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New versions of curvature and torsion formulas for the complete lifting of a linear connection to Weil bundles

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Abstract. New versions of Slovák's formulas expressing the covariant derivative and curvature of the linear connection $\mathcal{T}_A\Gamma$ are presented.

1. Introduction. Let T_A be a Weil functor and consider a linear connection Γ on a vector bundle (E, M, π); one defines (see [3] or [7]) the linear connection $\mathcal{T}_A\Gamma$ on $(T_AE, T_AM, T_A\pi)$ by

$$\mathcal{T}_A\Gamma = \kappa_E \circ T_A\Gamma \circ (\kappa_M^{-1} \times_{T_AM} \mathrm{id}_{T_AE}) : TT_AM \times_{T_AM} T_AE \to TT_AE,$$

where $\kappa : T_AT \to TT_A$ is the canonical flow natural equivalence.

The main results of this paper are Propositions 6 and 7 giving new versions of formulas expressing the covariant derivative and curvature of $\mathcal{T}_A\Gamma$. In the case $E = TM$, we obtain some results of [2] and [1] (Corollaries 3 and 4).

2. Weil functor

2.1. *Weil algebra*

Definition 1. A *Weil algebra* is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C_0^\infty(\mathbb{R}^p, \mathbb{R})$ $(p \in \mathbb{N}^*)$.

We denote by $\mathcal{M}_p$ the maximal ideal of $\mathcal{E}_p$.

Example 1. (1) $\mathbb{R}$ is a Weil algebra since it is canonically isomorphic to the quotient $\mathcal{E}_p/\mathcal{M}_p$.

(2) $J_0^r(\mathbb{R}^p, \mathbb{R}) = \mathcal{E}_p/\mathcal{M}_p^{r+1}$ is a Weil algebra.

2.2. *Covariant description of a Weil functor $T_A : \mathcal{M}f \to \mathcal{FM}$.* We write $\mathcal{M}f$ for the category of differentiable manifolds and mappings of class C^∞;

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furthermore, $\mathcal{FM}$ is the category of fibered manifolds and fibered manifold morphisms.

Let $A=\mathcal{E}_p/I$ be a Weil algebra and consider a manifold M. In $C^\infty(\mathbb{R}^p, M)$ one defines an equivalence relation $\mathcal{R}$ by: $\varphi\mathcal{R}\psi$ if and only if $\varphi(0) = \psi(0) = x$ and for any $[h]_x \in C^\infty_x(M,\mathbb{R})$, $[h]_x \circ [\psi]_0 - [h]_x \circ [\varphi]_0 \in I$.

The equivalence class of φ is denoted by $j_A\varphi$ and is called the *A-velocity* at 0 of φ; the class $j_A\varphi$ depends only on the germ of φ at 0. The quotient $C^\infty(\mathbb{R}^p, M)/\mathcal{R}$ is denoted by T_AM.

The mapping $\pi_{A,M} : T_AM \to M$, $j_A\varphi \mapsto \varphi(0)$, defines a bundle structure on T_AM and for any differentiable mapping $f : M \to N$, one defines a bundle morphism $T_Af : T_AM \to T_AN$ (over f) by $T_Af(j_A(\varphi)) = j_A(f \circ \varphi)$.

The correspondence $T_A : \mathcal{M}f \to \mathcal{FM}$ is a product preserving bundle functor (see [3]).

Example 2. If $A = J^r_0(\mathbb{R}^p, \mathbb{R})$, then T_A is equivalent to the functor T^r_p of (p, r)-velocities, and if $A = \mathcal{E}_p/\mathcal{M}^2_p$, then $T_A = T$, the tangent bundle functor.

2.3. *The canonical flow-natural equivalence.* Let T_A, T_B be two Weil functors. Our purpose here is to make explicit a natural equivalence

$$\kappa : T_A \circ T_B \to T_B \circ T_A.$$

Lemma 1 ([3]). *Let M be a manifold. For any $\zeta = j_A\varphi \in T_AT_BM$, there is a differentiable mapping $\Phi : \mathbb{R}^p \times \mathbb{R}^q \to M$ such that $\varphi(z) = j_B\Phi_z$ in a neighbourhood of* $0 \in \mathbb{R}^p$.

By this lemma, one defines $\kappa : T_A \circ T_B \to T_B \circ T_A$ as follows:

$$\kappa_M(\zeta) = j_B\eta,$$

where $\eta : \mathbb{R}^q \to T_AM$, $t \mapsto j_A\Phi^t$. It is a well-defined natural equivalence. In particular, for $T_B = T$, we obtain the canonical flow-natural equivalence.

3. Prolongations of tensor fields of type $(1, s)$.

In this section, A is a Weil algebra, i.e. $\mathcal{E}_p/I$ with $\mathcal{M}_p \supset I \supset \mathcal{M}^{r+1}_p$ and r minimal; $\mathcal{VB}$ is the category of vector bundles and vector bundle homomorphisms. The module of differentiable sections of a vector bundle (E, M, π) is denoted here by $\mathcal{S}ec(M, E)$.

3.1. *The functor $T_A : \mathcal{VB} \to \mathcal{VB}$.* It is defined as follows:

(1) $$T_A(E, M, \pi) = (T_AE, T_AM, T_A\pi), \qquad T_A(\bar{f}, f) = (T_A\bar{f}, T_Af)$$

see ([2] and [4]).

3.2. *Natural transformations $\chi_\alpha : T_A \to T_A$.* Consider a vector bundle (E, M, π). For any multi-index $\alpha \in \mathbb{N}^p$ such that $|\alpha| \le r$, we put

(2) $$(\chi_\alpha)_E(j_Af) = j_A(z^\alpha f),$$

where $f : \mathbb{R}^p \to E$ is C^∞ and $z^\alpha f : \mathbb{R}^p \to E,\ z \mapsto z^\alpha f(z) \in E_{\pi(f(0))}$. One defines in this way some natural transformations $\chi_\alpha : T_A \to T_A$, since each $(\chi_\alpha)_E$ is a vector bundle morphism over $\mathrm{id}_{T_A M}$.

Proposition 1. *For any multi-index $\alpha \in \mathbb{N}^p$ such that $|\alpha| \le r$, the diagram*

$$\begin{array}{ccc} T_A T E & \xrightarrow{(\chi_\alpha)_{TE}} & T_A T E \\ {\scriptstyle \kappa_E}\downarrow & & \downarrow{\scriptstyle \kappa_E} \\ T T_A E & \xrightarrow{T((\chi_\alpha)_E)} & T T_A E \end{array}$$

is commutative, where $\kappa : T_A T \to T T_A$ is the canonical flow-natural equivalence.

3.3. *Prolongation of tensor fields of type* $(1,0)$. Consider a $\mathcal{VB}$-object (E, M, π) and a differentiable section $S : M \to E$. One defines the following prolongations of S on $(T_A E, T_A M, T_A \pi)$:

$$S^{(0)} = T_A S \text{ (since } (\chi_0)_E = \mathrm{id}_{T_A E}), \qquad S^{(\alpha)} = (\chi_\alpha)_E \circ T_A S,\ 1 \le |\alpha| \le r, \tag{3}$$

where χ_α is the natural transformation (2). If $|\alpha| > r$, then $S^{(\alpha)} := 0_{T_A E}$.

Let $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialisation of E and $\varepsilon_j(x) = \varphi^{-1}(x, e_j)$, $1 \le j \le n$, a basis of sections of E over U associated to φ (here (e_j), $1 \le j \le n$, is the usual basis of $\mathbb{R}^n$). Using the identification $T_A(U \times \mathbb{R}^n) \cong T_A U \times T_A \mathbb{R}^n$, one defines a family of sections

$$(\varepsilon_{j,\alpha}), \qquad |\alpha| \le r,\ 1 \le j \le n,$$

of $T_A E$ over $T_A U$ by

$$\varepsilon_{j,\alpha}(\widetilde{x}) = T_A \varphi^{-1}(\widetilde{x}, e_{j\alpha}), \tag{4}$$

where $e_{j\alpha} = j_A(z^\alpha e_j)$. Then

$$\varepsilon_j^{(\alpha)} = \varepsilon_{j,\alpha}, \qquad 1 \le j \le n \text{ and } |\alpha| \le r,$$

and we deduce

Proposition 2. *The $S^{(\alpha)}$, $|\alpha| \le r$, and $S \in \mathcal{S}ec(M, E)$ generate the $C^\infty(T_A M)$-module $\mathcal{S}ec(T_A M, T_A E)$.*

3.4. *Prolongation of vector fields.* Let T_A be a Weil functor and $\kappa : T_A \circ T \to T \circ T_A$ the canonical flow-natural equivalence. If X is a vector field on M, one defines $\binom{p+r}{r}$ vector fields on $T_A M$ by

$$X^{\mathrm{c}} = \kappa_M \circ T_A X, \qquad X^{(\alpha)} = \kappa_M \circ (\chi_\alpha)_{TM} \circ T_A X \quad \text{for } 1 \le |\alpha| \le r. \tag{5}$$

Proposition 3. *Let T_A be a Weil functor and M a manifold.*

(i) *The vector fields $X^{(\alpha)}$ for $|\alpha| \le r$ and $X \in \mathfrak{X}(M)$ generate $\mathfrak{X}(T_A M)$ over $C^\infty(T_A M)$.*

(ii) *For any* $X, Y \in \mathfrak{X}(M)$, *we have*

$$[X^{(\alpha)}, Y^{(\beta)}] = \begin{cases} [X,Y]^{(\alpha+\beta)} & \text{if } 0 \le |\alpha+\beta| \le r, \\ 0 & \text{if } |\alpha+\beta| > r. \end{cases}$$

Proof. This is a modification of some result of [2]. ∎

3.5. *Natural transformations* $\overline{\chi}_\alpha : T_A \circ \otimes_s^1 \to \otimes_s^1 \circ T_A$. They are defined as follows:

$$(\overline{\chi}_\alpha)_E(j_A\varphi)(j_A\eta_1, \ldots, j_A\eta_s) = (\chi_\alpha)_E(j_A(\varphi * (\eta_1, \ldots, \eta_s))) \tag{6}$$

for any vector bundle E, where $\varphi : \mathbb{R}^p \to \otimes_s^1 E$, $\eta_1, \ldots, \eta_s : \mathbb{R}^p \to E$ are C^∞ and

$$\varphi * (\eta_1, \ldots, \eta_s) : \mathbb{R}^p \to E, \quad z \mapsto \varphi(z)(\eta_1(z), \ldots, \eta_s(z)).$$

3.6. *Prolongation of tensor fields of type* $(1, s)$. Let φ be a tensor field of type $(1, s)$ on E. We put

$$\varphi^{(\alpha)} = (\overline{\chi}_\alpha)_E \circ T_A\varphi, \quad 0 \le |\alpha| \le r; \tag{7}$$

then $\varphi^{(\alpha)}$ is a tensor field of type $(1, s)$ on T_AE. In particular if $E = TM$, we put

$$\overline{\varphi}^{(\alpha)} = (\otimes_s^1 \kappa_M) \circ (\overline{\chi}_\alpha)_{TM} \circ T_A\varphi, \quad 0 \le |\alpha| \le r; \tag{8}$$

then $\overline{\varphi}^{(\alpha)}$ is a tensor field of type $(1, s)$ on T_AM.

PROPOSITION 4. *Let* φ *be a tensor field on* E *of type* $(1, s)$; *then* $\varphi^{(\alpha)}$, $0 \le |\alpha| \le r$, *is the unique tensor field on* $(T_AE, T_AM, T_A\pi)$ *of type* $(1, s)$ *such that*

$$\varphi^{(\alpha)}(S_1^{(\alpha_1)}, \ldots, S_s^{(\alpha_s)}) = (\varphi(S_1, \ldots, S_s))^{(\alpha+\alpha_1+\ldots+\alpha_s)} \tag{9}$$

for any $S_1, \ldots, S_s \in \mathcal{S}ec(M, E)$ *and* $\alpha_1, \ldots, \alpha_s \in \mathbb{N}^p$ *with* $0 \le |\alpha_1|, \ldots, |\alpha_s| \le r$.

Proof. $\varphi^{(\alpha)}$ is unique by Proposition 2; moreover

$$\begin{aligned}
&\varphi^{(\alpha)}(S_1^{(\alpha_1)}, \ldots, S_s^{(\alpha_s)})(j_A\eta) \\
&\quad = \varphi^{(\alpha)}(j_A\eta)(S_1^{(\alpha_1)}(j_A\eta), \ldots, S_s^{(\alpha_s)}(j_A\eta)) \\
&\quad = (\overline{\chi}_\alpha)_E(j_A(\varphi \circ \eta))(S_1^{(\alpha_1)}(j_A\eta), \ldots, S_s^{(\alpha_s)}(j_A\eta)) && \text{by (7)} \\
&\quad = (\overline{\chi}_\alpha)_E(j_A(\varphi \circ \eta))(j_A(z^{\alpha_1}S_1 \circ \eta), \ldots, j_A(z^{\alpha_s}S_s \circ \eta)) && \text{by (2), (3)} \\
&\quad = (\chi_\alpha)_E(j_A((\varphi \circ \eta) * (z^{\alpha_1}S_1 \circ \eta, \ldots, z^{\alpha_s}S_s \circ \eta))) && \text{by (6)} \\
&\quad = (\chi_\alpha)_E(j_A(z^{\alpha_1+\ldots+\alpha_s}(\varphi \circ \eta) * (S_1 \circ \eta, \ldots, S_s \circ \eta)))
\end{aligned}$$

$$
\begin{aligned}
&= j_A(z^{\alpha+\alpha_1+\ldots+\alpha_s}(\varphi\circ\eta)*(S_1\circ\eta,\ldots,S_s\circ\eta)) \quad \text{by (2)}\\
&= j_A(z^{\alpha+\alpha_1+\ldots+\alpha_s}\varphi(S_1,\ldots,S_s)\circ\eta)\\
&= (\chi_{\alpha+\alpha_1+\ldots+\alpha_s})_E(j_A(\varphi(S_1,\ldots,S_s)\circ\eta)) \quad \text{by (2)}\\
&= (\chi_{\alpha+\alpha_1+\ldots+\alpha_s})_E(T_A\varphi(S_1,\ldots,S_s)(j_A\eta))\\
&= (\varphi(S_1,\ldots,S_s))^{(\alpha+\alpha_1+\ldots+\alpha_s)}(j_A\eta). \ \blacksquare
\end{aligned}
$$

COROLLARY 1. *Let φ be a tensor field on M of type $(1,s)$; then $\overline{\varphi}^{(\alpha)}$, $0\le|\alpha|\le r$, is the unique tensor field on T_AM of type $(1,s)$ such that*

$$\overline{\varphi}^{(\alpha)}(X_1^{(\alpha_1)},\ldots,X_s^{(\alpha_s)}) = (\varphi(X_1,\ldots,X_s))^{(\alpha+\alpha_1+\ldots\alpha_s)} \tag{10}$$

for any $X_1,\ldots,X_s\in\mathfrak{X}(M)$ and $\alpha_1,\ldots,\alpha_s\in\mathbb{N}^p$ satisfying $0\le|\alpha_1|,\ldots,|\alpha_s|\le r$.

3.7. Prolongations of sections of the vector bundle $\bigwedge^s T^*M\otimes(\otimes_1^1 E)$ over M

3.7.1. *Canonical morphisms $\chi_{\alpha,E}: T_A(\bigwedge^s T^*M\otimes(\otimes_1^1E))\to\bigwedge^s T^*T_AM\otimes(\otimes_1^1T_AE)$.* Let (E,M,π) be a vector bundle. One defines some vector bundle morphisms over T_AM,

$$\chi_{\alpha,E}: T_A(\textstyle\bigwedge^sT^*M\otimes(\otimes_1^1E))\to\bigwedge^sT^*T_AM\otimes(\otimes_1^1T_AE),\quad 0\le|\alpha|\le r,$$

with natural transformations

$$\overline{\chi}_\alpha: T_A\circ\otimes_1^1\to\otimes_1^1\circ T_A,\quad 0\le|\alpha|\le r,$$

by

$$\chi_{\alpha,E}(j_A\Phi)(\kappa_M(j_A\varphi_1),\ldots,\kappa_M(j_A\varphi_s)) = (\overline{\chi}_\alpha)_E(j_A(\Phi*(\varphi_1,\ldots,\varphi_s))), \tag{11}$$

where $\Phi:\mathbb{R}^p\to\bigwedge^sT^*M\otimes(\otimes_1^1E)$, $\varphi_i:\mathbb{R}^p\to TM$, $1\le i\le s$, are C^∞ and

$$\Phi*(\varphi_1,\ldots,\varphi_s):\mathbb{R}^p\to\otimes_1^1E,\quad z\mapsto\Phi(u)(\varphi_1(z),\ldots,\varphi_s(z)).$$

3.7.2. *Prolongation of sections.* Let (E,M,π) be a vector bundle and R an $E^*\otimes E$-valued differential form on M of degree s. One defines a $(T_AE)^*\otimes T_AE$-valued differential form $R^{(\alpha)}$ on T_AM of degree s by

$$R^{(\alpha)} = \chi_{\alpha,E}\circ T_AR,\quad 0\le|\alpha|\le r. \tag{12}$$

REMARK 1. Assume that $E=TM$. We put

$$
\begin{aligned}
\overline{R}^{(\alpha)} &= \textstyle\bigwedge^s \mathrm{id}_{T^*T_AM}\otimes(\otimes_1^1\kappa_M)\circ\chi_{\alpha,TM}\circ T_AR\\
&= \textstyle\bigwedge^s \mathrm{id}_{T^*T_AM}\otimes(\otimes_1^1\kappa_M)\circ R^{(\alpha)}.
\end{aligned}
\tag{13}
$$

This is a $T^*T_AM\otimes TT_AM$-valued differential form on T_AM of degree s. We denote $\overline{R}^{(0)}$ by R^{c} and call it the *canonical lift* (or *complete lift*) *of R to T_AM.*

Proposition 5. *$R^{(\alpha)}$ is the unique $(T_AE)^* \otimes T_AE$-valued differential form on T_AM of degree s such that*

$$R^{(\alpha)}(X_1^{(\beta_1)},\dots,X_s^{(\beta_s)})S^{(\gamma)} = (R(X_1,\dots,X_s)S)^{(\alpha+\beta_1+\dots+\beta_s+\gamma)}$$

for any $X_1,\dots,X_s \in \mathfrak{X}(M)$, $S \in \mathcal{S}ec(M,E)$ and any multi-indices $\beta_1,\dots,\beta_s$, $\gamma \in \mathbb{N}^p$ satisfying $0 \le |\beta_1|,\dots,|\beta_s|,|\gamma| \le r$.

Proof. We just deal with the case $s = 2$. Put

$$K = \chi_{\alpha,E}(j_A(R\circ\eta))(\kappa_M(j_A(u^{\beta_1}X_1\circ\eta)),\kappa_M(j_A(u^{\beta_2}X_2\circ\eta)))S^{(\gamma)}(j_A\eta);$$

then

$$\begin{aligned}
&(R^{(\alpha)}(X_1^{(\beta_1)},X_2^{(\beta_2)})S^{(\gamma)})(j_A\eta)\\
&= R^{(\alpha)}(j_A\eta)(X_1^{(\beta_1)}(j_A\eta),X_2^{(\beta_2)}(j_A\eta))S^{(\gamma)}(j_A\eta)\\
&= K && \text{by (2), (5)}\\
&= (\overline{\chi}_\alpha)_E(j_A(R\circ\eta * (z^{\beta_1}X_1\circ\eta, z^{\beta_2}X_2\circ\eta)))(j_A(z^\gamma S\circ\eta)) && \text{by (11)}\\
&= (\chi_\alpha)_E(j_A((R\circ\eta * (z^{\beta_1}X_1\circ\eta, z^{\beta_2}X_2\circ\eta)) * z^\gamma S\circ\eta)) && \text{by (6)}\\
&= (\chi_\alpha)_E(j_A(z^{\beta_1+\beta_2+\gamma}R(X_1,X_2)S\circ\eta))\\
&= (\chi_\alpha)_E((\chi_{\beta_1+\beta_2+\gamma})_E(j_A(R(X_1,X_2)S\circ\eta)))\\
&= (\chi_{\alpha+\beta_1+\beta_2+\gamma})_E\circ T_A(R(X_1,X_2)S)(j_A\eta)\\
&= (R(X_1,X_2)S)^{(\alpha+\beta_1+\beta_2+\gamma)}
\end{aligned}$$

for any $j_A\eta \in T_AM$. The uniqueness of $R^{(\alpha)}$ follows from Proposition 3. ∎

Corollary 2. *Let R be an s-differential form on T_AM with values in $T^*M \otimes TM$. Then $\overline{R}^{(\alpha)}$ is the unique $T^*T_AM \otimes TT_AM$-differential form on T_AM of degree s such that*

$$\overline{R}^{(\alpha)}(X_1^{(\beta_1)},\dots,X_s^{(\beta_s)})Y^{(\gamma)} = (R(X_1,\dots,X_s)Y)^{(\alpha+\beta_1+\dots\beta_s+\gamma)}$$

for any $X_1,\dots,X_s,Y \in \mathfrak{X}(M)$ and any multi-indices $\beta_1,\dots\beta_s,\gamma \in \mathbb{N}^p$ satisfying $0 \le |\beta_1|,\dots,|\beta_s|,|\gamma| \le r$.

4. Main results. We denote by Φ the vertical projection and by K the connector of a linear connection Γ on a vector bundle (E,M,π).

Proposition 6. *Let Γ be a linear connection on a vector bundle (E,M,π), ∇ the covariant derivative associated to Γ and $\widetilde{\nabla}$ the covariant derivative associated to $\mathcal{T}_A\Gamma$. Then $\mathcal{T}_A\Gamma$ is the unique linear connection on $(T_AE,T_AM,T_A\pi)$ such that*

$$\widetilde{\nabla}_{X^{(\alpha)}}S^{(\beta)} = \begin{cases}(\nabla_X S)^{(\alpha+\beta)}, & \alpha,\beta\in\mathbb{N}^p,\ 0\le|\alpha+\beta|\le r,\\ 0, & \alpha,\beta\in\mathbb{N}^p,\ |\alpha+\beta|>r,\end{cases}$$

where $S \in \mathcal{S}ec(M,E)$ and $X \in \mathfrak{X}(M)$.

Proof. $\mathcal{T}_A\Gamma$ is unique by Propositions 2 and 3(i); moreover

$$\begin{aligned}
\widetilde{\nabla}_{X^{(\alpha)}}S^{(\beta)} &= T_AK\circ\kappa_E^{-1}\circ T(S^{(\beta)})\circ X^{(\alpha)}\\
&= T_AK\circ\kappa_E^{-1}\circ T(S^{(\beta)})\circ\kappa_M\circ(\chi_\alpha)_{TM}\circ T_AX \qquad \text{by (5)}\\
&= T_AK\circ\kappa_E^{-1}\circ T((\chi_\beta)_E\circ T_AS)\circ\kappa_M\circ(\chi_\alpha)_{TM}\circ T_AX \qquad \text{by (3)}\\
&= T_AK\circ\kappa_E^{-1}\circ T((\chi_\beta)_E)\circ T(T_AS)\circ\kappa_M\circ(\chi_\alpha)_{TM}\circ T_AX\\
&= T_AK\circ\kappa_E^{-1}\circ T((\chi_\beta)_E)\circ\kappa_E\circ T_A(TS)\circ(\chi_\alpha)_{TM}\circ T_AX\\
&\qquad\qquad \text{by the definition of } \kappa\\
&= T_AK\circ(\chi_\beta)_{TE}\circ T_A(TS)\circ(\chi_\alpha)_{TM}\circ T_AX \qquad \text{by Proposition 1}\\
&= T_AK\circ(\chi_\beta)_{TE}\circ(\chi_\alpha)_{TE}\circ T_A(TS)\circ T_AX\\
&= (\chi_\beta)_E\circ T_AK\circ(\chi_\alpha)_{TE}\circ T_A(TS)\circ T_AX\\
&= (\chi_\beta)_E\circ(\chi_\alpha)_E\circ T_AK\circ T_A(TS)\circ T_AX\\
&= (\chi_{\alpha+\beta})_E\circ T_A(\nabla_XS). \ \blacksquare
\end{aligned}$$

PROPOSITION 7. *Let R_∇ be the curvature tensor of a linear connection Γ on (E,M,π) and $\widetilde{\nabla}$ the covariant derivative associated to $\mathcal{T}_A\Gamma$. Then the curvature tensor $R_{\widetilde{\nabla}}$ of $\widetilde{\nabla}$ satisfies*

$$R_{\widetilde{\nabla}} = (R_\nabla)^{(0)}.$$

Proof. Since $R_{\widetilde{\nabla}}(\overline{X},\overline{Y})\overline{S} = \widetilde{\nabla}_{\overline{X}}\widetilde{\nabla}_{\overline{Y}}\overline{S}-\widetilde{\nabla}_{\overline{Y}}\widetilde{\nabla}_{\overline{X}}\overline{S}-\widetilde{\nabla}_{[\overline{X},\overline{Y}]}\overline{S}$ for any $\overline{X},\overline{Y}\in\mathfrak{X}(T_AM)$ and $\overline{S}\in\mathcal{S}ec(T_AM,T_AE)$, we apply Propositions 3(ii) and 6 to show that $R_{\widetilde{\nabla}}$ satisfies the conclusion of Proposition 5. ■

REMARK. In particular, let Γ be a linear connection on M (i.e. $E = TM$); one can define a linear connection Γ^{c} on T_AM by

$$\Gamma^{\mathrm{c}} = T\kappa_M\circ\mathcal{T}_A\Gamma\circ(\mathrm{id}_{TT_AM}\times_{T_AM}\kappa_M^{-1}),$$

which is called the *canonical lift* (or *complete lift*) *of Γ to T_AM*. The restriction of Γ^{c} to $P^1M\subset T^1_mM$ (the frame bundle of M), $m=\dim M$, was studied in [1].

COROLLARY 3. *Let Γ be a linear connection on M, ∇ the covariant derivative associated to Γ, Γ^{c} the canonical lift of Γ to T_AM, and ∇^{c} the covariant derivative associated to Γ^{c}. Then Γ^{c} is the unique linear connection on T_AM satisfying the identities*

$$\nabla^{\mathrm{c}}_{X^{(\alpha)}}Y^{(\beta)} = \begin{cases}(\nabla_XY)^{(\alpha+\beta)}, & \alpha,\beta\in\mathbb{N}^p,\ 0\le|\alpha+\beta|\le r,\\ 0, & \alpha,\beta\in\mathbb{N}^p,\ |\alpha+\beta|>r,\end{cases}$$

where $X,Y\in\mathfrak{X}(M)$.

COROLLARY 4. *Let T_∇ and R_∇ be the torsion and curvature tensors, respectively, of a linear connection Γ on M, and let ∇^{c} be the covariant derivative of Γ^{c}. Then the torsion $T_{\nabla^{\mathrm{c}}}$ and curvature $R_{\nabla^{\mathrm{c}}}$ tensors of ∇^{c} are the canonical lifts of T_∇ and R_∇ respectively, that is,*

$$T_{\nabla^{\mathrm{c}}} = (T_\nabla)^{\mathrm{c}}, \qquad R_{\nabla^{\mathrm{c}}} = (R_\nabla)^{\mathrm{c}}.$$

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