\alpha)}$, by

$$\omega^{(\alpha)} = \varepsilon^r_M \circ \chi^{(r-\alpha)}_{T^*M} \circ T^r\omega.$$

When $\alpha = r$, $\omega^{(\alpha)}$ is called the *complete lift* of ω and denoted by $\omega^{(c)}$.

In local coordinates, if $\omega = \omega_i dx^i$, then

$$\omega^{(\alpha)} = (\omega_i)^{(\alpha-\beta)} dx^i_\beta.$$

Proposition 3.2.

(i) *For any $X \in \mathfrak{X}(M)$ and $\beta = 0, \dots, r$, we have*

$$\omega^{(\alpha)}(X^{(\beta)}) = [\omega(X)]^{(\alpha-\beta)}.$$

(ii) *For any $X \in \mathfrak{X}(M)$ and $\beta = 0, \dots, r$, we have*

$$(d\omega)^{(\alpha)} = d(\omega)^{(\alpha)} \quad \text{and} \quad \mathcal{L}_{X^{(\beta)}}\omega^{(\alpha)} = (\mathcal{L}_X\omega)^{(\alpha-\beta)}.$$

(iii) *The set $\{\omega^{(\alpha)} \mid \omega \in \Omega^1(M),\ \alpha = 0, \dots, r\}$ generates the $C^\infty(T^rM)$-module $\Omega^1(T^rM)$.*

The proofs of Propositions 3.1 and 3.2 can be found in [MO].

3.3. Canonical Nijenhuis tensors on higher order tangent bundles. Let M be a smooth manifold. Multiplication of tangent vectors by real numbers is a map $\mathfrak{m}_M : \mathbb{R} \times TM \to TM$. Applying the functor T^r, we obtain $T^r(\mathfrak{m}_M) : J^r_0(\mathbb{R}, \mathbb{R}) \times T^rTM \to T^rTM$. Then

$$\mathcal{T}^r(\mathfrak{m}_M) = \kappa^r_M \circ T^r(\mathfrak{m}_M) \circ (\mathrm{id}_{J^r_0(\mathbb{R},\mathbb{R})} \times (\kappa^r_M)^{-1}) : J^r_0(\mathbb{R}, \mathbb{R}) \times TT^rM \to TT^rM$$

and we define, for each $\alpha \in \{0, \dots, r\}$, the tensor field

$$S_\alpha = \mathcal{T}^r(\mathfrak{m}_M)(e_\alpha, \cdot) : TT^rM \to TT^rM,$$

where $(e_0, \dots, e_r)$ is the canonical basis of $J^r_0(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^{r+1}$.

Definition 3.1. The $(1,1)$-tensor field S_α is called the *canonical Nijenhuis tensor* on T^rM of degree α.

Proposition 3.3.

(i) *For any $X \in \mathfrak{X}(M)$ and $\beta \in \{0, \dots, r\}$, we have*

$$S_\alpha(X^{(\beta)}) = X^{(\alpha+\beta)}.$$

(ii) *Denote by S^*_α the dual map of S_α. Then for any $\omega \in \Omega^1(M)$ and $\beta \in \{0, \dots, r\}$, we have*

$$S^*_\alpha(\omega^{(\beta)}) = \omega^{(\beta-\alpha)}.$$

Proof. (i) Let $X \in \mathfrak{X}(M)$. We know that $X^{(\beta)} = \kappa^r_M \circ T^r(\mathfrak{m}_M)(e_\beta, T^r X)$, and it follows that

$$\begin{aligned} S_\alpha(X^{(\beta)}) &= \kappa^r_M \circ T^r(\mathfrak{m}_M)(e_\alpha, (\kappa^r_M)^{-1}) \circ \kappa^r_M \circ T^r(\mathfrak{m}_M)(e_\beta, T^r X) \\ &= \kappa^r_M \circ T^r(\mathfrak{m}_M)(e_\alpha, T^r(\mathfrak{m}_M)(e_\beta, T^r X)) \\ &= \kappa^r_M \circ T^r(\mathfrak{m}_M)(e_{\alpha+\beta}, T^r X) = X^{(\alpha+\beta)}. \end{aligned}$$

(ii) For any $X \in \mathfrak{X}(M)$ and $\gamma \in \{0, \ldots, r\}$, we have

$$\begin{aligned} S^*_\alpha(\omega^{(\beta)})(X^{(\gamma)}) &= \omega^{(\beta)}(S_\alpha(X^{(\gamma)})) = \omega^{(\beta)}(X^{(\gamma+\alpha)}) \\ &= (\omega(X))^{(\beta-\alpha-\gamma)} = \omega^{(\beta-\alpha)}(X^{(\gamma)}). \end{aligned}$$

Therefore, $S^*_\alpha(\omega^{(\beta)}) = \omega^{(\beta-\alpha)}$. ∎

Let (U, x^i) be a local coordinate system of M. We denote by (x^i, x^i_β) the local coordinate system of $T^r M$ over $T^r U$. The local expression of the tensor field S_α is

$$S_\alpha = dx^i_\beta \otimes \frac{\partial}{\partial x^i_{\alpha+\beta}}.$$

COROLLARY 3.1. *Denote by T_α the torsion of the $(1,1)$-tensor S_α. Then $T_\alpha = 0$.*

Proof. Let $X, Y \in \mathfrak{X}(M)$ and $\beta, \gamma \in \{0, \ldots, r\}$. We have

$$\begin{aligned} T_\alpha(X^{(\beta)}, Y^{(\gamma)}) &= [S_\alpha X^{(\beta)}, S_\alpha Y^{(\gamma)}] - S_\alpha([S_\alpha X^{(\beta)}, Y^{(\gamma)}]) \\ &\quad + S_\alpha([X^{(\beta)}, S_\alpha Y^{(\gamma)}]) - S_{2\alpha}([X^{(\beta)}, Y^{(\gamma)}]) \\ &= [X, Y]^{(\beta+\gamma+2\alpha)} - S_\alpha([X, Y]^{(\beta+\gamma+\alpha)}). \end{aligned}$$

As $T_\alpha(X^{(\beta)}, Y^{(\gamma)}) = 0$ for any $X, Y \in \mathfrak{X}(M)$ and $\beta, \gamma = 0, \ldots, r$, we deduce that $T_\alpha = 0$. ∎

From this corollary, we deduce that the pair $(T^r M, S_\alpha)$ is a Nijenhuis manifold, called the canonical Nijenhuis manifold on $T^r M$.

COROLLARY 3.2.

(i) *For any $\alpha, \beta \in \{0, \ldots, r\}$, we have*

$$S_\alpha \circ S_\beta = S_\beta \circ S_\alpha = S_{\alpha+\beta}.$$

(ii) *Let p_α be a natural number such that $\alpha \cdot p_\alpha > r$. Then*

$$\underbrace{S_\alpha \circ \cdots \circ S_\alpha}_{p_\alpha \text{ times}} = 0.$$

In particular, when $r = \alpha = 1$ we obtain the canonical $(1,1)$-tensor on TM and we have the famous formula

$$S_{\mathrm{can}} \circ S_{\mathrm{can}} = 0.$$

Proof. Let $X \in \mathfrak{X}(M)$ and $\gamma \in \{0, \dots, r\}$. We have

$$S_\alpha \circ S_\beta(X^{(\gamma)}) = S_\alpha(X^{(\beta+\gamma)}) = X^{(\alpha+\beta+\gamma)} = S_{\alpha+\beta}(X^{(\gamma)}).$$

Therefore $S_\alpha \circ S_\beta = S_{\alpha+\beta}$. ∎

REMARK 3.1. Let $S : TM \to TM$ be a $(1,1)$-tensor field. For each $\beta \in \{0, \dots, r\}$ we put

$$S^{(\beta)} = \kappa^r_M \circ \chi^{(\beta)}_{TM} \circ T^r S \circ (\kappa^r_M)^{-1}. \tag{3.2}$$

Then $S^{(\beta)}$ is a $(1,1)$-tensor field on $T^r M$; when $\beta = 0$, it is called the *complete lift* of S and denoted by $S^{(c)}$. We verify easily that for any $X \in \mathfrak{X}(M)$ and $\gamma \le r$,

$$S^{(\beta)}(X^{(\gamma)}) = (SX)^{(\beta+\gamma)}.$$

From this equality, it follows that

$$S_\alpha \circ S^{(\beta)} = S^{(\beta)} \circ S_\alpha = S^{(\alpha+\beta)}.$$

In particular,

$$[S^{(\beta)}, S_\alpha] = 0 \quad \text{and} \quad (S^{(c)} \circ S_\alpha)^{p_\alpha} = 0.$$

We show easily that if $T_S = 0$ then $T_{S^{(\beta)}} = 0$, so that $(T^r M, S^{(\beta)})$ is a Nijenhuis manifold.

4. Canonical Poisson–Nijenhuis manifolds

4.1. Higher order tangent lifts of Poisson manifolds. We recall in this subsection the notion of higher order tangent lifts of Poisson manifolds. For each natural number $q \ge 2$, we consider the natural transformations $\bigwedge^q : \bigoplus^q T^* \to \bigwedge^q T^*$ defined for any smooth manifold M by

$$\textstyle\bigwedge^q_M : \bigoplus^q T^* M \to \bigwedge^q T^* M, \qquad \xi_1 \oplus \cdots \oplus \xi_q \mapsto \xi_1 \wedge \cdots \wedge \xi_q.$$

The bundle map

$$T^r(\textstyle\bigwedge^q_M) \circ (\bigoplus^q \alpha^r_M) : \bigoplus^q T^* T^r M \to T^r(\bigwedge^q T^* M)$$

is a well-defined and skew-symmetric fibred morphism over $\mathrm{id}_{T^r M}$. Therefore, there is a unique bundle morphism

$$\alpha^{r,q}_M : \textstyle\bigwedge^q T^* T^r M \to T^r(\bigwedge^q T^* M)$$

over $\mathrm{id}_{T^r M}$ such that

$$\alpha^{r,q}_M \circ \textstyle\bigwedge^q_{T^r M} = T^r(\bigwedge^q_M) \circ (\bigoplus^q \alpha^r_M).$$

For $q = 1$, we put $\alpha^{r,1}_M = \alpha^r_M$ and the local expression for $\alpha^{r,q}_M$ is given in [KWN]. We denote by $\kappa^{r,q}_M$ the vector bundle morphism

$$\kappa^{r,q}_M : T^r(\textstyle\bigwedge^q TM) \to \bigwedge^q T T^r M$$

such that, for any $u\oplus v\in T^r(\bigwedge^q TM)\oplus\bigwedge^q(T^*T^rM)$,

$$\langle u,\alpha_M^{r,q}(v)\rangle'^q_{T^rM}=\langle\kappa_M^{r,q}(u),v\rangle^q_{T^rM},$$

where $\langle\cdot,\cdot\rangle^q_M:\bigwedge^q TM\times_M\bigwedge^q T^*M\to\mathbb{R}$ is the canonical pairing and $\langle\cdot,\cdot\rangle'^q_{T^rM}=\tau_r\circ T^r(\langle\cdot,\cdot\rangle^q_M):T^r(\bigwedge^q TM)\times_{T^rM}T^r(\bigwedge^q T^*M)\to\mathbb{R}$. So, we have the natural transformation (see [KWN])

$$\kappa^{r,q}:T^r\circ(\textstyle\bigwedge^q T)\to\bigwedge^q T\circ T^r.$$

For any manifold M of dimension m, we have locally

$$\kappa_M^{r,q}(x^i_\beta,\Pi^{i_1\cdots i_q}_\beta)=(x^i_\beta,\widetilde{\Pi}^{i_1,\beta_1\cdots i_q,\beta_q})$$

with

$$\widetilde{\Pi}^{i_1,\beta_1\cdots i_q,\beta_q}=\sum_{\gamma_1+\cdots+\gamma_q+\gamma=r}\delta^{r-\gamma_1}_{\beta_1}\cdots\delta^{r-\gamma_q}_{\beta_q}\Pi^{i_1\cdots i_q}_\gamma.$$

Let Π be a multivector field of degree q on M. We put

$$\Pi^{(c)}=\kappa_M^{r,q}\circ T^r(\Pi):T^rM\to\textstyle\bigwedge^q TT^rM.$$

Then $\Pi^{(c)}$ is a multivector field of degree q on T^rM. Let $(x^1,\ldots,x^m)$ be a local coordinate system of M such that

$$\Pi=\sum_{1\le i_1<\cdots<i_q\le m}\Pi^{i_1\cdots i_q}\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_q}}.$$

Then

$$\Pi^{(c)}=\sum_{\beta_1+\cdots+\beta_q+\beta=r}(\Pi^{i_1\cdots i_q})^{(\beta)}\frac{\partial}{\partial x^{i_1}_{r-\beta_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_q}_{r-\beta_q}}.$$

In the particular case where $q=2$ and $\Pi=\Pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$, we have

$$\Pi^{(c)}=(\Pi^{ij})^{(\beta+\gamma-r)}\frac{\partial}{\partial x^i_\beta}\wedge\frac{\partial}{\partial x^j_\gamma}.$$

PROPOSITION 4.1 (see [KWN]). *If Π is a simple multivector field of degree k (i.e. $\Pi=X_1\wedge\cdots\wedge X_k$ with $X_1,\ldots,X_k\in\mathfrak{X}(M)$), then*

$$\Pi^{(c)}=\sum_{\beta_1+\cdots+\beta_k=r}X_1^{(r-\beta_1)}\wedge\cdots\wedge X_k^{(r-\beta_k)}.\tag{4.1}$$

REMARK 4.1. For $r=1$, we have

$$\Pi^{(c)}=\sum_{i=1}^k X_1^{(v)}\wedge\cdots\wedge X_i^{(c)}\wedge\cdots\wedge X_k^{(v)},$$

where $X_j^{(v)}$ is the vertical lift of the vector field X_j from M to TM. Thus, we obtain the result of [GU].

By the formulas (3.1) and (4.1), we deduce that for any $\Phi \in \mathfrak{X}^p(M)$ and $\Psi \in \mathfrak{X}^q(M)$, we have

$$[\Phi^{(c)}, \Psi^{(c)}] = [\Phi, \Psi]^{(c)}.$$

So, if (M, Π) is a Poisson manifold then so is $(T^r M, \Pi^{(c)})$. This induced Poisson structure on $T^r M$ is called the *tangent lifting* of the Poisson structure of order r.

PROPOSITION 4.2 (see [KWN]). *Let (M, Π) be a Poisson manifold.*

(i) *If $\sharp_\Pi$ is the anchor map induced by Π, we have*

$$\sharp_{\Pi^{(c)}} = \kappa^r_M \circ T^r(\sharp_\Pi) \circ \alpha^r_M. \tag{4.2}$$

(ii) *For any $\omega \in \Omega^1(M)$ and $\beta \in \{0, \dots, r\}$, we have*

$$\sharp_{\Pi^{(c)}}(\omega^{(\beta)}) = [\sharp_\Pi(\omega)]^{(r-\beta)}.$$

(iii) *For any $\omega, \varpi \in \Omega^1(M)$ and $\alpha, \beta \in \{0, \dots, r\}$, we have*

$$[\omega^{(\alpha)}, \varpi^{(\beta)}]_{\Pi^{(c)}} = ([\omega, \varpi]_\Pi)^{(\alpha+\beta-r)}.$$

.

4.2. The main result. Let (M, Π) be a Poisson manifold. The pair $(T^r M, \Pi^{(c)})$ is also a Poisson manifold and its sharp map is given by (4.2).

LEMMA 4.1. *For each $\alpha \in \{0, \dots, r\}$, we have*

$$\sharp_{\Pi^{(c)}} \circ S^*_\alpha = S_\alpha \circ \sharp_{\Pi^{(c)}}.$$

Proof. For any $\omega \in \Omega^1(M)$ and $\beta = 0, \dots, r$, we have

$$\sharp_{\Pi^{(c)}} \circ S^*_\alpha(\omega^{(\beta)}) = \sharp_{\Pi^{(c)}}(\omega^{(\beta-\alpha)}) = [\sharp_\Pi(\omega)]^{(r+\alpha-\beta)}.$$

In the same way,

$$S_\alpha \circ \sharp_{\Pi^{(c)}}(\omega^{(\beta)}) = S_\alpha([\sharp_\Pi(\omega)]^{(r-\beta)}) = [\sharp_\Pi(\omega)]^{(r+\alpha-\beta)}.$$

It follows that, for any $\omega \in \Omega^1(M)$ and $\beta = 0, \dots, r$,

$$\sharp_{\Pi^{(c)}} \circ S^*_\alpha(\omega^{(\beta)}) = S_\alpha \circ \sharp_{\Pi^{(c)}}(\omega^{(\beta)}).$$

Therefore, $\sharp_{\Pi^{(c)}} \circ S^*_\alpha = S_\alpha \circ \sharp_{\Pi^{(c)}}$. ■

REMARK 4.2. From this lemma, it follows that the vector bundle morphism $\sharp_{\Pi^{(c)}} \circ S^*_\alpha$ is skew-symmetric. It defines a bivector field denoted by Π^α on $T^r M$, and for $\alpha = 0$, we have $\Pi^0 = \Pi^{(c)}$.

LEMMA 4.2.

(i) *For any $\omega \in \Omega^1(M)$ and $\beta \in \{0, \dots, r\}$, we have*

$$\sharp_{\Pi^\alpha}(\omega^{(\beta)}) = [\sharp_\Pi(\omega)]^{(r-\beta+\alpha)}.$$

(ii) *For any $\omega, \varpi \in \Omega^1(M)$ and $\beta, \gamma = 0, \dots, r$,*

$$[\omega^{(\beta)}, \varpi^{(\gamma)}]_{\Pi^\alpha} = [\omega, \varpi]_\Pi^{(\beta+\gamma-\alpha-r)}.$$

Proof. (i) By (3.2), we have

$$\sharp_{\Pi^{(\alpha)}}(\omega^{(\beta)}) = S_\alpha \circ \sharp_{\Pi^{(c)}}(\omega^{(\beta)}) = S_\alpha([\sharp_\Pi(\omega)]^{(r-\beta)}) = [\sharp_\Pi(\omega)]^{(r-\beta+\alpha)}.$$

(ii) By the equality

$$[\omega^{(\beta)}, \varpi^{(\gamma)}]_{\Pi^{(\alpha)}} = \mathcal{L}_{\sharp_{\Pi^{(\alpha)}}(\omega^{(\beta)})}\varpi^{(\gamma)} - \mathcal{L}_{\sharp_{\Pi^{(\alpha)}}(\varpi^{(\beta)})}\omega^{(\gamma)} - d(\Pi^{(\alpha)}(\omega^{(\beta)}, \varpi^{(\gamma)}))$$

the result follows from the first part of the lemma and Propositions 3.1, 3.2 and 4.2. ∎

Theorem 4.1. *Let (M, Π) be a Poisson manifold. Then for each $\alpha \in \{0, \ldots, r\}$, $(T^r M, \Pi^{(c)}, S_\alpha)$ is a Poisson–Nijenhuis manifold.*

Proof. Let $\omega, \varpi \in \Omega^1(M)$ and $\beta, \gamma \in \{0, \ldots, r\}$. We have

$$\begin{aligned}
\nabla_{S_\alpha \Pi^{(c)}}(\omega^{(\beta)}, \varpi^{(\gamma)}) &= [\omega^{(\beta)}, \varpi^{(\gamma)}]_{\Pi^\alpha} - [S_\alpha^* \omega^{(\beta)}, \varpi^{(\gamma)}]_{\Pi^{(c)}} \\
&\quad - [\omega^{(\beta)}, S_\alpha^* \varpi^{(\gamma)}]_{\Pi^{(c)}} + S_\alpha^*[\omega^{(\beta)}, \varpi^{(\gamma)}]_{\Pi^{(c)}} \\
&= [\omega, \varpi]_\Pi^{(\beta+\gamma-\alpha-r)} - [\omega^{(\beta-\alpha)}, \varpi^{(\gamma)}]_{\Pi^{(c)}} \\
&\quad - [\omega^{(\beta)}, \varpi^{(\gamma-\alpha)}]_{\Pi^{(c)}} + S_\alpha^*([\omega, \varpi]_\Pi^{(\gamma+\beta-r)}) \\
&= [\omega, \varpi]_\Pi^{(\beta+\gamma-\alpha-r)} - [\omega, \varpi]_\Pi^{(\gamma+\beta-\alpha-r)} \\
&\quad - [\omega, \varpi]_\Pi^{(\gamma+\beta-\alpha-r)} + S_\alpha^*([\omega, \varpi]_\Pi^{(\gamma+\beta-r)}) \\
&= [\omega, \varpi]_\Pi^{(\beta+\gamma-\alpha-r)} - [\omega, \varpi]_\Pi^{(\gamma+\beta-\alpha-r)}.
\end{aligned}$$

It follows that $\nabla_{S_\alpha \Pi^{(c)}} = 0$. The rest follows from Lemma 4.1. ∎

Remark 4.3. In [KO], the author has shown that, if (M, Π, S) is a Poisson–Nijenhuis manifold, then the 2-vector field defined by the vector bundle morphism $S \circ \sharp_\Pi$ is a Poisson bivector. It follows that, for $\alpha = 1, \ldots, r$, the bivector Π^α is a Poisson bivector. This Poisson structure on $T^r M$ is called the *α-lift* of the Poisson manifold (M, Π).

Let (U, x^i) be a local coordinate system of M such that locally,

$$\Pi = \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

Then

$$\Pi^\alpha = (\Pi^{ij})^{(\beta+\gamma-\alpha-r)} \frac{\partial}{\partial x_\beta^i} \wedge \frac{\partial}{\partial x_\gamma^j}.$$

In particular, for $r = \alpha = 1$, we have

$$\Pi^1 = \Pi^{ij} \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j}.$$

So, we obtain the result of [KW].

4.3. Some properties of the α-lift of Poisson manifolds. In this subsection, we fix $\alpha \in \{1, \dots, r\}$.

THEOREM 4.2. *Let (M, Π) be a Poisson manifold.*

(i) *We have*

$$\sharp_{\Pi^\alpha} = \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r(\sharp_\Pi) \circ \alpha^r_M.$$

(ii) *For any $f \in C^\infty(M)$ and $\beta \in \{0, \dots, r\}$, we have*

$$X_{f^{(\beta)}} = (X_f)^{(r-\beta+\alpha)}. \tag{4.3}$$

(iii) *For $f, g \in C^\infty(M)$ and $\beta, \gamma \in \{0, \dots, r\}$, we have*

$$\{f^{(\beta)}, g^{(\gamma)}\}_{\Pi^\alpha} = (\{f, g\}_\Pi)^{(\beta+\gamma-\alpha-r)},$$

where $\{\cdot,\cdot\}_\Pi$ is a Poisson bracket on $C^\infty(M)$.

(iv) *If $f : (M, \Pi_M) \to (N, \Pi_N)$ is a Poisson morphism, then so is $T^r f : (T^r M, \Pi^\alpha_M) \to (T^r N, \Pi^\alpha_N)$. In particular, if (G, Π) is a Poisson–Lie group, then $(T^r G, \Pi^\alpha)$ is a Poisson–Lie group.*

Proof. (i) Let $\omega \in \Omega^1(M)$ and $\beta = 0, \dots, r$. We know that

$$\sharp_{\Pi^\alpha}(\omega^{(\beta)}) = [\sharp_\Pi(\omega)]^{(r-\beta+\alpha)}.$$

We put $\kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r(\sharp_\Pi) \circ \alpha^r_M = (\sharp_\Pi)^{(\alpha)}$. Then

$$\begin{aligned}(\sharp_\Pi)^{(\alpha)}(\omega^{(\beta)}) &= \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r(\sharp_\Pi) \circ \chi^{(r-\beta)}_{T^*M} \circ T^r\omega \\ &= \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ \chi^{(r-\beta)}_{TM} \circ T^r(\sharp_\Pi(\omega)) \\ &= \kappa^r_M \circ \chi^{(r+\alpha-\beta)}_{TM} \circ T^r(\sharp_\Pi(\omega)) = (\sharp_\Pi(\omega))^{(r+\alpha-\beta)}.\end{aligned}$$

(ii) Let $f \in C^\infty(M)$. Then

$$X_{f^{(\beta)}} = \sharp_{\Pi^\alpha}(df^{(\beta)}) = [\sharp_\Pi(df)]^{(r+\alpha-\beta)} = (X_f)^{(r+\alpha-\beta)}.$$

(iii) Let $f, g \in C^\infty(M)$ and $\beta, \gamma = 0, \dots, r$. Then

$$\{f^{(\beta)}, g^{(\gamma)}\}_{\Pi^\alpha} = X_{f^{(\beta)}}(g^{(\gamma)}) = (X_f)^{(r+\alpha-\beta)}(g^{(\gamma)}) = (\{f, g\}_\Pi)^{(\gamma+\beta-\alpha-r)}.$$

(iv) We use the properties of the natural transformations of κ^r_M and α^r_M:

$$\begin{aligned}TT^r f \circ \sharp_{\Pi^\alpha_M} \circ T^*T^r f &= TT^r f \circ \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r(\sharp_{\Pi_M}) \circ \alpha^r_M \circ T^*T^r f \\ &= \kappa^r_N \circ \chi^{(\alpha)}_{TN} \circ T^r T f \circ T^r(\sharp_{\Pi_M}) \circ T^r T^* f \circ \alpha^r_N \\ &= \kappa^r_N \circ \chi^{(\alpha)}_{TN} \circ T^r(Tf \circ \sharp_{\Pi_M} \circ T^* f) \circ \alpha^r_N = \sharp_{\Pi^\alpha_N}.\end{aligned}$$

Thus $T^r f$ is a Poisson morphism. ∎

REMARK 4.4. (i) By (4.3), if f is a Casimir function for (M, Π), then for each $\beta \in \{0, \dots, r\}$, $f^{(\beta)}$ is a Casimir function for $(T^r M, \Pi^\alpha)$. In particular, for any $\beta < \alpha$, $f^{(\beta)}$ is a Casimir function.

(ii) If Π is a regular Poisson bivector of rank $2d$, then Π^α is regular of rank $2d(r-\alpha+1)$.

REMARK 4.5. For $\beta \in \{0,\dots,r\}$, we have

$$\sharp_{\Pi^\alpha} \circ S^*_\beta = \sharp_{\Pi^{(c)}} \circ S^*_\alpha \circ S^*_\beta = \sharp_{\Pi^{(c)}} \circ S^*_{\alpha+\beta} = S_{\alpha+\beta} \circ \sharp_{\Pi^{(c)}} = S_\beta \circ \sharp_{\Pi^\alpha}.$$

By the procedure of Subsection 4.2, we verify easily that $(T^rM, \Pi^\alpha, S_\beta)$ is a Poisson–Nijenhuis manifold. This structure is the same as the structure obtained from the canonical Nijenhuis tensor $S_{\alpha+\beta}$ on the Poisson manifold $(T^rM, \Pi^{(c)})$.

COROLLARY 4.1. *For any $\alpha, \beta \in \{0,\dots,r\}$, Π^α and Π^β are compatible, so*

$$[\Pi^\alpha, \Pi^\beta] = 0.$$

Proof. Apply [V2, Theorem 1.3] and Remark 4.5. ∎

5. Applications

5.1. Other prolongations of Lie algebroids. For any vector bundle (E, M, π), we define the β-prolongation of a section u, denoted $u^{(\beta)}$, by

$$u^{(\beta)} = \chi_E^{(\beta)} \circ T^r u, \qquad 0 \le \beta \le r,$$

where $\chi_E^{(\beta)} : T^rE \to T^rE$ is a smooth map defined in Subsection 3.1. For convenience, we put $u^{(\beta)} = 0$ for $\beta \notin \{0,\dots,r\}$.

We denote by (x^i, y^j) a local coordinate system of E; it induces local coordinate systems

$$\begin{array}{ll} (x^i, \pi_j) & \text{in } E^*, \\ (x^i, y^j, x^i_\beta, y^j_\beta) & \text{in } T^rE, \\ (x^i, \pi_j, x^i_\beta, \pi^\beta_j) & \text{in } T^rE^*, \\ (x^i, \widetilde{\pi}_j, x^i_\beta, \widetilde{\pi}^\beta_j) & \text{in } (T^rE)^*. \end{array}$$

We recall that there exists a natural bundle isomorphism

$$I^r_{E^*} : T^rE^* \to (T^rE)^*$$

such that locally,

$$I^r_{E^*}(x^i, \pi_j, x^i_\gamma, \pi^\gamma_j) = (x^i, \widetilde{\pi}_j, x^i_\gamma, \widetilde{\pi}^\gamma_j) \quad \text{with} \quad \begin{cases} \widetilde{\pi}_j = \pi^r_j, \\ \widetilde{\pi}^\gamma_j = \pi^{r-\gamma}_j. \end{cases}$$

With these notations, we deduce the following result:

THEOREM 5.1. *Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid and $\alpha \in \{0,\dots,r\}$. There is a unique Lie algebroid structure on the bundle $T^rE \to T^rM$ with anchor map*

$$\rho^{(\alpha)} = \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r\rho$$

such that for any $u, v \in \Gamma(E)$ *and* $\beta, \gamma = 0, \ldots, r$,

$$[u^{(\beta)}, v^{(\gamma)}] = [u, v]^{(\alpha+\beta+\gamma)}.$$

This structure is called the α*-lift of the Lie algebroid* E*.*

Proof. Since $(E, [\cdot,\cdot], \rho)$ is a Lie algebroid, it induces a linear Poisson bivector Π_{E^*} on E^*. So, the map $\sharp_{\Pi_{E^*}} : T^*E^* \to TE^*$ is a morphism of double vector bundles. By Theorem 4.2(1), $\sharp_{\Pi^{\alpha}_{E^*}}$ is a morphism of double vector bundles. Therefore, $(T^rE^*, \Pi^{\alpha}_{E^*})$ is a linear Poisson bivector and it follows that $(T^rE^*)^*$ is a Lie algebroid. We endow T^rE with the structure of Lie algebroid such that $I^r_E : T^rE \to (T^rE^*)^*$ is an isomorphism of Lie algebroids. The rest of the proof is similar to the proof of [KWN, Theorem 3]. ∎

Remark 5.1. Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid and u a smooth section of E. For $\beta \in \{0, 1, \ldots, r\}$, we have $\rho^{(\alpha)}(u^{(\beta)}) = [\rho(u)]^{(\alpha+\beta)}$.

Corollary 5.1. *Let* $(E, [\cdot,\cdot], \rho)$ *be a Lie algebroid. Then the vector bundle morphism* $\chi^{(\alpha)}_E : T^rE \to T^rE$ *is a morphism of Lie algebroids between the* α*-lift of the Lie algebroid denoted by* $(T^rE, [\cdot,\cdot], \rho^{(\alpha)})$ *and the tangent lift of order* r *of the Lie algebroid denoted by* $(T^rE, [\cdot,\cdot], \rho^{(r)})$ (see [KWN]).

Proof. We know that for any $u \in \Gamma(E)$ and $\beta = 0, \ldots, r$, we have $\chi^{(\alpha)}_E(u^{(\beta)}) = u^{(\alpha+\beta)}$. It follows that

$$\begin{aligned}\chi^{(\alpha)}_E[u^{(\beta)}, v^{(\gamma)}] &= \chi^{(\alpha)}_E([u, v]^{(\alpha+\beta+\gamma)}) = [u, v]^{(2\alpha+\beta+\gamma)}\\ &= [\chi^{(\alpha)}_E(u^{(\beta)}), \chi^{(\alpha)}_E(v^{(\gamma)})]\end{aligned}$$

for any $u, v \in \Gamma(E)$ and $\beta, \gamma = 0, \ldots, r$. We deduce our result from

$$\rho^{(r)} \circ \chi^{(\alpha)}_E = \kappa^r_M \circ T^r\rho \circ \chi^{(\alpha)}_E = \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r\rho.$$

Thus $\rho^{(r)} \circ \chi^{(\alpha)}_E = \rho^{(\alpha)}$. ∎

Corollary 5.2. *Let* (M, Π) *be a Poisson manifold, let* T^rT^*M *designate the* α*-lift of the Lie algebroid* $(T^*M, [\cdot,\cdot]_\Pi, \sharp_\Pi)$*, and let* T^*T^rM *be the Lie algebroid defined by the Poisson bivector* Π^α*. The canonical mapping* $\alpha^r_M : T^*T^rM \to T^rT^*M$ *is an isomorphism of Lie algebroids.*

Proof. This follows by a calculation in local coordinates. ∎

Example 5.1. We know that since (T^rM, S_α) is a Nijenhuis manifold, it induces a Lie algebroid structure on TT^rM such that the bracket is given for $X, Y \in \mathfrak{X}(T^rM)$ by

$$[X, Y]_{S_\alpha} = [S_\alpha X, Y] + [X, S_\alpha Y] - S_\alpha[X, Y].$$

We denote by $(T^rTM, [\cdot,\cdot]_\alpha)$ the α-lift of the canonical Lie algebroid on TM. The vector bundle isomorphism κ^r_M is an isomorphism of Lie algebroids between $(T^rTM, [\cdot,\cdot]_\alpha)$ and $(TT^rM, [\cdot,\cdot]_{S_\alpha})$.

EXAMPLE 5.2. Let $\mathfrak{g}$ be a Lie algebra; it is a Lie algebroid over a point. Let $\{e_1, \dots, e_m\}$ be a basis of $\mathfrak{g}$. For all $i, j \in \{1, \dots, m\}$, we have

$$[e_i, e_j] = c_{ij}^k e_k.$$

Here the c_{ij}^k are constant functions, so that $(c_{ij}^k)^{(\nu)} = 0$ for all $\nu \geq 1$. The α-lift of the Lie algebroid $\mathfrak{g}$ is such that for any $i, j \in \{1, \dots, m\}$ and $\beta, \gamma \in \{0, \dots, r\}$,

$$[e_i^\beta, e_j^\gamma] = c_{ij}^k e_k^{\alpha+\beta+\gamma}.$$

In particular, when $r = 1$, the vertical lift of the Lie algebra is such that

$$[\dot{e}_i, \dot{e}_j] = [\dot{e}_i, e_j] = [e_i, \dot{e}_j] = 0 \quad \text{and} \quad [e_i, e_j] = c_{ij}^k \dot{e}_k.$$

When $\alpha = 0$, we obtain the usual tangent lift of order r of Poisson manifolds and Lie algebroids.

REMARK 5.2. Let $(E, [\cdot,\cdot], \rho)$ be a Lie algebroid over M, and $J : E \to E$ a morphism of vector bundles over M. For $u, v \in \Gamma(E)$, we put

$$[u, v]_J = [Ju, v] + [u, Jv] - J[u, v],$$
$$T_J(u, v) = [Ju, Jv] - J([Ju, v] + [u, Jv] - J[u, v]).$$

We easily verify that if $T_J = 0$, then $(E, [\cdot,\cdot]_J)$ is a Lie algebroid over M with anchor map $\rho_J = \rho \circ J$. We thus obtain a J-deformation of the initial Lie algebroid $(E, [\cdot,\cdot], \rho)$.

Consider the canonical vector bundle morphism $J_\alpha = \chi_E^{(\alpha)}$. By Corollary 5.1, the α-prolongation of the Lie algebroid on $T^r E$ coincides with the J_α-deformation of the Lie algebroid $(T^r E, [\cdot,\cdot], \rho^{(r)})$.

5.2. Higher order tangent lifts of Poisson–Nijenhuis manifolds. Let $S : TM \to TM$ be a tensor. We put

$$(S^*)^{(c)} = \varepsilon_M^r \circ T^r(S^*) \circ \alpha_M^r,$$

where S^* designates the dual map of S.

LEMMA 5.1. *Let (M, S) be a Nijenhuis manifold. Then*

$$(S^{(c)})^* = (S^*)^{(c)}.$$

Proof. For any $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned}\langle X^{(\alpha)}, (S^{(c)})^*(\omega^{(\beta)})\rangle_{T^rM} &= \langle S^{(c)}(X^{(\alpha)}), \omega^{(\beta)}\rangle_{T^rM} = \langle (SX)^{(\alpha)}, \omega^{(\beta)}\rangle_{T^rM} \\ &= (\langle SX, \omega\rangle_M)^{(\beta-\alpha)} = (\langle X, S^*\omega\rangle_M)^{(\beta-\alpha)} \\ &= \langle X^{(\alpha)}, (S^*\omega)^{(\beta)}\rangle_{T^rM} = \langle X^{(\alpha)}, (S^*)^{(c)}(\omega^{(\beta)})\rangle_{T^rM}.\end{aligned}$$

Therefore $(S^{(c)})^*(\omega^{(\beta)}) = (S^*)^{(c)}(\omega^{(\beta)})$, thus $(S^{(c)})^* = (S^*)^{(c)}$. ∎

LEMMA 5.2. *Let (M, Π, S) be a Poisson–Nijenhuis manifold. Then*

$$\sharp_{\Pi^{(c)}} \circ (S^{(c)})^* = S^{(c)} \circ \sharp_{\Pi^{(c)}}.$$

Proof. We compute

$$\begin{aligned}\sharp_{\Pi^{(c)}} \circ (S^{(c)})^* &= \sharp_{\Pi^{(c)}} \circ (S^*)^{(c)} = \kappa^r_M \circ T^r(\sharp_\Pi) \circ \alpha^r_M \circ \varepsilon^r_M \circ T^r S^* \circ \alpha^r_M \\ &= \kappa^r_M \circ T^r(\sharp_\Pi \circ S^*) \circ \alpha^r_M = \kappa^r_M \circ T^r(S \circ \sharp_\Pi) \circ \alpha^r_M \\ &= S^{(c)} \circ \sharp_{\Pi^{(c)}}. \ \blacksquare\end{aligned}$$

Let (M, Π, S) be a Poisson–Nijenhuis manifold. We denote by Π_S the bivector defined by $S \circ \sharp_\Pi$. By Lemma 5.2, we deduce that

$$\sharp_{\Pi_S^{(c)}} = S^{(c)} \circ \sharp_{\Pi^{(c)}}.$$

Therefore, for any $\omega, \varpi \in \Omega^1(M)$ and $\alpha, \beta \in \{0, \ldots, r\}$, we have

$$[\omega^{(\alpha)}, \varpi^{(\beta)}]_{\Pi_S^{(c)}} = [\omega, \varpi]_{\Pi_S}^{(\alpha+\beta-r)}. \tag{5.1}$$

THEOREM 5.2. *Let (M, Π, S) be a Poisson–Nijenhuis manifold. For any $\omega, \varpi \in \Omega^1(M)$ and $\alpha, \beta = 0, \ldots, r$, we have*

$$\nabla_{\Pi^{(c)} S^{(c)}}(\omega^{(\alpha)}, \varpi^{(\beta)}) = (\nabla_{\Pi S}(\omega, \varpi))^{(\alpha+\beta-r)}.$$

In particular, $(T^r M, \Pi^{(c)}, S^{(c)})$ is a Poisson–Nijenhuis manifold.

Proof. This follows from Lemma 5.2, Proposition 4.2 and equation (5.1). ∎

COROLLARY 5.3. *Let (M, Π, S) be a Poisson–Nijenhuis manifold. Recall that for $\alpha \in \{0, \ldots, r\}$, $S^{(\alpha)} = \kappa^r_M \circ \chi^{(\alpha)}_{TM} \circ T^r S \circ (\kappa^r_M)^{-1}$.*

- (i) *For each $\alpha \in \{0, \ldots, r\}$, $(T^r M, \Pi^{(c)}, S^{(\alpha)})$ is a Poisson–Nijenhuis manifold.*
- (ii) *For each $\alpha, \beta \in \{0, \ldots, r\}$, $(T^r M, \Pi^{\alpha}, S^{(\beta)})$ is a Poisson–Nijenhuis manifold.*

Proof. This follows from the equalities $S_\alpha \circ S^{(c)} = S^{(\alpha)} = S^{(c)} \circ S_\alpha$. ∎

REMARK 5.3. Let (M, Π, S) be a Poisson–Nijenhuis manifold. For any $k \geq 2$, we put

$$S^{\langle k \rangle} = \underbrace{S \circ \cdots \circ S}_{k \text{ times}} \quad \text{and} \quad S^{\langle 1 \rangle} = S.$$

In the same way, $\Pi^{\langle k \rangle}$ is the Poisson bivector defined by the vector bundle morphism $S \circ \sharp_{\Pi^{\langle k-1 \rangle}}$ with $\Pi^{\langle 1 \rangle} = \Pi$. The sequence $(S^{\langle k \rangle}, \Pi^{\langle k \rangle})_{k \geq 2}$ is the hierarchy of the Poisson–Nijenhuis manifold (M, Π, S), so that for $k, p \geq 1$ we have

$$[\Pi^{\langle k \rangle}, \Pi^{\langle p \rangle}] = 0.$$

From the equalities

$$(S^{(c)})^{\langle k\rangle} \circ S_\alpha = S_\alpha \circ (S^{(c)})^{\langle k\rangle} = (S_\alpha \circ S)^{\langle k\rangle} = (S^{(\alpha)})^{\langle k\rangle} \quad (k \geq 1),$$

it follows that $(\Pi^{\langle k\rangle})^\alpha = (\Pi^\alpha)^{\langle k\rangle}$ where the sequence $(\Pi^\alpha)^{\langle k\rangle}$ is defined by $(S^{(\alpha)})^{\langle k\rangle}$.

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References

[CK] A. Cabras and I. Kolář, *Prolongation of projectable tangent valued forms*, Arch. Math. (Brno) 38 (2002), 243–257.

[CS] F. Cantrijn, M. Crampin, W. Sarlet and D. Saunders, *The canonical isomorphism between T^kT^* and T^*T^k*, C. R. Acad. Sci. Paris 309 (1989), 1509–1514.

[C] T. Courant, *Tangent Lie algebroids*, J. Phys. A 23 (1994), 4527–4536.

[GMP] J. Gancarzewicz, W. Mikulski and Z. Pogoda, *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. 135 (1994), 1–41.

[GU] J. Grabowski and P. Urbański, *Tangent lifts of Poisson and related structures*, J. Phys. A 28 (1995), 6743–6777.

[K] I. Kolář, *Functorial prolongations of Lie algebroids*, in: Differential Geometry and Its Applications, Matfyzpress, Praha, 2005, 305–314.

[KMS] I. Kolář, P. Michor and J. Slovák, *Natural Operations in Differential Geometry*, Springer, Berlin, 1993.

[KO] Y. Kosmann-Schwarzbach, *The Lie bialgebroid of a Poisson–Nijenhuis manifold*, Lett. Math. Phys. 38 (1996), 421–428.

[KM1] Y. Kosmann-Schwarzbach and F. Magri, *On the modular class of Poisson–Nijenhuis manifolds*, arXiv:math/0611202v1 [math. SG] (2006).

[KM2] Y. Kosmann-Schwarzbach and F. Magri, *Poisson–Nijenhuis structures*, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990), 35–81.

[KW] P. M. Kouotchop Wamba, *Canonical Poisson–Nijenhuis structures on the tangent bundles*, to appear.

[KWN] P. M. Kouotchop Wamba, A. Ntyam and J. Wouafo Kamga, *Tangent lift of higher order of multivector fields and applications*, J. Math. Sci. Adv. Appl. 15 (2012), 89–112.

[M] K. C. H. Mackenzie,*General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, Cambridge, 2005.

[MX] K. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 73 (1998), 415–452.

[MV] G. Mitric and I. Vaisman, *Poisson structures on tangent bundles*, Differential Geom. Appl. 18 (2003), 207–228.

[MO] A. Morimoto, *Lifting of some type of tensors fields and connections to tangent bundles of p^r-velocities*, Nagoya Math. J. 40 (1970), 13–31.

[V1] I. Vaisman, *A lecture on Poisson–Nijenhuis manifold structures*, The Erwin Schrödinger International Institute for Mathematical Physics, Wien, 1994.

[V2] I. Vaisman, *The Poisson–Nijenhuis manifolds revisited*, Rend. Semin. Mat. Univ. Politec. Torino 52 (1994), 377–394.

[V3] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progr. Math. 118, Birkhäuser, 1994.

[W] J. Wouafo Kamga, *Global prolongation of geometric objects to some jet spaces*, International Centre for Theoretical Physics, Trieste, 1997.

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