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Some subclasses of meromorphic and multivalent functions

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Abstract. The authors introduce two new subclasses $F_{p,k}(\lambda, A, B)$ and $G_{p,k}(\lambda, A, B)$ of meromorphically multivalent functions. Distortion bounds and convolution properties for $F_{p,k}(\lambda, A, B)$, $G_{p,k}(\lambda, A, B)$ and their subclasses with positive coefficients are obtained. Some inclusion relations for these function classes are also given.

1. Introduction and preliminaries. Throughout this paper, we assume that

$$N = \{1, 2, 3, \ldots\}, \quad p \in \mathbb{N}, \quad k \in \mathbb{N} \setminus \{1\}, \\ -1 \le B < 0, \quad B < A \le -B, \quad 0 \le \lambda \le 1. \tag{1.1}$$

For functions $f(z)$ and $g(z)$ analytic in the open unit disk $U = \{z : |z| < 1\}$, we say that $f(z)$ is *subordinate* to $g(z)$ in U and write $f(z) \prec g(z)$ $(z \in U)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| \le |z|$ and $f(z) = g(w(z))$ $(z \in U)$.

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N}), \tag{1.2}$$

which are analytic and p-valent in the punctured open unit disk $U_0 = U \setminus \{0\}$.

The following lemma will be required in our investigation.

Lemma 1.1. *Let $f(z) \in \Sigma_p$ defined by* (1.2) *satisfy*

$$\sum_{n=p}^{\infty} [n(1 - B) + p\lambda\delta_{n,p,k}(1 - A)]|a_n| \le p(A - B). \tag{1.3}$$

Then

$$-\frac{zf'(z)}{(1 - \lambda)z^{-p} + \lambda f_{p,k}(z)} \prec p\frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{1.4}$$

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where

$$\delta_{n,p,k} = \begin{cases} 1 & ((n+p)/k \in \mathbb{N}), \\ 0 & ((n+p)/k \notin \mathbb{N}). \end{cases} \tag{1.5}$$

for $n \geq p$ *and*

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} f(\varepsilon_k^j z), \qquad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right). \tag{1.6}$$

Proof. For $f(z) \in \Sigma_p$ defined by (1.2), the function $f_{p,k}(z)$ in (1.6) can be expressed as

$$f_{p,k}(z) = z^{-p} + \sum_{n=p}^{\infty} \delta_{n,p,k} a_n z^n \tag{1.7}$$

with

$$\delta_{n,p,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(n+p)} = \begin{cases} 1 & ((n+p)/k \in \mathbb{N}), \\ 0 & ((n+p)/k \notin \mathbb{N}). \end{cases}$$

In view of (1.1) and (1.5), we see that

$$pA\lambda\delta_{n,p,k} + nB \leq -pB\lambda\delta_{n,p,k} + pB \leq 0 \quad (n \geq p). \tag{1.8}$$

Let the inequality (1.3) be satisfied. Then from (1.7) and (1.8) we deduce that

$$\begin{aligned} \left| \frac{\frac{zf'(z)}{(1-\lambda)z^{-p}+\lambda f_{p,k}(z)} + p}{pA + \frac{Bzf'(z)}{(1-\lambda)z^{-p}+\lambda f_{p,k}(z)}} \right| &= \left| \frac{\sum_{n=p}^{\infty}(n+p\lambda\delta_{n,p,k})a_n z^{n+p}}{p(A-B) + \sum_{n=p}^{\infty}(pA\lambda\delta_{n,p,k}+nB)a_n z^{n+p}} \right| \\ &\leq \frac{\sum_{n=p}^{\infty}(n+p\lambda\delta_{n,p,k})|a_n|}{p(A-B) + \sum_{n=p}^{\infty}(pA\lambda\delta_{n,p,k}+nB)|a_n|} \\ &\leq 1 \quad (|z|=1). \end{aligned}$$

Hence, by the maximum modulus theorem, we arrive at (1.4). ∎

We now consider the following two subclasses of Σ_p.

Definition 1.2. A function $f(z) \in \Sigma_p$ defined by (1.2) is said to be in the *class* $F_{p,k}(\lambda, A, B)$ if it satisfies the coefficient inequality (1.3).

It follows from Lemma 1.1 that, if $f(z) \in F_{p,k}(\lambda, A, B)$, then the subordination relation (1.4) holds. In particular, we see that each function in the class $F_{p,k}(\lambda, A, B)$ with $\lambda = 1$ is meromorphically p-valent starlike with respect to k-symmetric points. A number of properties for analytic (and meromorphic) functions which are starlike with respect to symmetric points and related functions have been studied by several authors (see, e.g., [1, 2, 4–10]).

DEFINITION 1.3. A function $f(z) \in \Sigma_p$ defined by (1.2) is said to be in the *class* $G_{p,k}(\lambda, A, B)$ if

$$\sum_{n=p}^{\infty} n[n(1-B)+p\lambda\delta_{n,p,k}(1-A)]|a_n| \leq p^2(A-B). \tag{1.9}$$

For $f(z) \in \Sigma_p$ defined by (1.2), we have

$$2z^{-p}+\frac{zf'(z)}{p} = z^{-p}+\sum_{n=p}^{\infty}\frac{n}{p}a_n z^n,$$

which implies that

$$f(z) \in G_{p,k}(\lambda, A, B) \quad \text{if and only if} \quad 2z^{-p}+\frac{zf'(z)}{p} \in F_{p,k}(\lambda, A, B). \tag{1.10}$$

If we write

$$\begin{aligned} \alpha_n &= \alpha_{n,p,k}(\lambda, A, B) = \frac{n(1-B)+p\lambda\delta_{n,p,k}(1-A)}{p(A-B)}, \\ \beta_n &= \frac{n}{p}\alpha_n \qquad (n \geq p), \end{aligned} \tag{1.11}$$

then it is easy to verify that

$$\frac{\partial\beta_n}{\partial\lambda} = \frac{n}{p}\frac{\partial\alpha_n}{\partial\lambda} \geq 0, \quad \frac{\partial\beta_n}{\partial A} = \frac{n}{p}\frac{\partial\alpha_n}{\partial A} < 0 \quad \text{and} \quad \frac{\partial\beta_n}{\partial B} = \frac{n}{p}\frac{\partial\alpha_n}{\partial B} \geq 0.$$

Hence we have the following inclusion relations. If

$$0 \leq \lambda_0 \leq \lambda \leq 1, \quad -1 \leq B_0 \leq B < 0, \quad B < A \leq -B, \quad A \leq A_0 \leq -B_0,$$

then

$$\begin{aligned} G_{p,k}(\lambda, A, B) \subset F_{p,k}(\lambda, A, B) &\subseteq F_{p,k}(\lambda_0, A, B) \subseteq F_{p,k}(\lambda_0, A_0, B_0) \\ &\subseteq F_{p,k}(0,1,-1) \\ &= t\{f(z) \in \Sigma_p : -\operatorname{Re}\{z^{p+1}f'(z)\} > 0 \ (z \in U)\} \end{aligned}$$

and

$$G_{p,k}(\lambda, A, B) \subseteq G_{p,k}(\lambda_0, A_0, B_0) \subseteq G_{p,k}(0,1,-1).$$

Let

$$f_j(z) = z^{-p}+\sum_{n=p}^{\infty} a_{n,j}z^n \in \Sigma_p \qquad (j=1,2).$$

Then the *Hadamard product* (or *convolution*) of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p}+\sum_{n=p}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 * f_1)(z).$$

In the present paper, we obtain distortion bounds and convolution properties for the classes $F_{p,k}(\lambda, A, B)$ and $G_{p,k}(\lambda, A, B)$ and their subclasses with positive coefficients. Some inclusion relations for these function classes are also provided.

2. Distortion bounds

THEOREM 2.1. *Let* $2p/k \in \mathbb{N}$ *and* $\lambda \leq \frac{1-B}{p(1-A)}$.

(i) *If* $f(z) \in F_{p,k}(\lambda, A, B)$, *then for* $z \in U_0$,

$$|z|^{-p} - \frac{A-B}{1-B+\lambda(1-A)}|z|^p \leq |f(z)| \leq |z|^{-p} + \frac{A-B}{1-B+\lambda(1-A)}|z|^p. \tag{2.1}$$

(ii) *If* $f(z) \in G_{p,k}(\lambda, A, B)$, *then for* $z \in U_0$,

$$p\left(|z|^{-p-1} - \frac{A-B}{1-B+\lambda(1-A)}|z|^{p-1}\right) \leq |f'(z)| \leq p\left(|z|^{-p-1} + \frac{A-B}{1-B+\lambda(1-A)}|z|^{p-1}\right). \tag{2.2}$$

The bounds in (2.1) *and* (2.2) *are sharp.*

Proof. Let $2p/k \in \mathbb{N}$. For $n \geq p$ $(n \in \mathbb{N})$ and $(n+p)/k \in \mathbb{N}$, we have

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} \geq \frac{1-B+\lambda(1-A)}{A-B}. \tag{2.3}$$

For $n \geq p+1$ and $(n+p)/k \notin \mathbb{N}$, we have $\delta_{n,p,k} = \delta_{p+1,p,k} = 0$ and

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} \geq \frac{(p+1)(1-B)}{p(A-B)}. \tag{2.4}$$

From the assumptions of the theorem we obtain

$$\frac{(p+1)(1-B)}{p(A-B)} \geq \frac{1-B+\lambda(1-A)}{A-B}. \tag{2.5}$$

(i) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in F_{p,k}(\lambda, A, B)$, then it follows from (2.3) to (2.5) that

$$\frac{1-B+\lambda(1-A)}{A-B}\sum_{n=p}^{\infty}|a_n| \leq 1.$$

Hence we have

$$|f(z)| \leq |z|^{-p} + |z|^p \sum_{n=p}^{\infty}|a_n| \leq |z|^{-p} + \frac{A-B}{1-B+\lambda(1-A)}|z|^p$$

and

$$|f(z)| \geq |z|^{-p} - |z|^p \sum_{n=p}^{\infty} |a_n| \geq |z|^{-p} - \frac{A-B}{1-B+\lambda(1-A)}|z|^p > 0$$

for $z \in U_0$.

(ii) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B)$, then (2.3)–(2.5) yield

$$\frac{1-B+\lambda(1-A)}{p(A-B)} \sum_{n=p}^{\infty} n|a_n| \leq 1.$$

This leads to (2.2).

Furthermore, the bounds in (2.1) and (2.2) are best possible as can be seen for the function

$$f(z) = z^{-p} + \frac{A-B}{1-B+\lambda(1-A)} z^p \in G_{p,k}(\lambda, A, B) \subset F_{p,k}(\lambda, A, B). \tag{2.6}$$ ■

THEOREM 2.2. *Let* $2p/k \notin \mathbb{N}$.

(i) *If* $f(z) \in F_{p,k}(\lambda, A, B)$, *then for* $z \in U_0$,

$$|z|^{-p} - \frac{A-B}{1-B}|z|^p \leq |f(z)| \leq |z|^{-p} + \frac{A-B}{1-B}|z|^p. \tag{2.7}$$

(ii) *If* $f(z) \in G_{p,k}(\lambda, A, B)$, *then for* $z \in U_0$,

$$p\left(|z|^{-p-1} - \frac{A-B}{1-B}|z|^{p-1}\right) \leq |f'(z)| \leq p\left(|z|^{-p-1} + \frac{A-B}{1-B}|z|^{p-1}\right). \tag{2.8}$$

The bounds in (2.7) *and* (2.8) *are sharp.*

Proof. Let $2p/k \notin \mathbb{N}$. For $n \geq p$ and $(n+p)/k \notin \mathbb{N}$, we have $\delta_{n,p,k} = \delta_{p,p,k} = 0$ and so for $0 \leq \lambda \leq 1$,

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} \geq \frac{1-B}{A-B}. \tag{2.9}$$

For $n \geq p$ and $(n+p)/k \in N$, we have

$$\delta_{n,p,k} = 1, \quad n = k\left(\left[\frac{2p}{k}\right] + l\right) - p > p \quad (l \in N)$$

and

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} > \frac{1-B+\lambda(1-A)}{A-B} \geq \frac{1-B}{A-B}. \tag{2.10}$$

(i) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in F_{p,k}(\lambda, A, B)$, then it follows from (2.9) and (2.10) that

$$\frac{1-B}{A-B}\sum_{n=p}^{\infty}|a_n| \le 1,$$

which leads to (2.7).

(ii) If $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in G_{p,k}(\lambda, A, B)$, then (2.9) and (2.10) yield

$$\frac{1-B}{p(A-B)}\sum_{n=p}^{\infty} n|a_n| \le 1,$$

which gives (2.8).

Furthermore, the function $f(z)$ defined by

$$f(z) = z^{-p} + \frac{A-B}{1-B} z^p \in G_{p,k}(\lambda, A, B) \subset F_{p,k}(\lambda, A, B) \tag{2.11}$$

shows that the bounds in (2.7) and (2.8) are best possible. ∎

3. Inclusion relation between $F_{p,k}(\lambda, C, D)$ and $G_{p,k}(\lambda, A, B)$. We now generalize the above-mentioned inclusion relation $G_{p,k}(\lambda, A, B) \subset F_{p,k}(\lambda, A, B)$.

THEOREM 3.1. *If $-1 \le D \le B$, then*

$$G_{p,k}(\lambda, A, B) \subset F_{p,k}(\lambda, C(D), D),$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{1-B}.$$

The number $C(D)$ is the smallest possible for each D.

Proof. Since $B < A \le -B$ and $-1 \le D \le B < 0$, we see that

$$D < C(D) \le D - \frac{2B(1-D)}{1-B} \le -D.$$

Let $f(z) \in G_{p,k}(\lambda, A, B)$. In order to prove that $f(z) \in F_{p,k}(\lambda, C(D), D)$, we need only find the smallest C $(D < C \le -D)$ such that

$$\frac{n(1-D) + p\lambda\delta_{n,p,k}(1-C)}{p(C-D)} \le \frac{n[n(1-B) + p\lambda\delta_{n,p,k}(1-A)]}{p^2(A-B)} \tag{3.1}$$

for all $n \ge p$, that is, that

$$\frac{(1-D)(n + p\lambda\delta_{n,p,k})}{p(C-D)} - \lambda\delta_{n,p,k} \le \frac{n}{p}\left(\frac{(1-B)(n + p\lambda\delta_{n,p,k})}{p(A-B)} - \lambda\delta_{n,p,k}\right) \quad (n \ge p). \tag{3.2}$$

For $n \geq p$ and $(n+p)/k \in \mathbb{N}$, (3.2) is equivalent to

$$C \geq D + \frac{1-D}{\frac{\lambda(p-n)}{n+p\lambda} + \frac{n(1-B)}{p(A-B)}} = \varphi(\lambda, n) \quad \text{(say)}. \tag{3.3}$$

Since

$$\begin{aligned}\varphi(\lambda, n) &\geq D + \frac{p(1-D)(A-B)}{n(1-B)} \\ &\geq D + \frac{(1-D)(A-B)}{1-B} =: C(D) \quad (n = p, p+1, \ldots),\end{aligned}$$

by (3.3) we have $f(z) \in F_{p,k}(\lambda, C(D), D)$.

Furthermore, for $2p/k \in \mathbb{N}$ and $D < C_0 < C(D)$, we have

$$\begin{aligned}&\frac{1-D+\lambda(1-C_0)}{C_0 - D} \cdot \frac{A-B}{1-B+\lambda(1-A)} \\ &\qquad > \frac{1-D+\lambda(1-C(D))}{C(D)-D} \cdot \frac{A-B}{1-B+\lambda(1-A)} = 1,\end{aligned}$$

which implies that the function $f(z) \in G_{p,k}(\lambda, A, B)$ defined by (2.6) is not in the class $F_{p,k}(\lambda, C_0, D)$. Also, for $2p/k \notin \mathbb{N}$ and $D < C_0 < C(D)$, we have

$$\frac{1-D}{C_0-D} \cdot \frac{A-B}{1-B} > \frac{1-D}{C(D)-D} \cdot \frac{A-B}{1-B} = 1,$$

which implies that the function $f(z) \in G_{p,k}(\lambda, A, B)$ defined by (2.11) is not in the class $F_{p,k}(\lambda, C_0, D)$. The proof of Theorem 3.1 is thus complete. ■

4. Convolution properties. In this section, we assume that

$$-1 \leq B_j < 0 \quad \text{and} \quad B_j < A_j \leq -B_j \quad (j = 1, 2). \tag{4.1}$$

THEOREM 4.1. *Let $f_j(z) \in F_{p,k}(\lambda, A_j, B_j)$ $(j = 1, 2)$ with $2p/k \in \mathbb{N}$ and $-1 \leq B \leq \max\{B_1, B_2\}$.*

(i) *If $(1 - B_1)(1 - B_2) \geq p(1 - A_1)(1 - A_2)$ and $0 \leq \lambda \leq 1$, then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, A(B), B)$, where*

$$A(B) = B + \frac{(1-B)(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2) + \lambda(1-A_1)(1-A_2)}. \tag{4.2}$$

The number $A(B)$ is the smallest possible for each B.

(ii) *If $(1 - B_1)(1 - B_2) < p(1 - A_1)(1 - A_2)$ and*

$$0 \leq \lambda \leq \lambda_1 = \frac{(1-B_1)(1-B_2)}{p(1-A_1)(1-A_2)}, \tag{4.3}$$

then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, A(B), B)$ and the number $A(B)$ is the smallest possible for each B.

(iii) *If* $(1-B_1)(1-B_2) < p(1-A_1)(1-A_2)$ *and* $\lambda_1 < \lambda \leq 1$, *then* $(f_1 * f_2)(z) \in F_{p,k}(\lambda, \widetilde{A_1}(B), B)$, *where*

$$\widetilde{A_1}(B) = B + \frac{p(1-B)}{p+1} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}. \tag{4.4}$$

The number $\widetilde{A_1}(B)$ *is the smallest possible for each* B.

Proof. Suppose that $-1 \leq B \leq \max\{B_1, B_2\}$. It follows from (4.1) and (4.2) that

$$\frac{1-B}{A(B)-B} \geq \frac{1-B_j}{A_j - B_j} \geq -\frac{1-B_j}{2B_j} \geq -\frac{1-B}{2B} > 0,$$

$$\frac{1-B}{\widetilde{A_1}(B)-B} = \left(1+\frac{1}{p}\right) \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} > -\frac{1-B}{2B} > 0.$$

Hence $B < A(B) \leq -B$ and $B < \widetilde{A_1}(B) < -B$.

Let $f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j} z^n \in F_{p,k}(\lambda, A_j, B_j)$ $(j = 1, 2)$ and $2p/k \in \mathbb{N}$. Then

$$\sum_{n=p}^{\infty} \left\{ \prod_{j=1}^{2} \frac{n(1-B_j) + p\lambda\delta_{n,p,k}(1-A_j)}{p(A_j - B_j)} \right\} |a_{n,1} a_{n,2}| \leq \prod_{j=1}^{2} \left\{ \sum_{n=p}^{\infty} \frac{n(1-B_j) + p\lambda\delta_{n,p,k}(1-A_j)}{p(A_j - B_j)} |a_{n,j}| \right\} \leq 1. \tag{4.5}$$

Also, $(f_1 * f_2)(z) \in F_{p,k}(\lambda, A, B)$ if and only if

$$\sum_{n=p}^{\infty} \frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} |a_{n,1} a_{n,2}| \leq 1. \tag{4.6}$$

In order to prove Theorem 4.1, it follows from (4.5) and (4.6) that we need only find the smallest A such that

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} \leq \prod_{j=1}^{2} \frac{n(1-B_j) + p\lambda\delta_{n,p,k}(1-A_j)}{p(A_j - B_j)} \qquad (n \geq p). \tag{4.7}$$

For $n \geq p$ and $(n+p)/k \in \mathbb{N}$, (4.7) is equivalent to

$$A \geq B + \frac{1-B}{\frac{n+p\lambda}{p} \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} - \lambda \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j} + \frac{p\lambda(\lambda+1)}{n+p\lambda}} = \varphi_1(\lambda, n) \quad \text{(say)}. \tag{4.8}$$

It is easy to verify that $\varphi_1(\lambda, n)$ $(0 \leq \lambda \leq 1)$ is decreasing with respect to n and so, in view of $2p/k \in \mathbb{N}$,

$$\begin{aligned}(4.9)\quad \varphi_1(\lambda, n) \leq \varphi_1(\lambda, p) &= B + \frac{1-B}{(1+\lambda)\prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - \lambda \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \lambda} \\ &= B + \frac{(1-B)(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)+\lambda(1-A_1)(1-A_2)}.\end{aligned}$$

For $n \geq p+1$ and $(n+p)/k \notin \mathbb{N}$, (4.7) simplifies to

$$A \geq B + \frac{1-B}{\frac{n}{p}\prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j}} = \varphi_1(0, n) \tag{4.10}$$

and we have

$$\varphi_1(0, n) \leq \varphi_1(0, p+1) = B + \frac{p(1-B)}{p+1}\prod_{j=1}^{2} \frac{A_j-B_j}{1-B_j}. \tag{4.11}$$

Now

$$\begin{aligned}(4.12)\quad &\varphi_1(\lambda, p) - \varphi_1(0, p+1) \\ &= \frac{(1-B)(A_1-B_1)(A_2-B_2)[(1-B_1)(1-B_2) - p\lambda(1-A_1)(1-A_2)]}{(p+1)(1-B_1)(1-B_2)[(1-B_1)(1-B_2)+\lambda(1-A_1)(1-A_2)]}.\end{aligned}$$

Therefore, if p, λ, A_j and B_j $(j = 1, 2)$ satisfy (i) or (ii), then from (4.7) to (4.12) we conclude that

$$\varphi_1(0, p+1) \leq \varphi_1(\lambda, p) = A(B), \quad (f_1 * f_2)(z) \in F_{p,k}(\lambda, A(B), B),$$

and the number $A(B)$ is sharp for the functions

$$f_j(z) = z^{-p} + \frac{A_j - B_j}{1 - B_j + \lambda(1-A_j)} z^p \in F_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2). \tag{4.13}$$

Also, if p, λ, A_j and B_j $(j = 1, 2)$ satisfy (iii), then

$$\varphi_1(\lambda, p) < \varphi_1(0, p+1) = \widetilde{A_1}(B), \quad (f_1 * f_2)(z) \in F_{p,k}(\lambda, \widetilde{A_1}(B), B),$$

and the number $\widetilde{A_1}(B)$ is sharp for the functions

$$f_j(z) = z^{-p} + \frac{p(A_j - B_j)}{(p+1)(1-B_j)} z^{p+1} \in F_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2). \ \blacksquare \tag{4.14}$$

COROLLARY 4.2. *Let* $f_1(z) \in F_{p,k}(\lambda, A_1, B_1)$, $f_2(z) \in G_{p,k}(\lambda, A_2, B_2)$, $2p/k \in \mathbb{N}$, $-1 \leq B \leq \max\{B_1, B_2\}$, *and let* $A(B), \widetilde{A_1}(B), \lambda_1$ *be as in Theorem* 4.1.

- (i) *If* $(1-B_1)(1-B_2) \geq p(1-A_1)(1-A_2)$ *and* $0 \leq \lambda \leq 1$, *then* $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ *and the number* $A(B)$ *is the smallest possible for each* B.

(ii) *If* $(1-B_1)(1-B_2) < p(1-A_1)(1-A_2)$ *and* $0 \le \lambda \le \lambda_1$, *then* $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ *and the number* $A(B)$ *is the smallest possible for each* B.

(iii) *If* $(1-B_1)(1-B_2) < p(1-A_1)(1-A_2)$ *and* $\lambda_1 < \lambda \le 1$, *then* $(f_1 * f_2)(z) \in G_{p,k}(\lambda, \widetilde{A_1}(B), B)$ *and the number* $\widetilde{A_1}(B)$ *is the smallest possible for each* B.

Proof. Since

$$f_1(z) \in F_{p,k}(\lambda, A_1, B_1), \qquad 2z^{-p} + \frac{zf_2'(z)}{p} \in F_{p,k}(\lambda, A_2, B_2)$$

(see (1.10)), and

$$f_1(z) * \left(2z^{-p} + \frac{zf_2'(z)}{p}\right) = 2z^{-p} + \frac{z(f_1 * f_2)'(z)}{p} \qquad (z \in U_0),$$

an application of Theorem 4.1 yields Corollary 4.2. ∎

THEOREM 4.3. *Let* $f_j(z) \in F_{p,k}(\lambda, A_j, B_j)$ $(j = 1, 2)$ *with* $2p/k \notin \mathbb{N}$ *and* $-1 \le B \le \max\{B_1, B_2\}$. *Then* $(f_1 * f_2)(z) \in F_{p,k}(\lambda, \widetilde{A}(B), B)$, *where*

$$\widetilde{A}(B) = B + \frac{(1-B)(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}, \tag{4.15}$$

and the number $\widetilde{A}(B)$ *is the smallest possible for each* B.

Proof. It is easy to see that $B < \widetilde{A}(B) \le -B$. Proceeding as in the proof of Theorem 4.1, we have (4.5)–(4.8) and (4.10). Noting that $2p/k \notin \mathbb{N}$, we find that

$$\begin{aligned} \varphi_1(\lambda, n) &\le \varphi_1\left(\lambda, k\left(\left[\frac{2p}{k}\right] + 1\right) - p\right) \\ &= B + \frac{1-B}{(n_{p,k} + \lambda - 1)\prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} - \lambda \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j} + \frac{\lambda(\lambda+1)}{n_{p,k}+\lambda-1}} \\ &\qquad (n \ge p, (n+p)/k \in N), \end{aligned}$$

where $n_{p,k} = (k/p)([2p/k] + 1) > 2$, and

$$\varphi_1(0, n) \le \varphi_1(0, p) = B + \frac{1-B}{\prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j}} \qquad (n \ge p, (n+p)/k \notin \mathbb{N}).$$

Since

$$(n_{p,k} + \lambda - 1)\left[(n_{p,k} + \lambda - 1)\prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} - \lambda \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j} - \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j}\right] + \lambda(\lambda + 1)$$

$$= (n_{p,k} + \lambda - 1)\left[(n_{p,k} - 2)\prod_{j=1}^{2}\frac{1-B_j}{A_j - B_j} - \lambda + \lambda\prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j}\right] + \lambda(\lambda+1)$$
$$\geq (n_{p,k} + \lambda - 1)(n_{p,k} - 2 - \lambda) + \lambda(\lambda+1)$$
$$= (n_{p,k} - 1)(n_{p,k} - 2) > 0 \quad (0 \leq \lambda \leq 1),$$

it follows that

$$\varphi_1(0,p) > \varphi_1(\lambda, k([2p/k]+1) - p).$$

Hence, if we take $A = \varphi_1(0,p) = \widetilde{A}(B)$, then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, \widetilde{A}(B), B)$ and the number $\widetilde{A}(B)$ is best possible for the functions

$$f_j(z) = z^{-p} + \frac{A_j - B_j}{1 - B_j} z^p \in F_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2). \ \blacksquare$$

Corollary 4.4. *Let $f_1(z) \in F_{p,k}(\lambda, A_1, B_1)$, $f_2(z) \in G_{p,k}(\lambda, A_2, B_2)$, $2p/k \notin \mathbb{N}$ and $-1 \leq B \leq \max\{B_1, B_2\}$. Then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, \widetilde{A}(B), B)$, where $\widetilde{A}(B)$ is as in Theorem* 4.3*, and it is the smallest possible for each B.*

Theorem 4.5. *Let $f_1(z) \in F_{p,k}(\lambda, A_1, B_1)$, $f_2(z) \in G_{p,k}(\lambda, A_2, B_2)$, $2p/k \in \mathbb{N}$ and $-1 \leq B \leq \max\{B_1, B_2\}$.*

(i) *If $(2p+1)(1-B_1)(1-B_2) \geq p^2(1-A_1)(1-A_2)$ and $0 \leq \lambda \leq 1$, then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, A(B), B)$, where $A(B)$ is as in Theorem* 4.1*, and it is the smallest possible for each B.*

(ii) *If $(2p+1)(1-B_1)(1-B_2) < p^2(1-A_1)(1-A_2)$ and*

$$0 \leq \lambda \leq \lambda_2 = \frac{(2p+1)(1-B_1)(1-B_2)}{p^2(1-A_1)(1-A_2)}, \tag{4.16}$$

then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, A(B), B)$ and the number $A(B)$ is the smallest possible for each B.

(iii) *If $(2p+1)(1-B_1)(1-B_2) < p^2(1-A_1)(1-A_2)$ and $\lambda_2 < \lambda \leq 1$, then $(f_1 * f_2)(z) \in F_{p,k}(\lambda, \widetilde{A_2}(B), B)$, where*

$$\widetilde{A_2}(B) = B + \frac{p^2(1-B)}{(p+1)^2}\prod_{j=1}^{2}\frac{A_j - B_j}{1 - B_j}, \tag{4.17}$$

and the number $\widetilde{A_2}(B)$ is the smallest possible for each B.

Proof. It is easy to see that $B < \widetilde{A_2}(B) < -B$. In order to prove Theorem 4.5, we need only find the smallest A such that

$$\frac{n(1-B) + p\lambda\delta_{n,p,k}(1-A)}{p(A-B)} \leq \frac{n}{p}\prod_{j=1}^{2}\frac{n(1-B_j) + p\lambda\delta_{n,p,k}(1-A_j)}{p(A_j - B_j)} \quad (n \geq p) \tag{4.18}$$

For $n \geq p$ and $(n+p)/k \in \mathbb{N}$, (4.18) is equivalent to

$$A \geq B + \frac{1-B}{\frac{n(n+p\lambda)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{n\lambda}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{\lambda(p+n\lambda)}{n+p\lambda}} = \varphi_2(\lambda, n) \quad \text{(say)}. \tag{4.19}$$

Defining the function $\psi(\lambda, x)$ by

$$\psi(\lambda, x) = \frac{x(x+p\lambda)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{\lambda x}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{\lambda(p+\lambda x)}{x+p\lambda} \qquad (x \geq p,\ 0 \leq \lambda \leq 1),$$

we obtain

$$\begin{aligned}\frac{\partial\psi(\lambda,x)}{\partial x} &= \frac{2x+p\lambda}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{\lambda}{p}\left(\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j} + 1\right) \\ &\quad - \frac{\lambda p(1-\lambda^2)}{(x+p\lambda)^2} \\ &\geq \frac{2}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{\lambda}{p} - \frac{\lambda(1-\lambda)}{p(1+\lambda)} \\ &\geq \frac{2-\lambda}{p} - \frac{\lambda(1-\lambda)}{p(1+\lambda)} > 0 \qquad (x \geq p,\ 0 \leq \lambda \leq 1),\end{aligned}$$

which implies that $\varphi_2(\lambda, n)$ defined by (4.19) is decreasing with respect to n $(n \geq p)$. Hence, in view of $2p/k \in \mathbb{N}$, we have

$$\varphi_2(\lambda, n) \leq \varphi_2(\lambda, p) = B + \frac{1-B}{\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \lambda\prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j}} = A(B).$$

For $n \geq p+1$ and $(n+p)/k \notin \mathbb{N}$, (4.18) becomes

$$A \geq B + \frac{1-B}{\left(\frac{n}{p}\right)^2\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \varphi_2(0, n)$$

and, in view of $2p/k \in \mathbb{N}$, we obtain

$$\varphi_2(0, n) \leq \varphi_2(0, p+1) = B + \frac{1-B}{\left(1+\frac{1}{p}\right)^2\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \widetilde{A}_2(B).$$

Now,

$$\begin{aligned}&\left(\left(1+\frac{1}{p}\right)^2 - 1\right)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \lambda\prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j} \\ &\qquad = \frac{(2p+1)(1-B_1)(1-B_2) - p^2\lambda(1-A_1)(1-A_2)}{p^2(A_1-B_1)(A_2-B_2)}.\end{aligned}$$

The remaining part of the proof is much akin to that of Theorem 4.1.

Furthermore, the number $A(B)$ is sharp for the functions

$$f_1(z) = z^{-p} + \frac{A_1 - B_1}{1 - B_1 + \lambda(1 - A_1)} z^p \in F_{p,k}(\lambda, A_1, B_1),$$

$$f_2(z) = z^{-p} + \frac{A_2 - B_2}{1 - B_2 + \lambda(1 - A_2)} z^p \in G_{p,k}(\lambda, A_2, B_2),$$

and the number $\widetilde{A_2}(B)$ is sharp for the functions

$$f_1(z) = z^{-p} + \frac{p(A_1 - B_1)}{(p+1)(1 - B_1)} z^{p+1} \in F_{p,k}(\lambda, A_1, B_1),$$

$$f_2(z) = z^{-p} + \frac{p^2(A_2 - B_2)}{(p+1)^2(1 - B_2)} z^{p+1} \in G_{p,k}(\lambda, A_2, B_2). \blacksquare$$

Corollary 4.6. *Let $f_j(z) \in G_{p,k}(\lambda, A_j, B_j)$ $(j = 1, 2)$ with $2p/k \in \mathbb{N}$ and $-1 \le B \le \max\{B_1, B_2\}$, and let $A(B), \widetilde{A_2}(B), \lambda_2$ be as in Theorem 4.5.*

(i) *If $(2p+1)(1 - B_1)(1 - B_2) \ge p^2(1 - A_1)(1 - A_2)$ and $0 \le \lambda \le 1$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ and $A(B)$ cannot be decreased.*
(ii) *If $(2p+1)(1 - B_1)(1 - B_2) < p^2(1 - A_1)(1 - A_2)$ and $0 \le \lambda \le \lambda_2$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, A(B), B)$ and $A(B)$ cannot be decreased.*
(iii) *If $(2p+1)(1-B_1)(1-B_2) < p^2(1-A_1)(1-A_2)$ and $\lambda_2 < \lambda \le 1$, then $(f_1 * f_2)(z) \in G_{p,k}(\lambda, \widetilde{A_2}(B), B)$ and $\widetilde{A_2}(B)$ cannot be decreased.*

Theorem 4.7. *Let $f(z) \in F_{p,k}(\lambda, A, B)$. Then*

$$(f * h_\sigma)(z) \neq 0 \quad (z \in U_0,\ \sigma \in \mathbb{C},\ |\sigma| = 1), \tag{4.20}$$

where

$$h_\sigma(z) = z^{-p} + \frac{1 + B\sigma}{p\sigma(A - B)} \left(\frac{pz^p}{1 - z} + \frac{z^{p+1}}{(1 - z)^2} \right) + \frac{\lambda(1 + A\sigma)}{\sigma(A - B)} g(z)$$

and

$$g(z) = \begin{cases} \dfrac{z^p}{1 - z^k} & (2p/k \in \mathbb{N}), \\ \dfrac{z^{k([2p/k]+1)-p}}{1 - z^k} & (2p/k \notin \mathbb{N}). \end{cases}$$

Proof. For $f(z) \in F_{p,k}(\lambda, A, B)$, from Lemma 1.1 we have (1.4), which is equivalent to

$$-\frac{zf'(z)}{(1 - \lambda)z^{-p} + \lambda f_{p,k}(z)} \neq p\frac{1 + A\sigma}{1 + B\sigma} \quad (z \in U,\ \sigma \in \mathbb{C},\ |\sigma| = 1,\ 1 + B\sigma \neq 0),$$

or to

$$p(1 + A\sigma)[(1 - \lambda)z^{-p} + \lambda f_{p,k}(z)] + (1 + B\sigma)zf'(z) \neq 0 \quad (z \in U_0,\ \sigma \in \mathbb{C},\ |\sigma| = 1). \tag{4.21}$$

Note that

$$(4.22)\qquad -\frac{zf'(z)}{p} = f(z) * \left(z^{-p} - \frac{1}{p}\sum_{n=p}^{\infty} nz^n\right) = f(z) * \left(z^{-p} - \frac{z^p}{1-z} - \frac{z^{p+1}}{p(1-z)^2}\right).$$

If we set

$$(4.23)\qquad f_{p,k}(z) = f(z) * (z^{-p} + g(z)),$$

then for $2p/k \in \mathbb{N}$,

$$(4.24)\qquad g(z) = \sum_{n=p}^{\infty} \delta_{n,p,k} z^n = \sum_{l=0}^{\infty} z^{p+lk} = \frac{z^p}{1-z^k},$$

and for $2p/k \notin \mathbb{N}$,

$$(4.25)\qquad g(z) = \sum_{l=1}^{\infty} z^{k([2p/k]+l)-p} = \frac{z^{k([2p/k]+1)-p}}{1-z^k}.$$

Now, making use of (4.21)–(4.25), we arrive at

$$f(z) * \left\{p(1+A\sigma)[(1-\lambda)z^{-p} + \lambda(z^{-p} + g(z))] + (1+B\sigma)\left(-pz^{-p} + \frac{pz^p}{1-z} + \frac{z^{p+1}}{(1-z)^2}\right)\right\} \neq 0$$

for $z \in U_0$, $\sigma \in \mathbb{C}$ and $|\sigma| = 1$. This gives the desired result (4.20). ∎

Corollary 4.8. *Let $f(z) \in G_{p,k}(\lambda, A, B)$. Then*

$$(4.26)\qquad f(z) * \left(2z^{-p} + \frac{zh'_\sigma(z)}{p}\right) \neq 0 \qquad (z \in U_0,\ \sigma \in \mathbb{C},\ |\sigma| = 1),$$

where $h_\sigma(z)$ is as in Theorem 4.7.

Proof. Since $f(z) \in G_{p,k}(\lambda, A, B)$ if and only if

$$2z^{-p} + \frac{zf'(z)}{p} \in F_{p,k}(\lambda, A, B),$$

it follows from Theorem 4.7 that

$$f(z) * \left(2z^{-p} + \frac{zh'_\sigma(z)}{p}\right) = \left(2z^{-p} + \frac{zf'(z)}{p}\right) * h_\sigma(z) \neq 0 \qquad (z \in U_0,\ \sigma \in \mathbb{C},\ |\sigma| = 1).$$ ∎

5. Functions with positive coefficients. Let $\varSigma_p^*$ denote the subclass of $\varSigma_p$ consisting of all functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{5.1}$$

Further let

$$\begin{aligned} F_{p,k}^*(\lambda, A, B) &= \varSigma_p^* \cap F_{p,k}(\lambda, A, B), \\ G_{p,k}^*(\lambda, A, B) &= \varSigma_p^* \cap G_{p,k}(\lambda, A, B). \end{aligned} \tag{5.2}$$

Lemma 5.1. *A function $f(z) \in \varSigma_p^*$ is in the class $F_{p,k}^*(\lambda, A, B)$ if and only if it satisfies the subordination relation* (1.4).

Proof. If $f(z) \in F_{p,k}^*(\lambda, A, B)$, then Lemma 1.1 implies that (1.4) holds true. Conversely, suppose that $f(z) \in \varSigma_p^*$ defined by (5.1) satisfies (1.4). Then, in view of $\operatorname{Re} w \leq |w|$ $(w \in \mathbb{C})$, we easily see that

$$\operatorname{Re}\left\{\frac{\sum_{n=p}^{\infty}(n + p\lambda\delta_{n,p,k})a_n z^{n+p}}{p(A-B) + \sum_{n=p}^{\infty}(pA\lambda\delta_{n,p,k} + nB)a_n z^{n+p}}\right\} < 1 \quad (z \in U). \tag{5.3}$$

By letting $z = \operatorname{Re} z \to 1$, (5.3) leads to

$$\sum_{n=p}^{\infty}[n(1-B) + p\lambda\delta_{n,p,k}(1-A)]a_n \leq p(A-B).$$

Hence $f(z) \in F_{p,k}^*(\lambda, A, B)$. ■

By using Lemma 5.1, we observe that

$$F_{p,k}^*(0, A, B) = \left\{f(z) \in \varSigma_p^* : -z^{p+1}f'(z) \prec p\frac{1+Az}{1+Bz} \ (z \in U)\right\}$$

coincides with the class $H^*(p; -A, -B)$ introduced and studied by Mogra [3].

It is worth noting that our Theorems 2.1, 2.2, 3.1, 4.1, 4.3, 4.5 and Corollaries 4.2, 4.4, 4.6 are still true if the class $F_{p,k}(\lambda, A, B)$ is replaced by $F_{p,k}^*(\lambda, A, B)$ and the class $G_{p,k}(\lambda, A, B)$ is replaced by $G_{p,k}^*(\lambda, A, B)$. Moreover, by using Lemma 5.1, (the proof of) Theorem 4.7 and Corollary 4.8, we get the following result.

Theorem 5.2. *Let $f(z) \in \varSigma_p^*$ and $h_\sigma(z)$ be as in Theorem* 4.7. *Then:*

(i) *$f(z) \in F_{p,k}^*(\lambda, A, B)$ if and only if* (4.20) *holds.*
(ii) *$f(z) \in G_{p,k}^*(\lambda, A, B)$ if and only if* (4.26) *holds.*

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