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On the extreme points of subordination families

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Abstract. We investigate extreme points of some classes of analytic functions defined by subordination and classes of functions with varying argument of coefficients. By using extreme point theory we obtain coefficient estimates and distortion theorems in these classes of functions. Some integral mean inequalities are also pointed out.

1. Introduction. Let $\widetilde{\mathcal{A}}$ denote the class of functions which are *analytic* in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We consider the usual topology on $\widetilde{\mathcal{A}}$ (see [9]) defined by a metric in which a sequence $\{f_n\}$ in $\widetilde{\mathcal{A}}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathcal{U}$. It follows from the theorems of Weierstrass and Montel that this topological space is complete.

Let $\mathcal{F}$ be a subclass of $\widetilde{\mathcal{A}}$. A function $f \in \mathcal{F}$ is called an *extreme point* of $\mathcal{F}$ if the condition

$$f = \gamma g + (1-\gamma)h \quad (g, h \in \mathcal{F},\ 0 < \gamma < 1)$$

implies $g = h$. We write $E\mathcal{F}$ for the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is *locally uniformly bounded* if for each r, $0 < r < 1$, there is a real constant $M = M(r)$ such that

$$|f(z)| \le M \quad (f \in \mathcal{F},\ |z| \le r).$$

We say that a class $\mathcal{F}$ is *convex* if

$$\gamma f + (1-\gamma)g \in \mathcal{F} \quad (f, g \in \mathcal{F},\ 0 \le \gamma \le 1).$$

Moreover, we define the *closed convex hull* of $\mathcal{F}$ as the intersection of all closed convex subsets of $\widetilde{\mathcal{A}}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $H\mathcal{F}$.

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If $J : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ is a linear homeomorphism which maps $\mathcal{F}$ onto $J(\mathcal{F})$, then it is easy to verify that

(1)
$$HJ(\mathcal{F}) = J(H\mathcal{F}).$$

Likewise, if the class

$$\mathcal{F} = \{f_n \in \widetilde{\mathcal{A}} : n \in \mathbb{N} = \{1, 2, \ldots, \}\}$$

is locally uniformly bounded, then

(2)
$$H\mathcal{F} = \Big\{\sum_{n=1}^{\infty} \gamma_n f_n : \sum_{n=1}^{\infty} \gamma_n = 1,\ \gamma_n \geq 0\,(n \in \mathbb{N}\Big\}.$$

A functional $\mathcal{J} : \widetilde{\mathcal{A}} \to \mathbb{R}$ is called *convex* on a convex class $\mathcal{F} \subset \widetilde{\mathcal{A}}$ if

$$\mathcal{J}(\gamma f + (1-\gamma)g) \leq \gamma\mathcal{J}(f) + (1-\gamma)\mathcal{J}(g) \quad (f, g \in \mathcal{F},\ 0 \leq \gamma \leq 1).$$

For each fixed $m, n \in \mathbb{N}$, $z \in \mathcal{U}$ the following real-valued functionals are continuous and convex on $\widetilde{\mathcal{A}}$:

(3)
$$\mathcal{J}(f) = |a_n|, \quad \mathcal{J}(f) = |f(z)|, \quad \mathcal{J}(f) = |f^{(m)}(z)| \quad (f \in \widetilde{\mathcal{A}}).$$

Moreover, for $\lambda > 0,\ 0 < r < 1$, the real-valued functional

(4)
$$\mathcal{J}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^{\lambda}\, d\theta\right)^{1/\lambda} \quad (f \in \widetilde{\mathcal{A}})$$

is continuous on $\widetilde{\mathcal{A}}$. For $\lambda \geq 1$, by Minkowski's inequality it is also convex on $\widetilde{\mathcal{A}}$.

The extreme point theory for analytic functions was intensively investigated by Hallenbeck and MacGregor [9] (see also [3], [7], [8] and [16]).

Let Ω denote the class of $\omega \in \widetilde{\mathcal{A}}$ such that

$$|\omega(z)| \leq |z| \quad (z \in \mathcal{U}).$$

We say that a function $f \in \widetilde{\mathcal{A}}$ is *subordinate* to a function $F \in \widetilde{\mathcal{A}}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a function $\omega \in \Omega$ such that

$$f(z) = F(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

Also, we denote

$$s(F) := \{f \in \widetilde{\mathcal{A}} : f \prec F\}.$$

For functions $f, g \in \widetilde{\mathcal{A}}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

we denote by $f * g$ their *Hadamard product* (or *convolution*), defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

We denote by $\mathcal{A}$ the class of functions $f \in \widetilde{\mathcal{A}}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{5}$$

Also, let $\mathcal{T}_\eta$ ($\eta \in \mathbb{R}$) denote the class of functions $f \in \mathcal{A}$ of the form (5) for which

$$\arg(a_n) = \pi + (1-n)\eta \quad (n = 2, 3, \ldots). \tag{6}$$

In particular, for $\eta = 0$ we obtain the class $\mathcal{T}_0$ of functions with negative coefficients. Moreover, we define

$$\mathcal{T} := \bigcup_{\eta \in \mathbb{R}} \mathcal{T}_\eta. \tag{7}$$

The class $\mathcal{T}$ was introduced by Silverman [17] (see also [20]). It is called the class of *functions with varying argument of coefficients*.

Let A, B be real parameters, $-1 \leq A < B \leq 1$, and let $\varphi, \phi \in \mathcal{A}$ be given functions of the form

$$\varphi(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad \phi(z) = z + \sum_{n=2}^{\infty} \beta_n z^n \quad (z \in \mathcal{U}), \tag{8}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are real and

$$0 \leq \alpha_n < \beta_n \quad (n = 2, 3, \ldots).$$

Moreover, we assume

$$d_n := (1+B)\beta_n - (1+A)\alpha_n \quad (n = 2, 3, \ldots), \qquad \liminf_{n \to \infty} \sqrt[n]{d_n} \geq 1. \tag{9}$$

We denote by $\mathcal{W}(\phi, \varphi; A, B)$ the class of functions $f \in \mathcal{A}$ such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{10}$$

and we set

$$\mathcal{W}(\varphi; A, B) := \mathcal{W}(z\varphi'(z), \varphi(z); A, B),$$
$$\mathcal{W}(\varphi; \alpha) := \mathcal{W}(\varphi; 2\alpha - 1, 1) \quad (0 \leq \alpha < 1).$$

In particular,

$$\mathcal{S}^*(\alpha) := \mathcal{W}\left(\frac{z}{1-z}; \alpha\right), \quad \mathcal{S}^c(\alpha) := \mathcal{W}\left(\frac{z}{(1-z)^2}; \alpha\right)$$

are the well-known classes of starlike functions of order α and convex functions of order α, respectively.

If we denote

$$\mathcal{P}(A,B) := s\left(\frac{1+Az}{1+Bz}\right),$$

and

$$\mathcal{P}(\alpha) := \mathcal{P}(2\alpha-1,1), \qquad \mathcal{P} := \mathcal{P}(0) \qquad (0 \le \alpha < 1),$$

then we observe that

$$f \in \mathcal{W}(\phi,\varphi;A,B) \Leftrightarrow \frac{\phi * f}{\varphi * f} \in \mathcal{P}(A,B), \tag{11}$$

and

$$p \in \mathcal{P} \Leftrightarrow (1-\alpha)p(z) + \alpha \equiv q(z) \in \mathcal{P}(\alpha). \tag{12}$$

Finally, we define classes of functions with varying argument of coefficients related to the class $\mathcal{W}(\phi,\varphi;A,B)$. Let us denote

$$\mathcal{TW}(\phi,\varphi;A,B) := \mathcal{T} \cap \mathcal{W}(\phi,\varphi;A,B),$$
$$\mathcal{TW}_\eta(\phi,\varphi;A,B) := \mathcal{T}_\eta \cap \mathcal{W}(\phi,\varphi;A,B).$$

The families $\mathcal{W}(\phi,\varphi;A,B)$ and $\mathcal{TW}_\eta(\phi,\varphi;A,B)$ unify various new and also well-known classes of analytic functions. We list a few of them in the last section.

The object of the present paper is to investigate extreme points of the classes $\mathcal{W}(\varphi;A,B)$ and $\mathcal{TW}_\eta(\phi,\varphi;A,B)$. By using extreme point theory we obtain coefficient estimates and distortion theorems in these classes of functions. Some integral mean inequalities are also pointed out.

2. Extreme points. First we consider extreme points of $\mathcal{W}(\varphi;\alpha)$. Using the Herglotz formula for Carathéodory functions and the relationship (12) we obtain

LEMMA 1. *A function p belongs to the class $\mathcal{P}(\alpha)$ if and only if there is a probability measure μ on $\partial\mathcal{U}$ such that*

$$p(z) = \int\limits_{|x|=1} \frac{1+(2\alpha-1)xz}{1+xz}\, d\mu(x) \qquad (z \in \mathcal{U}). \tag{13}$$

The correspondence between $\mathcal{P}$ and probability measures μ on $\partial\mathcal{U}$ given trough (13) *is one-to-one.*

Using Lemma 1 we get the following lemma.

LEMMA 2. *A function f belongs to the class $\mathcal{S}^*(\alpha)$ if and only if there exists a probability measure μ on $\partial\mathcal{U}$ such that*

$$f(z) = z\exp \int\limits_{|x|=1} \{-2(1-\alpha)\log(1+xz)\}\, d\mu(x) \qquad (z \in \mathcal{U}).$$

The correspondence between $\mathcal{S}^*(\alpha)$ and probability measures μ on $\partial\mathcal{U}$ is one-to-one.

Let $f \in \mathcal{S}^*(\alpha)$. Then by Lemma 2 we have

$$\frac{f(z)}{z} = \exp \int_{|x|=1} \{-2(1-\alpha)\log(1+xz)\}\, d\mu(x) \qquad (z \in \mathcal{U}).$$

Since the function $k(z) = \log(1+z)$ is univalent and convex in $\mathcal{U}$, we have

$$\int_{|x|=1} \log(1+xz)\, d\mu(x) \prec k(z),$$

and consequently

$$\frac{f(z)}{z} = \exp\{-2(1-\alpha)\log(1+\omega(z))\} \qquad (z \in \mathcal{U})$$

for some $\omega \in \Omega$. Hence by definition of subordination we have the following well-known result.

LEMMA 3. *If $f \in \mathcal{S}^*(\alpha)$, then*

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)^{2(1-\alpha)}}.$$

LEMMA 4 ([9]). *Let*

$$F_a(z) = \frac{1}{(1+z)^a} \qquad (z \in \mathcal{U},\ a \geq 1). \tag{14}$$

A function $f \in \widetilde{\mathcal{A}}$ belongs to the class $Hs(F_a)$ if and only if it can be represented by the formula

$$f(z) = \int_{|x|=1} \frac{1}{(1+xz)^a}\, d\mu(x) \qquad (z \in \mathcal{U}),$$

where μ is a probability measure on $\partial\mathcal{U}$.

THEOREM 1. *Let $0 \leq \alpha \leq 1/2$. A function $f \in \mathcal{A}$ belongs to the class $H\mathcal{S}^*(\alpha)$ if and only if it can be represented by the formula*

$$f(z) = \int_{|x|=1} \frac{z}{(1+xz)^{2(1-\alpha)}}\, d\mu(x) \qquad (z \in \mathcal{U}), \tag{15}$$

where μ is a probability measure on $\partial\mathcal{U}$. Also,

$$EH\mathcal{S}^*(\alpha) = \left\{ \frac{z}{(1+xz)^{2(1-\alpha)}} : |x| = 1 \right\}. \tag{16}$$

Proof. Let $\mathcal{G}$ be the class of functions represented by (15). It is clear that $H\mathcal{G} = \mathcal{G}$. If $f \in H\mathcal{S}^*(\alpha)$, then according to Lemma 3 we have $f(z)/z \in$

$Hs(F_{2(1-\alpha)})$, where $F_{2(1-\alpha)}$ is defined by (14). Thus, by Lemma 4 we have (15), i.e.

$$H\mathcal{S}^*(\alpha) \subset H\mathcal{G} = \mathcal{G}.$$

Since $E\mathcal{G} \subset \mathcal{S}^*(\alpha)$, we get $\mathcal{G} \subset H\mathcal{S}^*(\alpha)$, and therefore $\mathcal{G} = H\mathcal{S}^*(\alpha)$. Moreover, $E\mathcal{G} = EH\mathcal{S}^*(\alpha)$ and we obtain (16). ∎

REMARK 1. We can represent the extreme points of the class $\mathcal{S}^*(\alpha)$ in the following form:

$$\frac{z}{(1+xz)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \frac{(2-2\alpha)_{n-1}}{(n-1)!}(-x)^{n-1}z^n \qquad (z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

THEOREM 2. *Let $\{\alpha_n\}$ be defined by* (8), $\alpha_n > 0$ $(n = 2, 3, \ldots)$, *and*

(17) $$\lim_{n\to\infty} \sqrt[n]{\alpha_n} = 1, \qquad 0 \le \alpha \le 1/2.$$

Then

(18) $$EH\mathcal{W}(\varphi;\alpha) = \{\varphi_x : |x| = 1\},$$

where

(19) $$\varphi_x(z) := z + \sum_{n=2}^{\infty} \frac{(2-2\alpha)_{n-1}}{(n-1)!\alpha_n}(-x)^{n-1}z^n \qquad (z \in \mathcal{U},\ |x| = 1).$$

Proof. Let

$$\varphi^\dagger(z) = z + \sum_{n=2}^{\infty} \frac{1}{\alpha_n} z^n \qquad (z \in \mathcal{U}).$$

Then the linear homeomorphism

$$I(f) = \varphi^\dagger * f \qquad (f \in \mathcal{A})$$

maps $\mathcal{S}^*(\alpha)$ onto $\mathcal{W}(\varphi;\alpha)$. Therefore, by (1) and Theorem 1 we obtain (18). ∎

Now, we consider extreme points of the class $\mathcal{TW}_\eta(\phi,\varphi;A,B)$. First we mention a sufficient condition for a function to belong to $\mathcal{W}(\phi,\varphi;A,B)$.

THEOREM 3. *Let $\{d_n\}$ be defined by* (9) *and* $-1 \le A < 1$, $0 \le B \le 1$. *If a function $f \in \mathcal{A}$ of the form* (5) *satisfies the condition*

(20) $$\sum_{n=2}^{\infty} d_n|a_n| \le B - A,$$

then $f \in \mathcal{W}(\phi,\varphi;A,B)$.

Proof. A function $f \in \mathcal{A}$ of the form (5) belongs to $\mathcal{W}(\phi, \varphi; A, B)$ if and only if there exists $\omega \in \Omega$ such that

$$\frac{(\phi * f)(z)}{(\varphi * f)(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathcal{U}),$$

or equivalently

$$\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{B(\phi * f)(z) - A(\varphi * f)(z)} \right| < 1 \quad (z \in \mathcal{U}). \tag{21}$$

Thus, it is sufficient to prove that

$$\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{z} \right| - \left| \frac{B(\phi * f)(z) - A(\varphi * f)(z)}{z} \right| < 0 \quad (z \in \mathcal{U}).$$

Indeed, letting $|z| = r$ $(0 < r < 1)$ we have

$$\begin{aligned} &\left| \frac{(\phi * f)(z) - (\varphi * f)(z)}{z} \right| - \left| \frac{B(\phi * f)(z) - A(\varphi * f)(z)}{z} \right| \\ &\quad = \left| \sum_{n=2}^{\infty} (\beta_n - \alpha_n) a_n z^{n-1} \right| - \left| (B - A) - \sum_{n=2}^{\infty} (B\beta_n - A\alpha_n) a_n z^{n-1} \right| \\ &\quad \le \sum_{n=2}^{\infty} (\beta_n - \alpha_n)|a_n| r^{n-1} - (B - A) + \sum_{n=2}^{\infty} (B\beta_n - A\alpha_n)|a_n| r^{n-1} \\ &\quad = \sum_{n=2}^{\infty} d_n |a_n| r^{n-1} - (B - A) < 0. \ \blacksquare \end{aligned}$$

THEOREM 4. *Let $f \in \mathcal{A}$ be a function of the form* (5), *satisfying* (6). *Then $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$ if and only if the condition* (20) *holds true.*

Proof. In view of Theorem 3 we need only show that if the function f is in $\mathcal{TW}_\eta(\phi, \varphi; A, B)$ then it satisfies (20). So assume $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$. Then, by (5) and (21) we have

$$\left| \frac{\sum_{n=2}^{\infty} (\beta_n - \alpha_n) a_n z^{n-1}}{B - A - \sum_{n=2}^{\infty} (B\beta_n - A\alpha_n) a_n z^{n-1}} \right| < 1 \quad (z \in \mathcal{U}).$$

Therefore, putting $z = re^{i\eta}$ $(0 \le r < 1)$ and applying (6) we obtain

$$\frac{\sum_{n=2}^{\infty} (\beta_n - \alpha_n)|a_n| r^{n-1}}{B - A - \sum_{n=2}^{\infty} (B\beta_n - A\alpha_n)|a_n| r^{n-1}} < 1. \tag{22}$$

It is clear that the denominator above cannot vanish for $r \in [0, 1)$. Moreover, it is positive for $r = 0$, and hence for $r \in [0, 1)$. Thus, by (22) we have

$$\sum_{n=2}^{\infty} [(1 + B)\beta_n - (1 + A)\alpha_n]|a_n| r^{n-1} < B - A,$$

which, upon letting $r \to 1^-$, readily yields (20). $\blacksquare$

Since the condition (20) is independent of η, Theorem 4 yields the following theorem.

THEOREM 5. *Let $f \in \mathcal{A}$ be a function of the form* (5), *satisfying* (6). *Then $f \in \mathcal{TW}(\phi, \varphi; A, B)$ if and only if the condition* (20) *holds true.*

Since $\widetilde{\mathcal{A}}$ is a complete metric space, Montel's theorem (see [13]) implies the following lemma.

LEMMA 5. *A class $\mathcal{F}$ contained in $\widetilde{\mathcal{A}}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

THEOREM 6. *The class $\mathcal{TW}_\eta(\phi, \varphi; A, B)$ is convex and compact.*

Proof. Let $f, g \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$, $0 \le \gamma \le 1$. Since

$$\begin{aligned}\gamma f(z) + (1-\gamma)g(z) &= \gamma\Big(z + \sum_{n=2}^{\infty} a_n z^n\Big) + (1-\gamma)\Big(z + \sum_{n=2}^{\infty} b_n z^n\Big)\\ &= z + \sum_{n=2}^{\infty} (\gamma a_n + (1-\gamma) b_n) z^n,\end{aligned}$$

by Theorem 4 we have

$$\begin{aligned}\sum_{n=2}^{\infty} d_n |\gamma a_n + (1-\gamma) b_n| &\le \gamma \sum_{n=2}^{\infty} d_n |a_n| + (1-\gamma) \sum_{n=2}^{\infty} d_n |b_n|\\ &\le \gamma(B-A) + (1-\gamma)(B-A) = B - A,\end{aligned}$$

and consequently $h = \gamma f + (1-\gamma) g \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$. Hence this class is convex. Furthermore, for $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$, $|z| \le r$, $0 < r < 1$, we have

$$|f(z)| \le r + \sum_{n=2}^{\infty} d_n a_n \frac{r^n}{d_n} \le r + (B-A) \sum_{n=2}^{\infty} \frac{r^n}{d_n}. \tag{23}$$

By (9) we have

$$\limsup_{n\to\infty} (r^n d_n^{-1})^{1/n} = \frac{r}{\liminf_{n\to\infty} \sqrt[n]{d_n}} \le r < 1. \tag{24}$$

Thus, the power series $\sum_{n=2}^{\infty} r^n d_n^{-1}$ converges and by (23) we conclude that the class $\mathcal{TW}_\eta(\phi, \varphi; A, B)$ is locally uniformly bounded. By Lemma 5, we only need to show that it is closed. Let $f_m \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$ $(m \in \mathbb{N})$ and $f_m \to f$. Suppose that

$$f_m(z) = z + \sum_{n=2}^{\infty} a_{n,m} z^n \quad (m \in \mathbb{N},\ z \in \mathcal{U})$$

and f is given by (5). Using Theorem 1 we have

$$\sum_{n=2}^{\infty} d_n |a_{n,m}| \le B - A \quad (m \in \mathbb{N}).$$

Since $f_m \to f$, we conclude that $a_{n,m} \to a_n$ as $m \to \infty$ $(n \in \mathbb{N})$. This gives the condition (20), and so $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$, which completes the proof. ∎

THEOREM 7.

$$E\mathcal{TW}_\eta(\phi, \varphi; A, B) = \{f_n : n \in \mathbb{N}\},$$

where $f_1(z) = z$ *and*

$$f_n(z) = f_{n,\eta}(z) = z - \frac{B-A}{d_n} e^{i(1-n)\eta} z^n \quad (n = 2, 3, \ldots,\ z \in \mathcal{U}). \tag{25}$$

Proof. By using (20) we easily verify that all functions of the form (25) are extreme points of $\mathcal{TW}_\eta(\phi, \varphi; A, B)$. Now, suppose $f \in E\mathcal{TW}_\eta(\phi, \varphi; A, B)$ and f is not of the form (25). If

$$f(z) = z - \gamma \frac{B-A}{d_n} z^n \quad (0 < \gamma < 1,\ n = 2, 3, \ldots,\ z \in \mathcal{U}),$$

then

$$f(z) = (1-\gamma) f_1(z) + \gamma f_n(z) \ \ (z \in \mathcal{U}),$$

and so f is not an extreme point of $\mathcal{TW}_\eta(\phi, \varphi; A, B)$. In the opposite case there exist $m, l \in \mathbb{N}$, $m \neq l$, so that the coefficients a_m and a_l do not vanish in the power series (5). Putting

$$g(z) = f(z) - a_l z^l + \frac{a_l}{d_m} z^m,$$
$$h(z) = f(z) - a_m z^m + \frac{a_m}{d_l} z^l,$$
$$\gamma = \frac{d_m a_m}{d_m a_m + d_l a_l},$$

we have

$$g, h \in \mathcal{TW}_\eta(\phi, \varphi; A, B), \quad g \neq h, \quad 0 < \gamma < 1 \quad \text{and} \quad f = \gamma g + (1-\gamma) h.$$

It follows that $f \notin E\mathcal{TW}_\eta(\phi, \varphi; A, B)$, and the proof is complete. ∎

3. Applications. The Krein–Milman theorem (see [10] and [15]) is fundamental in the theory of extreme points.

LEMMA 6 (Krein–Milman theorem). *If $\mathcal{F}$ is a compact convex subclass of $\mathcal{A}$, then $HE\mathcal{F} = \mathcal{F}$.*

In particular, the Krein–Milman implies the following important result.

LEMMA 7 ([9]). *Let $\mathcal{F}$ be a compact convex subclass of the class $\widetilde{\mathcal{A}}$ and $J : \widetilde{\mathcal{A}} \to \mathbb{R}$ be a continuous and convex functional on $\mathcal{F}$. Then*

$$\max\{\mathcal{J}(f) : f \in H\mathcal{F}\} = \max\{\mathcal{J}(f) : f \in \mathcal{F}\} = \max\{\mathcal{J}(f) : f \in EH\mathcal{F}\}.$$

LEMMA 8 ([11]). *Let $f, g \in \widetilde{\mathcal{A}}$. If $f \prec g$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |g(re^{i\theta})|^\lambda \, d\theta \qquad (0 < r < 1,\ \lambda > 0).$$

By using (2) and Theorems 6 and 7, the Krein–Milman theorem gives the following corollary.

COROLLARY 1.

$$\mathcal{TW}_\eta(\phi, \varphi; A, B) = \Big\{ \sum_{n=1}^{\infty} \gamma_n f_{n,\eta} : \sum_{n=2-1}^{\infty} \gamma_n = 1,\ \gamma_n \ge 0\,(n \in \mathbb{N}) \Big\},$$

where $f_1(z) = z$ and $f_{n,\eta}$ are defined by (25).

Moreover, by Theorem 2 we obtain

COROLLARY 2. *Let φ_x be defined by* (19), *and suppose* (17) *holds. Then the class $H\mathcal{W}(\varphi; \alpha)$ contains all functions $f \in \mathcal{A}$ represented by the formula*

$$f(z) = \int_{|x|=1} \varphi_x(z) \, d\mu(x) \qquad (z \in \mathcal{U}),$$

where μ is a probability measure on $\partial\mathcal{U}$.

Using the extremal points of the classes $\mathcal{W}(\varphi; \alpha)$ and $\mathcal{TW}_\eta(\phi, \varphi; A, B)$ we obtain some results listed below. Combining (3) with Lemma 7 yields the following three corollaries:

COROLLARY 3. *Let φ_x be defined by* (19), *and suppose* (17) *holds. If a function f of the form* (5) *belongs to the class $\mathcal{W}(\varphi; \alpha)$, then*

$$|a_n| \le \frac{(2 - 2\alpha)_{n-1}}{(n-1)!\alpha_n} \qquad (n = 2, 3, \ldots).$$

The result is sharp. The functions φ_x are the extremal functions.

COROLLARY 4. *If a function f of the form* (5) *belongs to the class $\mathcal{TW}_\eta(\phi, \varphi; A, B)$, then*

$$|a_n| \le \frac{B - A}{d_n} \qquad (n = 2, 3, \ldots), \tag{26}$$

where d_n is defined by (9). *The result is sharp. The functions $f_{n,\eta}$ of the form* (25) *are the extremal functions.*

COROLLARY 5. *Let φ_x be defined by* (19), *and suppose* (17) *holds. If $f \in \mathcal{W}(\varphi; \alpha)$, then*

$$\min_{|z|=r} \varphi_1^{(k)}(z) \le |f^{(k)}(z)| \le \varphi_1^{(k)}(r) \qquad (k = 0, 1, \ldots,\ |z| = r < 1).$$

The result is sharp. The functions φ_x are the extremal functions.

For the extreme points $f_{n,\eta}$ of the form (25) we have

$$\begin{aligned} f'_{n,\eta}(z) &= 1 - \frac{(B-A)n}{d_n}\, e^{i(1-n)\eta} z^{n-1}, \\ f^{(k)}_{n,\eta}(z) &= -\frac{(B-A)n!}{(n-k)!d_n}\, e^{i(1-n)\eta} z^{n-k} \qquad (k = 2, \dots, n), \\ f^{(k)}_{n,\eta}(z) &= 0 \qquad (k = n+1, n+2, \dots). \end{aligned} \tag{27}$$

Let $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 < r < 1$, and define the sequence $\{\delta^{(k)}_n\}$ by

$$\delta^{(k)}_n = \frac{(B-A)n!}{(n-k)!d_n}\, r^{n-k} \qquad (n \geq \max\{k, 2\}). \tag{28}$$

Applying (24) we obtain

$$\limsup_{n\to\infty} \delta^{(k)}_n = 0 \qquad (k \in \mathbb{N}_0).$$

Thus, there exist $n_k \in \mathbb{N}$ $(k \in \mathbb{N}_0)$ such that

$$\delta^{(k)}_{n_k} = \max\{\delta^{(k)}_n : n \geq \max\{k, 2\}\} \qquad (k \in \mathbb{N}_0). \tag{29}$$

Therefore, by Lemma 7 we have the following corollary.

Corollary 6. *If $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$ and $|z| = r < 1$, then*

$$r - \frac{B-A}{d_{n_0}} r^{n_0} \leq |f(z)| \leq r + \frac{B-A}{d_{n_0}}\, r^{n_0}, \tag{30}$$

$$1 - \frac{(B-A)n_1}{d_{n_1}} r^{n_1} \leq |f'(z)| \leq 1 + \frac{(B-A)n_1}{d_{n_1}}\, r^{n_1}, \tag{31}$$

$$|f^{(k)}(z)| \leq \frac{(B-A)(n_k)!}{(n_k-k)!d_{n_k}}\, r^{n_k-k} \qquad (k \geq 2), \tag{32}$$

where n_k is defined by (29). The result is sharp. The functions $f_{n_k,\eta}$ of the form (25) are the extremal functions.

From Corollary 6 we have the following corollary.

Corollary 7. *Let $f \in \mathcal{TW}_\eta(\phi, \varphi; A, B)$, $|z| = r < 1$, and $k \in \mathbb{N}_0$. If the sequence $\{\delta^{(k)}_n\}$ defined by (28) is nonincreasing with respect to n, then*

$$r - \frac{B-A}{d_2} r^2 \leq |f(z)| \leq r + \frac{B-A}{d_2}\, r^2 \qquad (k = 0), \tag{33}$$

$$1 - \frac{2(B-A)}{d_2} r^2 \leq |f'(z)| \leq 1 + \frac{2(B-A)}{d_2}\, r^2 \qquad (k = 1), \tag{34}$$

$$|f^{(k)}(z)| \leq \frac{(B-A)k!}{d_k} \qquad (k = 2, 3, \dots), \tag{35}$$

where n_k is defined by (29). The result is sharp. The functions $f_{n_k,\eta}$ of the form (25) are the extremal functions.

Now, we consider some integral mean inequalities. By (4), Lemma 7 yields the following corollary.

Corollary 8. *Let φ_x be defined by* (19), *and suppose* (17) *holds. If $f \in \mathcal{W}(\varphi;\alpha)$, then*

$$\int_0^{2\pi} |f^{(k)}(re^{i\theta})|^\lambda \, d\theta \le \int_0^{2\pi} |\varphi_1^{(k)}(re^{i\theta})|^\lambda \, d\theta \quad (0 < r < 1,\ \lambda \ge 1).$$

Corollary 9. *Let $0 < r < 1$, $\lambda \ge 1$, $k \in \mathbb{N}_0$ and assume that the sequence $\{\delta_n^{(k)}\}$ defined by* (28) *is nonincreasing with respect to n. If $f \in \mathcal{TW}_\eta(\phi,\varphi;A,B)$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(re^{i\theta})|^\lambda \, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f_{2,\eta}^{(k)}(re^{i\theta})|^\lambda \, d\theta \quad (k = 0,1), \tag{36}$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(re^{i\theta})|^\lambda \, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f_{k,\eta}^{(k)}(re^{i\theta})|^\lambda \, d\theta \quad (k = 2,3,\ldots), \tag{37}$$

where $f_{k,\eta}$ are the functions defined by (25).

Proof. Since

$$\frac{f_{n,\eta}}{z} \prec \frac{f_{2,\eta}}{z} \quad \text{and} \quad f'_{n,\eta} \prec f'_{2,\eta} \quad (n \in \mathbb{N}),$$

using (7) and Lemma 8 we have

$$\max\Big\{\int_0^{2\pi} |f_{n,\eta}^{(k)}(re^{i\theta})|^\lambda \, d\theta : n \in \mathbb{N}\Big\} = \int_0^{2\pi} |f_{2,\eta}^{(k)}(re^{i\theta})|^\lambda \, d\theta \quad (k = 0,1).$$

Thus, Lemma 7 yields (36). The inequality (37) is an immediate consequence of (35) and (27). ∎

Making use of (7) and Corollaries 4, 6, 7 and 9, we get the corollaries listed below.

Corollary 10. *If a function f of the form* (5) *belongs to the class $\mathcal{TW}(\phi,\varphi;A,B)$, then the coefficient estimates* (26) *hold true. The result is sharp. The functions $f_{n,\eta}$ of the form* (25) ($\eta \in \mathbb{R}$) *are the extremal functions.*

Corollary 11. *If $f \in \mathcal{TW}(\phi,\varphi;A,B)$ and $|z| = r < 1$, then the bounds* (30)–(32) *hold true. The results are sharp. The functions $f_{n_k,\eta}$ ($\eta \in \mathbb{R}$) of the form* (25) *are the extremal functions.*

Corollary 12. *Let $f \in \mathcal{TW}(\phi,\varphi;A,B)$, $|z| = r < 1$, and $k \in \mathbb{N}_0$. If the sequence $\{b_n^{(k)}\}$ defined by* (28) *is nonincreasing with respect to n, then the inequalities* (33)–(35) *hold true. The result is sharp. The functions $f_{n,\eta}$ ($\eta \in \mathbb{R}$) of the form* (25) *are the extremal functions.*

COROLLARY 13. *Let $0 < r < 1$, $\lambda \geq 1$, $k \in \mathbb{N}_0$ and assume that the sequence $\{\delta_n^{(k)}\}$ defined by* (28) *is nonincreasing with respect to n. If $f \in \mathcal{TW}(\phi, \varphi; A, B)$, then the inequalities* (36) *and* (37) *hold true.*

4. Concluding remarks. We conclude this paper by observing that, in view of the subordination relation (10), choosing the functions ϕ and φ appropriately, we can consider new and also well-known classes of functions. In particular, the class

$$\mathcal{W}_n(\varphi; A, B) := \mathcal{W}\Big(z\varphi'(z), \sum_{k=0}^{n-1} \varphi(x^k z); A, B\Big),$$

where $n \in \mathbb{N}$, $x^n = 1$, consists of the functions $f \in \mathcal{A}$ such that

$$\frac{z(\varphi * f)'(z)}{\sum_{k=0}^{n-1} (\varphi * f)(x^k z)} \prec \frac{1 + Az}{1 + Bz}.$$

It is related to the class of starlike functions with respect to n-symmetric points. Moreover, for $n = 1$, we obtain the class

$$\mathcal{W}(\varphi; A, B) = \mathcal{W}_1(\varphi; A, B)$$

defined by the following condition:

$$\frac{z(\varphi * f)'(z)}{(\varphi * f)(z)} \prec \frac{1 + Az}{1 + Bz}.$$

This class is related to the class of starlike functions.

Let λ be a convex parameter. A function $f \in \mathcal{A}$ belongs to the class

$$\mathcal{V}_\lambda(\varphi; A, B) := \mathcal{W}\left(\lambda \frac{\varphi(z)}{z} + (1 - \lambda)\varphi'(z), z; A, B\right)$$

if it satisfies the following condition:

$$\lambda \frac{(\varphi * f)(z)}{z} + (1 - \lambda)(\varphi * f)'(z) \prec \frac{1 + Az}{1 + Bz}.$$

Likewise, a function $f \in \mathcal{A}$ belongs to the class

$$\mathcal{U}_\lambda(\varphi; A, B) := \mathcal{W}\left(\lambda \frac{\varphi(z)}{z} + (1 - \lambda)\varphi'(z); A, B\right)$$

if it satisfies the following condition:

$$\frac{z(\varphi * f)'(z) + (1 - \lambda)z^2(\varphi * f)''(z)}{\lambda(\varphi * f)(z) + (1 - \lambda)z(\varphi * f)'(z)} \prec \frac{1 + Az}{1 + Bz}.$$

The classes $\mathcal{W}_n^p(\varphi; A, B), \mathcal{U}_\lambda(\varphi; A, B)$ and $\mathcal{V}_\lambda(\varphi; A, B)$ generalize well-known important classes, which were investigated in earlier works (see for example [1, 4, 14, 18, 19, 12]). Most of these classes were defined by using linear operators and special functions.

If we apply the results of this paper to the classes discussed above, we can get several additional new results. Some of these classes were obtained in earlier works (see for example [2, 5, 6, 21]).

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