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Stable harmonic maps between Finsler manifolds and Riemannian manifolds with positive Ricci curvature

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Abstract. We study the stability of harmonic maps between Finsler manifolds and Riemannian manifolds with positive Ricci curvature, and we prove that if M^n is a compact Einstein Riemannian minimal submanifold of a Riemannian unit sphere with Ricci curvature satisfying $\mathrm{Ric}^M > n/2$, then there is no non-degenerate stable harmonic map between M and any compact Finsler manifold.

1. Introduction. Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A *Finsler metric* on M is a function $F : TM \to [0, +\infty)$ with the following properties:

(i) regularity: $F(x, Y)$ is smooth in $TM \setminus 0$;
(ii) positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) strong convexity: the fundamental quadratic form $g = g_{ij} dx^i \otimes dx^j$ is positive-definite, where $g_{ij} = \frac{1}{2} \frac{\partial^2(F^2)}{\partial Y^i \partial Y^j}$.

Let $\phi : M \to \overline{M}$ be a non-degenerate (that is, $\ker(d\phi) = 0$) smooth map between Finsler manifolds. Harmonic maps between Finsler manifolds are defined as the critical points of energy functionals. The first and second variation formulas for non-degenerate harmonic maps between Finsler manifolds were given in [HS] and [SZ]. As for stability of harmonic maps between Finsler manifolds, the results of He–Shen [HS] and Shen–Zhang [SZ] show that there is no non-degenerate stable harmonic map $M \to N$ if either the domain M or the codomain N is a Riemannian unit sphere S^n $(n > 2)$. A direct generalization is to consider the non-existence of non-degenerate stable harmonic maps between arbitrary compact Finsler manifolds and

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Riemannian manifolds with positive Ricci curvature. In this paper, we prove the following

THEOREM 1.1. *Let M^n $(n \geq 3)$ be a compact Einstein Riemannian minimal submanifold of a Riemannian unit sphere. If the Ricci curvature satisfies* $\mathrm{Ric}^M > n/2$, *then there is no non-degenerate stable harmonic map between M and any compact Finsler manifold.*

REMARK. This theorem is obtained by He–Shen [HS] when M is a Riemannian unit sphere.

COROLLARY 1.2. *There is no non-degenerate stable harmonic map between any compact Finsler manifold and a minimal Clifford hypersurface* $S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2})$ $(m > 2)$.

COROLLARY 1.3. *There is no non-degenerate stable harmonic map between any compact Finsler manifold and the Riemannian product* $S^m(1/2) \times S^m(1/2) \times S^m(1/2) \times S^m(1/2)$ $(m > 2)$.

2. Preliminaries. We shall use the following convention on index ranges unless otherwise stated:

$$1 \leq i, j, \ldots \leq n; \quad 1 \leq \alpha, \beta, \ldots \leq m; \quad 1 \leq a, b, \ldots \leq n-1.$$

Let (M, F) be an n-dimensional Finsler manifold. The *Hilbert* form and *Cartan* tensor are defined as follows:

$$\omega^n = \frac{\partial F}{\partial Y^i} dx^i, \qquad A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \qquad A_{ijk} = F \frac{\partial g_{ij}}{\partial Y^k}.$$

It is well known that there is a unique connection, the *Chern connection*, ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying

$$X\langle U, V\rangle = \langle \nabla_X U, V\rangle + \langle U, \nabla_X V\rangle + 2C(U, V, \nabla_X(Fe_n)), \tag{2.1}$$

where $A_{ijk} = FC_{ijk}$ and $X, U, V \in \Gamma(\pi^* TM)$.

The curvature 2-forms of the Chern connection ∇ are

$$\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l + \frac{1}{F} P^i_{jkl} dx^k \wedge \delta Y^l, \tag{2.2}$$

where $\delta Y^i = dY^i + N^i_j dx^j$, $N^i_j = \gamma^i_{jk} Y^k - \frac{1}{F} A^i_{jk} \gamma^k_{st} Y^s Y^t$ and γ^i_{jk} are the formal Christoffel symbols of the second kind for g_{ij}.

Take a g-orthonormal frame $\{e_i = u_i^j \frac{\partial}{\partial x^j}\}$ with $e_n = \hat{e} = Y/F$ for each fibre of $\pi^* TM$ and let $\{\omega^i\}$ be its dual coframe. The collection $\{\omega^i, \omega^i_n\}$ forms an orthonormal basis for $T^*(TM \setminus \{0\})$ with respect to the Sasaki type metric $g_{ij} dx^i \otimes dx^j + g_{ij} \delta Y^i \otimes \delta Y^j$. The pull-back of the Sasaki metric from $TM \setminus \{0\}$ to SM is a Riemannian metric

$$\widehat{g} = g_{ij} dx^i \otimes dx^j + \delta_{ab} \omega_n^a \otimes \omega_n^b. \tag{2.3}$$

Then we have

LEMMA 2.1 ([M, Lemma 2.2]). *For $\psi = \psi_i \omega^i \in \Gamma(\pi^* T^* M)$, we have*

$$\operatorname{div}_{\widehat{g}} \psi = \sum_i \psi_{i|i} + \sum_{a,b} \psi_a P_{bba},$$

where $|$ denotes the horizontal covariant differential with respect to the Chern connection, $e_i^H = u_i^j \frac{\delta}{\delta x^j} = u_i^j \big(\frac{\partial}{\partial x^j} - N_j^k \frac{\partial}{\partial Y^k}\big)$ denotes the horizontal part of e_i and $P_{bba} = P_{bba}^n$.

Let $\phi : M^n \to \overline{M}^m$ be a non-degenerate smooth map between Finsler manifolds. The *energy density* of ϕ is the function $e(\phi) : SM \to \mathbb{R}$ defined by

$$e(\phi)(x,Y) = \frac{1}{2} g^{ij}(x,Y) \phi_i^\alpha \phi_j^\beta \overline{g}_{\alpha\beta}(\overline{x}, \overline{Y}), \tag{2.4}$$

where $d\phi\big(\frac{\partial}{\partial x^i}\big) = \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$ and $\overline{Y} = \overline{Y}^\alpha \frac{\partial}{\partial \overline{x}^\alpha} = Y^i \phi_i^\alpha \frac{\partial}{\partial \overline{x}^\alpha}$.

We define the *energy functional* $E(\phi)$ by

$$E(\phi) = \frac{1}{C_{n-1}} \int_{SM} e(\phi)\, dV_{SM}, \tag{2.5}$$

where $dV_{SM} = \Omega d\tau \wedge dx$, $\Omega = \det(g_{ij}/F)$, $d\tau = \sum_i (-1)^{i-1} Y^i dY^1 \wedge \cdots \wedge \widehat{dY^i} \wedge \cdots \wedge dY^n$, $dx = dx^1 \wedge \cdots \wedge dx^n$ and C_{n-1} denotes the volume of the unit Euclidean sphere S^{n-1}.

We call ϕ a *harmonic map* if it is a critical point of the energy functional. Let $\widetilde{\nabla}$ be the pullback Chern connection on $\pi^*(\phi^{-1} T\overline{M})$. We introduce the tension field τ of ϕ and a section $J(\overline{U}, d\phi, \overline{V}) \in C(T^* M \otimes \phi^{-1} T\overline{M})$ as follows:

$$\begin{aligned} \tau = {} & \sum_i (\widetilde{\nabla}_{e_i^H} d\phi) e_i + \sum_{i,\alpha} \{2\overline{C}(\overline{e}_\alpha, d\phi e_i, \widetilde{\nabla}_{e_i^H}(d\phi F e_n))\overline{e}_\alpha \\ & + (\widetilde{\nabla}_{F e_n^H} \overline{C})(d\phi e_i, d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha + 2\overline{C}(\widetilde{\nabla}_{F e_n^H}(d\phi e_i), d\phi e_i, \overline{e}_\alpha)\overline{e}_\alpha\} \\ & + \sum_{a,b} \langle \overline{e}_\alpha, d\phi e_b \rangle \overline{e}_\alpha P_{aab} \end{aligned} \tag{2.6}$$

and

$$J(\overline{U}, d\phi, \overline{V}) = \overline{R}(\overline{U}, d\phi)\overline{V} + \frac{F}{\overline{F}} \overline{P}(\overline{U}, \widetilde{\nabla}_{\hat{e}} d\phi)\overline{V} - \frac{F}{\overline{F}} \overline{P}(d\phi, \widetilde{\nabla}_{\hat{e}} \overline{U})\overline{V}. \tag{2.7}$$

We have

THEOREM 2.2 ([SZ, Theorem 2.1]). *A map ϕ is harmonic if and only if*

$$\int_{SM} \langle V, \tau \rangle \, dV_{SM} = 0$$

for any vector $V \in \Gamma(\phi^{-1} T\overline{M})$.

THEOREM 2.3 ([SZ, Theorem 3.1]). *Let $\phi : M^n \to \overline{M}^m$ be a non-degenerate harmonic map. Let ϕ_t be a smooth variation of ϕ with $\phi_0 = \phi$*

and $V = \frac{\partial \phi_t}{\partial t}|_{t=0}$. *Then the second variation of the energy functional for* ϕ *is*

$$I(V,V) = \frac{d^2}{dt^2} E(\phi_t)\bigg|_{t=0} = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5,$$

where

$$\Xi_1 = \frac{1}{C_{n-1}} \int\limits_{SM} \|\widetilde{\nabla} V\|^2 \, dV_{SM}, \tag{2.8}$$

$$\Xi_2 = \frac{1}{C_{n-1}} \int\limits_{SM} \mathrm{Tr}_g \langle J(V, d\phi, V), d\phi\rangle \, dV_{SM}, \tag{2.9}$$

$$\Xi_3 = \frac{1}{C_{n-1}} \int\limits_{SM} \mathrm{Tr}_g \, \overline{C}(d\phi, d\phi, J(V, d\phi(Fe_n), V)) \, dV_{SM}, \tag{2.10}$$

$$\Xi_4 = -\frac{1}{C_{n-1}} \int\limits_{SM} \langle \widetilde{\nabla}_V V, \tau\rangle \, dV_{SM}, \tag{2.11}$$

$$\begin{aligned} \Xi_5 = \frac{1}{C_{n-1}} \int\limits_{SM} &\{\mathrm{Tr}_g (\widetilde{\nabla}_{V^H} \overline{C})(d\phi, d\phi, \widetilde{\nabla}_{F\hat{e}} V) \\ &+ \mathrm{Tr}_g \, \overline{C}(d\phi, d\phi, \widetilde{\nabla}_{F\hat{e}} V, \widetilde{\nabla}_{F\hat{e}} V) \\ &+ 4 \, \mathrm{Tr}_g \, \overline{C}(\widetilde{\nabla} V, d\phi, \widetilde{\nabla}_{F\hat{e}} V)\} \, dV_{SM}. \end{aligned} \tag{2.12}$$

3. The stability. Let $x : M^n \to S^{n+p} \to E^{n+p+1}$ be a compact Riemannian minimal submanifold of the Riemannian unit sphere S^{n+p}. We choose a local field of orthonormal frames $\{e_0, \ldots, e_{n+p}\}$ in the Euclidean space E^{n+p+1} such that $\{e_1, \ldots, e_n\}$ are tangent to M and $x = e_0$. Then we have

$$\begin{aligned} de_i &= \omega_i^j e_j + h_{ij}^\alpha e_\alpha \omega^j - \omega^i x, \\ de_\mu &= -h_{ij}^\mu \omega^j e_i + \omega_\mu^\nu e_\nu \quad (\mu, \nu = n+1, \ldots, n+p), \end{aligned}$$

where $B = h_{ij}^\mu \omega^i \otimes \omega^j \otimes e_\mu$ is the second fundamental form of M in S^{n+P}. Let $\{\Lambda_1, \ldots, \Lambda_{n+p+1}\}$ be the constant orthonormal basis in E^{n+p+1} and let $V_A = \langle \Lambda_A, e_i\rangle e_i$, $A = 1, \ldots, n+p+1$. A straightforward computation shows

$$\nabla_{e_i} V_A = \sum_{\mu, j} v_A^\mu h_{ij}^\mu e_j - \langle \Lambda_A, e_0\rangle e_i \tag{3.1}$$

where $v_A^\mu = \langle \Lambda_A, e_\mu\rangle$.

First, we assume that the source manifold is M. The second variation formula for the harmonic map $\phi : M \to \overline{M}^m$ can be written as

$$\sum_A I(d\phi V_A, d\phi V_A) = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5,$$

where

$$(3.2)\qquad \varXi_1 = \sum_A \frac{1}{C_{n-1}} \int_{SM} \|\widetilde{\nabla}(d\phi V_A)\|^2 \, dV_{SM},$$

$$(3.3)\qquad \varXi_2 = \sum_A \frac{1}{C_{n-1}} \int_{SM} \mathrm{Tr}_g \langle J(d\phi V, d\phi, d\phi V), d\phi\rangle \, dV_{SM},$$

$$(3.4)\qquad \varXi_3 = \sum_A \frac{1}{C_{n-1}} \int_{SM} \mathrm{Tr}_g \, \overline{C}(d\phi, d\phi, J(d\phi V, d\phi(Fe_n), d\phi V)) \, dV_{SM},$$

$$(3.5)\qquad \varXi_4 = -\sum_A \frac{1}{C_{n-1}} \int_{SM} \langle \widetilde{\nabla}_{d\phi V_A}(d\phi V_A), \tau\rangle \, dV_{SM}$$

$$\begin{aligned}(3.6)\qquad \varXi_5 = \sum_A \frac{1}{C_{n-1}} \int_{SM} &\{\mathrm{Tr}_g(\widetilde{\nabla}_{(d\phi V_A)^H}\overline{C})(d\phi, d\phi, \widetilde{\nabla}_{V_A}(d\phi F\hat{e}))\\ &+ \mathrm{Tr}_g \, \overline{C}(d\phi, d\phi, \widetilde{\nabla}_{V_A}(d\phi F\hat{e}), \widetilde{\nabla}_{V_A}(d\phi F\hat{e}))\\ &+ 4\,\mathrm{Tr}_g \, \overline{C}(\widetilde{\nabla}(d\phi V_A), d\phi, \widetilde{\nabla}_{V_A}(d\phi F\hat{e}))\} \, dV_{SM}.\end{aligned}$$

We have [HS, (3.4)]

$$\begin{aligned}(3.7)\quad (\widetilde{\nabla}_{X^H}\widetilde{\nabla}_Z d\phi)Y = {} & - d\phi R(X,Y)Z + (\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_Z d\phi)X\\ &+ (\widetilde{\nabla}_Y d\phi)(\nabla_{X^H}Z) - (\widetilde{\nabla}_X d\phi)(\nabla_{Y^H}Z)\\ &+ \overline{R}(d\phi X, d\phi Y)d\phi Z + \frac{F}{\overline{\overline{F}}}\,\overline{P}(d\phi X, (\widetilde{\nabla}_{e_n}d\phi)Y)d\phi Z\\ &- \frac{F}{\overline{\overline{F}}}\,\overline{P}(d\phi Y, (\widetilde{\nabla}_{e_n}d\phi)X)d\phi Z.\end{aligned}$$

Set $X = Z = V$, $Y = e_i$ in (3.7). We obtain

$$\begin{aligned}(3.8)\qquad & - \langle \overline{R}(d\phi e_i, d\phi V)d\phi V, d\phi e_i\rangle + \frac{F}{\overline{\overline{F}}}\,\langle \overline{P}(d\phi V, (\widetilde{\nabla}_{e_i}d\phi)e_n)d\phi V, d\phi e_i\rangle\\ &\qquad - \frac{F}{\overline{\overline{F}}}\,\langle \overline{P}(d\phi e_i, (\widetilde{\nabla}_{e_n}d\phi)V)d\phi V, d\phi e_i\rangle\\ &= - \langle d\phi R(e_i, V)V, d\phi e_i\rangle + \langle (\widetilde{\nabla}_{V^H}\widetilde{\nabla}_V d\phi)e_i, d\phi e_i\rangle\\ &\qquad - \langle (\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_V d\phi)V, d\phi e_i\rangle - \langle (\widetilde{\nabla}_{e_i}d\phi)(\nabla_{V^H}V), d\phi e_i\rangle\\ &\qquad + \langle (\widetilde{\nabla}_V d\phi)(\nabla_{e_i^H}V), d\phi e_i\rangle.\end{aligned}$$

Substituting (3.8) into (3.3) yields

$$\begin{aligned}(3.9)\qquad \varXi_2 = \sum_A \frac{1}{C_{n-1}} \int_{SM} &\{-\langle d\phi R(e_i, V_A)V_A, d\phi e_i\rangle\\ &+ \langle (\widetilde{\nabla}_{V_A^H}\widetilde{\nabla}_{V_A} d\phi)e_i, d\phi e_i\rangle\\ &- \langle (\widetilde{\nabla}_{e_i^H}\widetilde{\nabla}_{V_A} d\phi)V_A, d\phi e_i\rangle - \langle (\widetilde{\nabla}_{e_i}d\phi)(\nabla_{V_A^H}V_A), d\phi e_i\rangle\\ &+ \langle (\widetilde{\nabla}_{V_A} d\phi)(\nabla_{e_i^H}V_A), d\phi e_i\rangle\} \, dV_{SM}.\end{aligned}$$

Similarly, set $X = Z = V_A$, $Y = Fe_n$ in (3.7). We also obtain

$$\begin{aligned}
(3.10)\qquad & -\overline{R}(d\phi Fe_n, d\phi V_A)d\phi V_A + \frac{F}{\overline{\overline{F}}}\overline{P}(d\phi V_A, (\widetilde{\nabla}_{Fe_n}d\phi)e_n)d\phi V_A \\
& \quad - \frac{F}{\overline{\overline{F}}}\overline{P}(d\phi Fe_n, (\widetilde{\nabla}_{e_n}d\phi)V_A)d\phi V_A \\
& = -d\phi R(Fe_n, V_A)V_A + (\widetilde{\nabla}_{V_A^H}\widetilde{\nabla}_{V_A}d\phi)Fe_n \\
& \quad - (\widetilde{\nabla}_{Fe_n^H}\widetilde{\nabla}_{V_A}d\phi)V_A - (\widetilde{\nabla}_{Fe_n}d\phi)(\nabla_{V_A^H}V_A) \\
& \quad + (\widetilde{\nabla}_{V_A}d\phi)(\nabla_{Fe_n^H}V_A).
\end{aligned}$$

Substituting (3.10) into (3.4) yields

$$\begin{aligned}
(3.11)\qquad \varXi_3 = & \sum_A \frac{1}{C_{n-1}} \int_{SM} \{-\overline{C}(d\phi e_i, d\phi e_i, d\phi R(Fe_n, V_A)V_A) \\
& + \overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A^H}\widetilde{\nabla}_{V_A}d\phi)Fe_n) \\
& - \overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{Fe_n^H}\widetilde{\nabla}_{V_A}d\phi)V_A) \\
& - \overline{C}(d\phi e_i, d\phi e_i + (\widetilde{\nabla}_{Fe_n}d\phi)(\nabla_{V_A^H}V_A)) \\
& + \overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A}d\phi)(\nabla_{Fe_n^H}V_A))\}\, dV_{SM}.
\end{aligned}$$

A direct calculation gives

LEMMA 3.1.

$$\begin{aligned}
\varXi_2 = & \sum_A \frac{1}{C_{n-1}} \int_{SM} \{-\langle d\phi R(e_i, V_A)V_A, d\phi e_i\rangle - \langle \widetilde{\nabla}_{e_i^H}\{(\widetilde{\nabla}_{V_A}d\phi)V_A\}, d\phi e_i\rangle \\
& + \langle \widetilde{\nabla}_{V_A^H}[(\widetilde{\nabla}_{V_A}d\phi)e_i], d\phi e_i\rangle - \langle (\widetilde{\nabla}_{V_A}d\phi)(\nabla_{V_A^H}e_i), d\phi e_i\rangle\}\, dV_{SM}.
\end{aligned}$$

LEMMA 3.2.

$$\begin{aligned}
\varXi_3 = & \sum_A \frac{1}{C_{n-1}} \int_{SM} \{-\overline{C}(d\phi e_i, d\phi e_i, d\phi R(Fe_n, V_A)V_A) \\
& - \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H}\{(\widetilde{\nabla}_{V_A}d\phi)V_A\}) \\
& + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H}[(\widetilde{\nabla}_{V_A}d\phi)Fe_n)]\}\, dV_{SM}.
\end{aligned}$$

We need the following

LEMMA 3.3.

$$\begin{aligned}
-\sum_A \int_{SM} & \{\langle \widetilde{\nabla}_{e_i^H}[(\widetilde{\nabla}_{V_A}d\phi)V_A], d\phi e_i\rangle \\
& + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H}[(\widetilde{\nabla}_{V_A}d\phi)V_A])\}\, dV_{SM} + \varXi_4 = 0.
\end{aligned}$$

Proof. Consider $\psi = \overline{C}(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi)V_A)F\omega^n$. We get

$$\begin{aligned} \operatorname{div}_{\widehat{g}} \psi = {} & (\widetilde{\nabla}_{Fe_n^H} \overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi)V_A) \\ & + 2\overline{C}(\widetilde{\nabla}_{Fe_n^H} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi)V_A) \\ & + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H} [(\widetilde{\nabla}_{V_A} d\phi)V_A]). \end{aligned} \tag{3.12}$$

Integrating (3.12) yields

$$\begin{aligned} \sum_A \int_{SM} & \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n^H} [(\widetilde{\nabla}_{V_A} d\phi)V_A])\, dV_{SM} \\ & = -\sum_A \int_{SM} \{(\widetilde{\nabla}_{Fe_n^H} \overline{C})(d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi)V_A) \\ & \qquad + 2\overline{C}(\widetilde{\nabla}_{Fe_n^H} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi)V_A)\}\, dV_{SM}. \end{aligned} \tag{3.13}$$

Similarly, we also have

$$\begin{aligned} \sum_A \int_{SM} & \langle \widetilde{\nabla}_{e_i^H} [(\widetilde{\nabla}_{V_A} d\phi)V_A], d\phi e_i \rangle\, dV_{SM} \\ & = -\sum_A \int_{SM} \{\langle (\widetilde{\nabla}_{V_A} d\phi)V_A, (\widetilde{\nabla}_{e_i^H} d\phi)e_i \rangle \\ & \qquad + 2\overline{C}((\widetilde{\nabla}_{V_A} d\phi)V_A, d\phi e_i, \widetilde{\nabla}_{e_i^H} d\phi F e_n)\}\, dV_{SM}. \end{aligned} \tag{3.14}$$

Because ϕ is a harmonic map, by (3.13) and (3.14), we immediately obtain

$$\begin{aligned} -\sum_A \int_{SM} & \{\overline{C}(d\phi e_i, d\phi e_i, \nabla_{Fe_n^H} [(\widetilde{\nabla}_{V_A} d\phi)V_A)] \\ & + \langle \widetilde{\nabla}_{e_i^H} [(\widetilde{\nabla}_{V_A} d\phi)V_A], d\phi e_i \rangle\}\, dV_{SM} + \Xi_4 = 0, \end{aligned} \tag{3.15}$$

which completes the proof of Lemma 3.3. ∎

If M is Einstein, then we have

$$\overline{C}(d\phi e_i, d\phi e_i, d\phi R(Fe_n, V_A)V_A) = 0. \tag{3.16}$$

It follows from (3.16) and Lemmas 3.1–3.3 that

$$\begin{aligned} \Xi_2 + \Xi_3 + \Xi_4 = \sum_A \frac{1}{C_{n-1}} \int_{SM} & \{-\langle d\phi R(e_i, V_A)V_A, d\phi e_i \rangle \\ & + \langle \widetilde{\nabla}_{V_A^H} [(\widetilde{\nabla}_{V_A} d\phi)e_i], d\phi e_i \rangle - \langle (\widetilde{\nabla}_{V_A} d\phi)(\nabla_{V_A^H} e_i), d\phi e_i \rangle \\ & + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H} [(\widetilde{\nabla}_{V_A} d\phi)Fe_n])\}\, dV_{SM}. \end{aligned} \tag{3.17}$$

Lemma 3.4.

$$\sum_A \frac{1}{C_{n-1}} \int_{SM} \langle \widetilde{\nabla}_{V_A^H}\{(\widetilde{\nabla}_{V_A} d\phi)e_i\}, d\phi e_i\rangle \, dV_{SM} + \Xi_1$$
$$= \frac{1}{C_{n-1}} \int_{SM} \Big\{ h_{ij}^{\mu} h_{ik}^{\mu} \langle d\phi e_j, d\phi e_k\rangle - \sum_A \langle d\phi(\nabla_{V_A^H} e_i), (\widetilde{\nabla}_{V_A} d\phi)e_i)\rangle$$
$$- \sum_A 2\overline{C}((\widetilde{\nabla}_{V_A} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H}(d\phi F e_n)) + |d\phi|^2 \Big\} \, dV_{SM}.$$

Proof. Let $\psi = \sum_{A,i,j} \langle (\widetilde{\nabla}_{V_A} d\phi)e_i, d\phi e_i\rangle v_A^j \omega^j$. We have

$$\operatorname{div}_{\widehat{g}} \psi = \sum_A \{\langle \nabla_{V_A^H}\{(\widetilde{\nabla}_{V_A} d\phi)e_i\}, d\phi e_i\rangle + \langle (\widetilde{\nabla}_{V_A} d\phi)e_i, \nabla_{V_A^H}(d\phi e_i)\rangle$$
$$+ 2\overline{C}((\widetilde{\nabla}_{V_A} d\phi)e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H}(d\phi F e_n))\}. \tag{3.18}$$

Integrating (3.18) implies

$$\sum_A \int_{SM} \langle \widetilde{\nabla}_{V_A^H}\{(\widetilde{\nabla}_{V_A} d\phi)e_i\}, d\phi e_i\rangle \, dV_{SM}$$
$$= -\sum_A \int_{SM} \{\langle (\widetilde{\nabla}_{V_A} d\phi)e_i, \nabla_{V_A^H}(d\phi e_i)\rangle$$
$$+ 2\overline{C}(\widetilde{\nabla}_{e_i}(d\phi V_A), d\phi e_i, \widetilde{\nabla}_{V_A^H}(d\phi F e_n))\} \, dV_{SM}. \tag{3.19}$$

On the other hand, we also have

$$\Xi_1 = \frac{1}{C_{n-1}} \int_{SM} \Big\{ \sum_A \langle \widetilde{\nabla}_{V_A^H}(d\phi e_i), (\widetilde{\nabla}_{V_A} d\phi)e_i)\rangle$$
$$- \sum_A \langle d\phi(\nabla_{V_A^H} e_i), (\widetilde{\nabla}_{V_A} d\phi)e_i)\rangle$$
$$+ h_{ij}^{\mu} h_{ik}^{\mu} \langle d\phi e_j, d\phi e_k\rangle + |d\phi|^2 \Big\} \, dV_{SM}. \tag{3.20}$$

Combining (3.19) and (3.20) shows Lemma 3.4. ∎

Substituting the formula of Lemma 3.4 into (3.17), we get

$$\Xi_1 + \Xi_2 + \Xi_3 + \Xi_4$$
$$= \frac{1}{C_{n-1}} \int_{SM} \Big\{ -\sum_A \langle d\phi R(e_i, V_A)V_A, d\phi e_i\rangle + h_{ij}^{\mu} h_{ik}^{\mu} \langle d\phi e_j, d\phi e_k\rangle$$
$$+ \sum_A \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H}[(\widetilde{\nabla}_{V_A} d\phi)F e_n]) + |d\phi|^2$$
$$- 2\overline{C}(\widetilde{\nabla}_{e_i}(d\phi V_A), d\phi e_i, \widetilde{\nabla}_{V_A^H}(d\phi F e_n)) \Big\} \, dV_{SM}. \tag{3.21}$$

Obviously, we can obtain

$$\sum_A \frac{1}{C_{n-1}} \int\limits_{SM} \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{V_A^H}[(\widetilde{\nabla}_{V_A} d\phi) F e_n])\, dV_{SM} \tag{3.22}$$

$$= -\sum_A \frac{1}{C_{n-1}} \int\limits_{SM} \{(\widetilde{\nabla}_{V_A^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_A)) + 2\overline{C}(\widetilde{\nabla}_{V_A} d\phi e_i, d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi) F e_n)\}\, dV_{SM}.$$

Substituting (3.22) into (3.21), we get

$$\begin{aligned} &\Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 \\ &= \frac{1}{C_{n-1}} \int\limits_{SM} \Big\{ -\sum_A \langle d\phi R(e_i, V_A) V_A, d\phi e_i \rangle + h^\mu_{ij} h^\mu_{ik} \langle d\phi e_j, d\phi e_k \rangle \\ &\quad - \sum_A (\widetilde{\nabla}_{V_A^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_A)) \\ &\quad - 4\sum_A \overline{C}(\widetilde{\nabla}_{e_i^H}(d\phi V_A), d\phi e_i, \widetilde{\nabla}_{V_A}(d\phi F e_n)) + |d\phi|^2 \Big\}\, dV_{SM}. \end{aligned} \tag{3.23}$$

On the other hand, we have

$$\begin{aligned} &\sum_A \frac{1}{C_{n-1}} \int\limits_{SM} \{(\widetilde{\nabla}_{(d\phi V_A)^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_A) \\ &\qquad + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_A, \widetilde{\nabla}_{Fe_n} d\phi V_A)\}\, dV_{SM} \\ &= \sum_A \frac{1}{C_{n-1}} \int\limits_{SM} \{(\widetilde{\nabla}_{v_A^k u_k^l \phi_l^\alpha (\frac{\partial}{\partial \overline{x}^\alpha} - \overline{N}_\alpha^\beta \frac{\partial}{\partial \overline{Y}^\beta})} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_A) \\ &\qquad + \overline{C}(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n} d\phi V_A, \widetilde{\nabla}_{v_A^k u_k^l (\frac{\partial}{\partial x^l} - N_l^j \frac{\partial}{\partial Y^j})} d\phi F e_n)\}\, dV_{SM} \\ &= \sum_A \frac{1}{C_{n-1}} \int\limits_{SM} (\widetilde{\nabla}_{V_A^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_A))\, dV_{SM}. \end{aligned} \tag{3.24}$$

It follows from (3.24) and (3.7) that

$$\begin{aligned} \Xi_5 = \sum_A \frac{1}{C_{n-1}} \int\limits_{SM} &\{(\widetilde{\nabla}_{V_A^H} \overline{C})(d\phi e_i, d\phi e_i, \widetilde{\nabla}_{Fe_n}(d\phi V_A)) \\ &+ 4\overline{C}(\widetilde{\nabla}_{e_i^H}(d\phi V_A), d\phi e_i, (\widetilde{\nabla}_{V_A} d\phi) F e_n)\}\, dV_{SM}. \end{aligned} \tag{3.25}$$

Combining (3.23) and (3.25) yields

$$\sum_A I(d\phi V_A, d\phi V_A) = \frac{1}{C_{n-1}} \int\limits_{SM} \{n|d\phi|^2 - 2R_{ijik} \langle d\phi e_j, d\phi e_k \rangle\}\, dV_{SM}. \tag{3.26}$$

Put $d\phi e_i = \sum_\alpha a_{\alpha i}\overline{e}_\alpha$ and $X_\alpha = \sum_i a_{\alpha i} e_i$. If $\mathrm{Ric}^M > Q$, then we have

$$\sum_A I(d\phi V_A, d\phi V_A) \le \frac{1}{C_{n-1}} \int\limits_{SM} (n-2Q)|d\phi|^2\, dV_{SM}. \tag{3.27}$$

From (3.27), we immediately obtain

Theorem 3.5. *Let M^n $(n \ge 3)$ be a compact Riemannian minimal submanifold of a Riemannian unit sphere with flat normal bundle. If* $\mathrm{Ric}^M >$ *$n/2$, then there is no non-degenerate stable harmonic map from M to any Finsler manifold.*

When the target manifold is Riemannian, let $\{\overline{e}_\alpha\}$ and $\{e_i\}$ be the orthonormal frame of $\overline{M}$ and M respectively. The second variation formula for the harmonic map $\phi : \overline{M}^m \to M^n$ can be written as

$$I(V,V) = \frac{1}{C_{n-1}} \int\limits_{S\overline{M}} \{\langle \widetilde{\nabla}_{\overline{e}_\alpha} V, \widetilde{\nabla}_{\overline{e}_\alpha} V\rangle - \langle R(d\phi \overline{e}_\alpha, V)V, d\phi\overline{e}_\alpha\rangle\}\, dV_{S\overline{M}}. \tag{3.28}$$

Let $d\phi\overline{e}_\alpha = a_{\alpha i} e_i$, by (3.1). We readily get

$$\begin{aligned} &\sum_A I(V_A, V_A) \\ &= \sum_A \frac{1}{C_{n-1}} \int\limits_{S\overline{M}} \{a_{\alpha i} a_{\alpha j} \langle \nabla_{e_i} V_A, \nabla_{e_j^H} V_A\rangle - a_{\alpha i} a_{\alpha j}\langle R(e_i, V_A)V_A, e_j\rangle\}\, dV_{S\overline{M}} \\ &= \frac{1}{C_{n-1}} \int\limits_{S\overline{M}} \{a_{\alpha i} a_{\alpha j} h^\alpha_{ik} h^\alpha_{jk} + |d\phi|^2 - a_{\alpha i} a_{\alpha j} \langle R(e_i, e_h)e_k, e_j\rangle\}\, dV_{S\overline{M}}. \end{aligned} \tag{3.29}$$

Putting $\mathrm{Ric}^M > Q$, using the Gauss equation and (3.29), we obtain

$$\sum_A I(V_A, V_A) \le \frac{1}{C_{n-1}} \int\limits_{S\overline{M}} (n-2Q)|d\phi|^2 dV_{S\overline{M}}. \tag{3.30}$$

From (3.30), we deduce

Theorem 3.6. *There is no non-degenerate stable harmonic map from any compact Finsler manifold to a Riemannian minimal submanifold M^n $(n \ge 3)$ of a Riemannian unit sphere with* $\mathrm{Ric}^M > n/2$.

Combining Theorems 3.1 and 3.2 shows Theorem 1.1.

Because the Riemannian product $S^m(1/2)\times S^m(1/2)\times S^m(1/2)\times S^m(1/2)$ is a compact minimal submanifold of $S^{4m+3}(1)$ with $\mathrm{Ric}^M = 4(m-1)$ and the Clifford torus $S^m(\sqrt{1/2})\times S^m(\sqrt{1/2})$ is a compact minimal hypersurface of $S^{2m+1}(1)$ with $\mathrm{Ric}^M = 2(m-1)$, we immediately derive

Corollary 3.7. *There is no non-degenerate stable harmonic map between any compact Finsler manifold and a minimal Clifford hypersurface $S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2})$ $(m > 2)$.*

COROLLARY 3.8. *There is no non-degenerate stable harmonic map between any compact Finsler manifold and the Riemannian product* $S^m(1/2) \times S^m(1/2) \times S^m(1/2) \times S^m(1/2)$ $(m > 2)$.

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