ANNALES
POLONICI MATHEMATICI
99.1 (2010)

Forced oscillation of third order nonlinear dynamic equations on time scales

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Abstract. Consider the third order nonlinear dynamic equation

$$x^{\Delta\Delta\Delta}(t)+p(t)f(x)=g(t), \tag{$*$}$$

on a time scale $\mathbb{T}$ which is unbounded above. The function $f\in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy $xf(x)>0$ for $x\neq 0$ and be nondecreasing. We study the oscillatory behaviour of solutions of $(*)$. As an application, we find that the nonlinear difference equation

$$\Delta^3 x(n)+n^\alpha|x|^\gamma \operatorname{sgn}(n)=(-1)^n n^c,$$

where $\alpha\geq -1$, $\gamma>0$, $c>3$, is oscillatory.

1. Introduction. Consider the third order nonlinear dynamic equation

$$x^{\Delta\Delta\Delta}(t)+p(t)f(x)=g(t), \tag{1.1}$$

and the second order nonlinear dynamic equation

$$x^{\Delta\Delta}(t)+p(t)f(x)=g(t), \tag{1.2}$$

on a time scale $\mathbb{T}$ which is unbounded above, and where $p(t),g(t)$ are real-valued, right-dense continuous functions on $\mathbb{T}$ and $p(t)$ is nonnegative but not eventually zero for large t. The function $f\in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy $xf(x)>0$ for $x\neq 0$ and be nondecreasing.

When $\mathbb{T}=\mathbb{R}$, the dynamic equation (1.2) is the second order nonlinear differential equation

$$x''(t)+p(t)f(x)=g(t). \tag{1.3}$$

In [5], James Wong studied the oscillatory behaviour of (1.3) and obtained the following

Theorem 1.1. *Assume that:*

(i) *There exists an $h\in C^2[0,\infty)$ such that $h''(t)=g(t)$ and $h(t)$ is oscillatory, i.e., it has unbounded zeros.*

2010 *Mathematics Subject Classification*: 34K11, 39A10, 39A99.
Key words and phrases: forced oscillation, dynamic equation, time scale.

DOI: 10.4064/ap99-1-7

(ii) *$h(t)$ satisfies*

$$\int_0^\infty p(t)f(h_+(t))\,dt = -\int_0^\infty p(t)f(h_-(t))\,dt = +\infty$$

where $h_+(t) = \max\{h(t), 0\}$ *and* $h_-(t) = \min\{h(t), 0\}$.

Then (1.3) *is oscillatory.*

In this paper, we extend this theorem to third order dynamic equations on time scales and as an application, we show that the nonlinear difference equation

$$\Delta^3 x(n) + n^\alpha |x|^\gamma \operatorname{sgn}(x) = (-1)^n n^c, \quad \gamma > 0, \tag{1.4}$$

is oscillatory, where $\alpha \geq -1$, $c > 3$. This equation is the discrete analog of the differential equation $x'''(t) + t^\alpha |x|^\gamma \operatorname{sgn}(n) = t^c \sin t$.

For completeness we recall some basic results for dynamic equations and the calculus on time scales (see [1] and [2]). Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. If $\sigma(t) = t$ we say t is *right-dense*, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is *left-dense*. The graininess function μ for a time scale $\mathbb{T}$ is defined by

$$\mu(t) = \sigma(t) - t,$$

and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ stands for $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is *differentiable* at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s}$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale), and when x is continuous at t and $\sigma(t) > t$,

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales.

2. Lemma and main theorem

LEMMA 2.1. *Suppose that there exists an* $h \in C^3(\mathbb{T}, \mathbb{R})$ *such that* $h^{\Delta\Delta\Delta}(t) = g(t)$ *and* $h(t)$ *is oscillatory. Let* $x(t)$ *be a solution of* (1.1). *Write* $x(t) = y(t) + h(t)$.

(i) *If* $x(t) > 0$ *for large* t, *then*

$$y(t) > 0, \quad y^{\Delta\Delta}(t) > 0 \quad \text{and} \quad y^{\Delta\Delta\Delta}(t) \le 0, \qquad \text{for large } t.$$

(ii) *If* $x(t) < 0$ *for large* t, *then*

$$y(t) < 0, \quad y^{\Delta\Delta}(t) < 0 \quad \text{and} \quad y^{\Delta\Delta\Delta}(t) \ge 0, \qquad \text{for large } t.$$

REMARK. Under the hypothesis of Lemma 2.1, (1.1) can be rewritten as a homogeneous dynamic equation

$$y^{\Delta\Delta\Delta}(t) + p(t)f(y(t) + h(t)) = 0.$$

Proof. (i) Suppose that $x(t) > 0$ for large t. Since $p(t) \ge 0$, from (1.1) we note that $y^{\Delta\Delta\Delta}(t) \le 0$ for large t. Hence $y^{\Delta\Delta}(t)$ is decreasing for large t. We claim that $y^{\Delta\Delta}(t) \ge 0$ for large t. Assume not; then there is a large t_1 such that $y^{\Delta\Delta}(t_1) < 0$.

Since $y^{\Delta\Delta\Delta}(t) \le 0$, we have $y^{\Delta\Delta}(t) \le y^{\Delta\Delta}(t_1) < 0$ for $t \ge t_1$. By the Mean Value Theorem (see Theorem 1.14 of [2]), we have $y^{\Delta}(t) \to -\infty$ as $t \to \infty$. So $y(t) \to -\infty$ as $t \to \infty$. But this together with $h(t)$ being oscillatory contradicts the assumption that $x(t) > 0$ for large t.

In the following, we show that $y^{\Delta\Delta}(t)$ is eventually positive, i.e., $y^{\Delta\Delta}(t) > 0$ for large t. Suppose that $y^{\Delta\Delta}(t_2) = 0$ for some large t_2. As $y^{\Delta\Delta\Delta}(t) \le 0$ and $y^{\Delta\Delta}(t) \ge 0$, this means that $y^{\Delta\Delta}(t) \equiv 0$. Returning to (1.1), this would imply $p(t) \equiv 0$ for large t, contradicting the assumption.

So $y^{\Delta}(t) > 0$ or $y^{\Delta}(t) < 0$ for large t. Therefore $y(t) > 0$ or $y(t) < 0$ for large t.

Since $h(t)$ is oscillatory and $x(t) > 0$, we have $y(t) > 0$.

Similarly, (ii) also holds. ∎

THEOREM 2.2. *Let* $h(t) = \int_{t_0}^{t}\{\int_{t_0}^{s}[\int_{t_0}^{\tau} g(u)\Delta u]\Delta\tau\}\Delta s$. *Assume that* $h(t)$ *is oscillatory and satisfies*

$$\int_{t_0}^{\infty} p(t)f(h_+(t))\Delta t = +\infty, \tag{2.1}$$

$$\int_{t_0}^{\infty} p(t)f(h_-(t))\Delta t = -\infty, \tag{2.2}$$

where $h_+(t) = \max\{h(t), 0\}$ *and* $h_-(t) = \min\{h(t), 0\}$. *Then* (1.1) *is oscillatory.*

Proof. Assume that the solution $x(t)$ of (1.1) is nonoscillatory. Without loss of generality we can assume that $x(t) > 0$ for large t. Let $x(t) = y(t) + h(t)$. By Theorem 1.74 of [1], we have $h^{\Delta\Delta\Delta}(t) = g(t)$. From (1.1),

$$y^{\Delta\Delta\Delta}(t) + p(t)f(y(t) + h(t)) = 0. \tag{2.3}$$

From Lemma 2.1, we have $y(t) > 0$, $y^{\Delta\Delta}(t) > 0$ and $y^{\Delta\Delta\Delta}(t) \le 0$, $t \in [t_1, \infty)$, for some $t_1 \in \mathbb{T}$. Integrating (2.3), we obtain

$$y^{\Delta\Delta}(t) - y^{\Delta\Delta}(t_1) + \int_{t_1}^{t} p(s)f(y(s) + h(s))\Delta s = 0. \tag{2.4}$$

Since $y^{\Delta\Delta\Delta}(t) \le 0$, $\lim_{t\to\infty} y^{\Delta\Delta}(t)$ exists and is finite; hence the integral in (2.4) converges as $t \to \infty$.

We note that for all $t \ge t_1$, $y(t) + h(t) > h_+(t)$. To see this, we write $y(t) + h(t) = y + h_+(t) + h_-(t)$ and observe that

(i) for $h_+(t) = 0$, $y(t) + h(t) = y(t) + h_-(t) = x(t) > 0 = h_+(t)$,
(ii) for $h_-(t) = 0$, $y(t) + h(t) = y(t) + h_+(t) > h_+(t)$, since $y(t) > 0$.

Since $f(x)$ is nondecreasing, we have $f(y(t) + h(t)) \ge f(h_+(t))$. Note that $p(t) \ge 0$. We now estimate as follows:

$$\int_{t_1}^{t} p(s)f(h_+(s))\Delta s \le \int_{t_1}^{t} p(s)f(y(s) + h(s))\Delta s < \infty. \tag{2.5}$$

By applying (2.1) to (2.5), we obtain the desired contradiction. ∎

Consider the nth order nonlinear dynamic equation

$$x^{\Delta^n}(t) + p(t)f(x) = g(t) \tag{2.6}$$

on a time scale $\mathbb{T}$ which is unbounded above, and where $p(t), g(t)$ are real-valued, right-dense continuous functions on $\mathbb{T}$ and $p(t)$ is nonnegative but not eventually zero for large t, $f(x)$ is a continuous and nondecreasing function of $x \in (-\infty, \infty)$, and $xf(x) > 0$ for $x \ne 0$.

Similar to Lemma 2.1, we have

LEMMA 2.3. *Suppose that there exists an $h \in C^n(\mathbb{T}, \mathbb{R})$ such that $h^{\Delta^n}(t) = g(t)$ and $h(t)$ is oscillatory. Let $x(t)$ be a solution of* (1.1) *and write $x(t) = y(t) + h(t)$.*

(i) *If $x(t) > 0$ for large t, then*

$$y(t) > 0, \quad y^{\Delta^{n-1}}(t) > 0 \quad \text{and} \quad y^{\Delta^n}(t) \le 0, \qquad \text{for large } t.$$

(ii) *If $x(t) < 0$ for large t, then*

$$y(t) < 0, \quad y^{\Delta^{n-1}}(t) < 0 \quad \text{and} \quad y^{\Delta^n}(t) \ge 0, \qquad \text{for large } t.$$

Using Lemma 2.3, we can generalize Theorem 2.2 to the nth order dynamic equation (2.6).

THEOREM 2.4. *Let* $h(t) = \int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{n-1}} g(\tau_n)\Delta\tau_n\Delta\tau_{n-1}\cdots\Delta\tau_1$. *Assume that* $h(t)$ *is oscillatory and satisfies*

$$\int_{t_0}^{\infty} p(t)f(h_+(t))\Delta t = +\infty, \qquad \int_{t_0}^{\infty} p(t)f(h_-(t))\Delta t = -\infty,$$

where $h_+(t) = \max\{h(t), 0\}$ *and* $h_-(t) = \min\{h(t), 0\}$. *Then* (2.6) *is oscillatory.*

3. Example

EXAMPLE 3.1. Consider the third order difference equation

$$\Delta^3 x(n) + p(n)|x|^\gamma \mathrm{sgn}(n) = g(n), \qquad \gamma > 0, \tag{3.1}$$

where $p(n) = n^\alpha$, $\alpha \geq -1$, $g(n) = (-1)^n n^c$, $c > 3$.

We need the following interesting lemma.

LEMMA 3.2. *For each real number* $c > 3$, *we have*

$$\lim_{m\to\infty} \frac{\sum_{i=1}^m i^c - \frac{m^{c+1}}{c+1} - \frac{m^c}{2} - \frac{c}{12}m^{c-1}}{m^{c-3}} = \frac{-c(c-1)(c-2)}{720}. \tag{3.2}$$

Proof. By Taylor's formula, we have

$$\begin{aligned}\left(1+\frac{1}{m}\right)^a &\\ = 1 + \frac{a}{m} &+ \frac{a(a-1)}{2m^2} + \frac{a(a-1)(a-2)}{6m^3}\\ &+ \frac{a(a-1)(a-2)(a-3)}{24m^4} + \frac{a(a-1)(a-2)(a-3)(a-4)}{120m^5} + o\left(\frac{1}{m^5}\right)\end{aligned} \tag{3.3}$$

for any real number a. For $c > 3$, by (3.3) and the Stolz–Cesàro Theorem (see Theorem 1.120 of [1] or Lemma 3.2 of [4]), it is easy to see that

$$\begin{aligned}&\lim_{m\to\infty} \frac{\sum_{i=1}^m i^c - \frac{m^{c+1}}{c+1} - \frac{m^c}{2} - \frac{c}{12}m^{c-1}}{m^{c-3}}\\ &= \lim_{m\to\infty} \frac{(m+1)^c - \frac{(m+1)^{c+1}}{c+1} - \frac{(m+1)^c}{2} - \frac{c(m+1)^{c-1}}{12} + \frac{m^{c+1}}{c+1} + \frac{m^c}{2} + \frac{cm^{c-1}}{12}}{(m+1)^{c-3} - m^{c-3}}\\ &= \lim_{m\to\infty} \frac{\frac{m^3}{2}(1+\frac{1}{m})^c - \frac{m^4}{c+1}(1+\frac{1}{m})^{c+1} - \frac{cm^2}{12}(1+\frac{1}{m})^{c-1} + \frac{m^4}{c+1} + \frac{m^3}{2} + \frac{cm^2}{12}}{(1+\frac{1}{m})^{c-3} - 1}.\end{aligned} \tag{3.4}$$

By (3.3), we have

$$\begin{aligned}\left(1+\frac{1}{m}\right)^c = 1 + \frac{c}{m} &+ \frac{c(c-1)}{2m^2} + \frac{c(c-1)(c-2)}{6m^3}\\ &+ \frac{c(c-1)(c-2)(c-3)}{24m^4} + o\left(\frac{1}{m^4}\right),\end{aligned} \tag{3.5}$$

$$\text{(3.6)}\quad \left(1+\frac{1}{m}\right)^{c+1}$$
$$= 1 + \frac{c+1}{m} + \frac{(c+1)c}{2m^2} + \frac{(c+1)c(c-1)}{6m^3}$$
$$+ \frac{(c+1)c(c-1)(c-2)}{24m^4} + \frac{(c+1)c(c-1)(c-2)(c-3)}{120m^5} + o\left(\frac{1}{m^5}\right),$$

$$\text{(3.7)}\quad \left(1+\frac{1}{m}\right)^{c-1}$$
$$= 1 + \frac{c-1}{m} + \frac{(c-1)(c-2)}{2m^2} + \frac{(c-1)(c-2)(c-3)}{6m^3} + o\left(\frac{1}{m^3}\right).$$

Using (3.5)–(3.7) in (3.4), it follows that (3.2) holds. ∎

Let $C_1 := -c(c-1)(c-2)/720$. Given $0 < \epsilon < 1$, for large m, we have the inequalities

$$\text{(3.8)}\qquad \sum_{i=1}^{m} i^c < \frac{m^{c+1}}{c+1} + \frac{m^c}{2} + \frac{cm^{c-1}}{12} + (C_1 + \epsilon)m^{c-3},$$

$$\text{(3.9)}\qquad \sum_{i=1}^{m} i^c > \frac{m^{c+1}}{c+1} + \frac{m^c}{2} + \frac{cm^{c-1}}{12} + (C_1 - \epsilon)m^{c-3}.$$

Therefore for $t = m$, by integrating by parts we have

$$\text{(3.10)}\qquad h(t) = \int_1^t\int_1^s\int_1^\tau g(u)\Delta u\Delta\tau\Delta s$$
$$= t\int_1^t\Big[\int_1^\tau p(u)\Delta u\Big]\Delta\tau - \int_1^t \sigma(s)\Big[\int_1^s g(u)\Delta u\Big]\Delta s$$
$$= t^2\int_1^t p(u)\Delta u - t\int_1^t \sigma(\tau)p(\tau)\Delta\tau$$
$$- \int_1^t \sigma(\tau)\Delta\tau\int_1^t p(u)\Delta u + \int_1^t\Big[\int_1^{\sigma(s)} \sigma(\tau)\Delta\tau\Big]g(s)\Delta s$$
$$= \left(\frac{m^2}{2} - \frac{3m}{2} + 1\right)\sum_{i=1}^{m-1}(-1)^i i^c - \left(m - \frac{3}{2}\right)\sum_{i=1}^{m-1}(-1)^i i^{c+1}$$
$$+ \frac{1}{2}\sum_{i=1}^{m-1}(-1)^i i^{c+2}.$$

Letting $m = 2k$, we get

$$
\begin{aligned}
(3.11)\qquad & h(2k)\\
&= (2k^2-3k+1)\sum_{i=1}^{2k-1}(-1)^i i^c - \left(2k-\frac{3}{2}\right)\sum_{i=1}^{2k-1}(-1)^i i^{c+1} + \frac{1}{2}\sum_{i=1}^{2k-1}(-1)^i i^{c+2}\\
&= (2k^2-3k+1)\sum_{i=1}^{2k-2}(-1)^i i^c - \left(2k-\frac{3}{2}\right)\sum_{i=1}^{2k-2}(-1)^i i^{c+1} + \frac{1}{2}\sum_{i=1}^{2k-2}(-1)^i i^{c+2}
\end{aligned}
$$

since the terms corresponding to $i = 2k-1$ cancel. From (3.8) and (3.9), it is easy to see that

$$
\begin{aligned}
(3.12)\qquad & \sum_{i=1}^{2k-2}(-1)^i i^c = -\sum_{i=1}^{2k-2} i^c + 2^{1+c}\sum_{i=1}^{k-1} i^c\\
&\le -\left[\frac{(2k-2)^{c+1}}{c+1} + \frac{(2k-2)^c}{2} + \frac{c(2k-2)^{c-1}}{12} + (C_1-\epsilon)(2k-2)^{c-3}\right]\\
&\quad + 2^{1+c}\left[\frac{(k-1)^{c+1}}{c+1} + \frac{(k-1)^c}{2} + \frac{c(k-1)^{c-1}}{12} + (C_1+\epsilon)(k-1)^{c-3}\right]\\
&= \frac{(2k-2)^c}{2} + \frac{c(2k-2)^{c-1}}{4} + [C_1(2^4-1)+\epsilon(2^4+1)](2k-2)^{c-3}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.13)\qquad \sum_{i=1}^{2k-2}(-1)^i i^c &\ge \frac{(2k-2)^c}{2} + \frac{c(2k-2)^{c-1}}{4}\\
&\quad + [C_1(2^4-1)-\epsilon(2^4+1)](2k-2)^{c-3}.
\end{aligned}
$$

From (3.11)–(3.13), we get

$$
\begin{aligned}
h(2k) &\le (2k^2-3k+1)\left[\frac{(2k-2)^c}{2} + \frac{c(2k-2)^{c-1}}{4}\right.\\
&\qquad \left. + [C_1(2^4-1)+\epsilon(2^4+1)](2k-2)^{c-3}\right]\\
&\quad - \left(2k-\frac{3}{2}\right)\left[\frac{(2k-2)^{c+1}}{2} + \frac{(c+1)(2k-2)^c}{4}\right.\\
&\qquad \left. + [C_1(2^4-1)-\epsilon(2^4+1)](2k-2)^{c-2}\right]\\
&\quad + \frac{1}{2}\left[\frac{(2k-2)^{c+2}}{2} + \frac{(c+2)(2k-2)^{c+1}}{4}\right.\\
&\qquad \left. + [C_1(2^4-1)+\epsilon(2^4+1)](2k-2)^{c-1}\right]\\
&= (2k-2)^c\left[-\frac{1}{8} + O\left(\frac{1}{k}\right)\right].
\end{aligned}
$$

Take $0 < A < 1/8$. Then

$$h(2k) \leq -A(2k-2)^c \quad \text{for large } k. \tag{3.14}$$

Letting $m = 2k+1$, from (3.10), (3.12) and (3.13) we get

$$\begin{aligned}
(3.15)\qquad & h(2k+1) \\
&= (2k^2-k)\sum_{i=1}^{2k}(-1)^i i^c - \left(2k-\frac{1}{2}\right)\sum_{i=1}^{2k}(-1)^i i^{c+1} + \frac{1}{2}\sum_{i=1}^{2k}(-1)^i i^{c+2} \\
&\geq (2k^2-k)\left[\frac{(2k)^c}{2} + \frac{c(2k)^{c-1}}{4} + [C_1(2^4-1) - \epsilon(2^4+1)](2k)^{c-3}\right] \\
&\quad - \left(2k-\frac{1}{2}\right)\left[\frac{(2k)^{c+1}}{2} + \frac{(c+1)(2k)^c}{4} + [C_1(2^4-1) + \epsilon(2^4+1)](2k)^{c-2}\right] \\
&\quad + \frac{1}{2}\left[\frac{(2k)^{c+2}}{2} + \frac{(c+2)(2k)^{c+1}}{4} + [C_1(2^4-1) - \epsilon(2^4+1)](2k)^{c-1}\right] \\
&= (2k)^c\left[\frac{1}{8} + O\left(\frac{1}{k}\right)\right].
\end{aligned}$$

Take $0 < B < 1/8$. Then

$$h(2k+1) \geq B(2k)^c \quad \text{for large } k. \tag{3.16}$$

From (3.14) and (3.16), we deduce that for large k,

$$\begin{aligned}
h_+(2k) &= 0, & h_+(2k+1) &\geq B(2k)^c, \\
h_-(2k) &\leq -A(2k-2)^c, & h_-(2k+1) &= 0.
\end{aligned}$$

So picking a large K, we have (note that $\alpha + c\gamma \geq \alpha \geq -1$)

$$\begin{aligned}
\int_K^\infty p(s)f(h_+(s))\Delta s &\geq \sum_{k=K}^\infty (2k+1)^\alpha B^\gamma (2k)^{c\gamma} = +\infty, \\
\int_K^\infty p(s)f(h_-(s))\Delta s &\leq -\sum_{k=K}^\infty (2k)^\alpha A^\gamma (2k-2)^{c\gamma} = -\infty.
\end{aligned}$$

From (3.14) and (3.16), we conclude that $h(t)$ is oscillatory. Therefore by Theorem 2.2, all solutions of equation (3.1) are oscillatory.

Remark. When $c = 3$, using the formulas

$$\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2,$$

$$\sum_{i=4}^{n} i^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1),$$

$$\sum_{i=5}^{n} i^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2+2n-1),$$

by (3.11), (3.15) and a complicated calculation, we can get

$$h(2k) = 51k^3 - \frac{55}{2}k^2 - \frac{3}{2}k, \qquad h(2k+1) = k^3 - \frac{7}{2}k^2 - \frac{1}{2}k.$$

It is easy to see that $h(m)$ does not satisfy the assumption of Theorem 2.2.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (No. 10971232).

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[2] M. Bohner and A. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[3] B. G. Jia, *Forced oscillation of second order nonlinear dynamic equations on time scales*, submitted.

[4] B. G. Jia, L. Erbe and A. Peterson, *Oscillation of sublinear Emden–Fowler dynamic equations on time scales*, J. Difference Equations Appl. 16 (2010), 217–226.

[5] J. S. W. Wong, *Second order nonlinear forced oscillations*, SIAM J. Math. Anal. 19 (1988), 667–675.

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Received 20.10.2009
and in final form 10.12.2009 (2105)