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Cauchy–Poisson transform and polynomial inequalities

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Abstract. We apply the Cauchy–Poisson transform to prove some multivariate polynomial inequalities. In particular, we show that if the pluricomplex Green function of a fat compact set E in $\mathbb{R}^N$ is Hölder continuous then E admits a Szegö type inequality with weight function $\operatorname{dist}(x, \partial E)^{-(1-\kappa)}$ with a positive κ. This can be viewed as a (nontrivial) generalization of the classical result for the interval $E = [-1, 1] \subset \mathbb{R}$.

1. Introduction. Let $\mathcal{P}(\mathbb{C}^N)$ denote the set of polynomials of N complex variables. An important role in pluripotential theory and approximation theory of many variables is played by the *Siciak extremal function* (or *polynomial extremal function*, see [Si1, Si2])

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p} : p \in \mathcal{P}(\mathbb{C}^N),\ \deg p \geq 1,\ \|p\|_E \leq 1\}, \quad z \in \mathbb{C}^N,$$

where E is a fixed compact subset of $\mathbb{C}^N$. By the Zakharyuta–Siciak theorem (see [Si2, Si3])

$$\log \Phi_E(z) = V_E(z), \quad z \in \mathbb{C}^N,$$

where

$$V_E(z) = \sup\{u(z) : u \in \mathrm{PSH}(\mathbb{C}^N),\ u \leq \mathrm{const} + \log(1 + \|z\|),\ u|_E \leq 0\}.$$

If $V_E^*(z) = \limsup_{w\to z} V_E(w)$ is locally bounded then it is called the *pluricomplex Green function.*

If E is a compact subset of $\mathbb{C}^N$ then, by the definition of Φ_E, we have the Bernstein–Walsh–Siciak type inequality

$$|p(z)| \leq \|p\|_E \cdot \Phi_E(z)^{\deg p}, \quad p \in \mathcal{P}(\mathbb{C}^N).$$

An important tool in the investigations of multivariate inequalities for derivatives of polynomials is provided by the following

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1.1. Proposition ([B2]). *If $E \subset \mathbb{R}^N$ and $x \in E$ then for all $p \in \mathcal{P}(\mathbb{C}^N)$ and all $v \in \mathbb{S}^{N-1}$,*

$$|D_v p(x)| \le (\deg p) \liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon v) \|p\|_E. \tag{1.1}$$

Moreover, if p has only real coefficients then we have a more precise inequality:

$$|D_v p(x)| \le (\deg p) \liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon v)(\|p\|_E^2 - p^2(x))^{1/2}. \tag{1.2}$$

1.2. Remark.

(1) If $E = [-1, 1]$ then $\liminf_{\varepsilon \to 0+} \varepsilon^{-1} V_E(x \pm i\varepsilon) = (1 - x^2)^{-1/2}$ and in this case (1.1) and (1.2) are generalizations of the well-known Bernstein and Szegö inequalities, respectively. (The Szegö inequality is also known as the van der Corput–Schaake inequality.)
(2) We shall see that the limit $\lim_{\varepsilon \to 0+} \varepsilon^{-1} V_E(x + i\varepsilon)$ always exists if $N = 1$, $x \in \operatorname{int}(E) \neq \emptyset$, and is equal to half the density $\varphi(x)$ of the equilibrium measure λ_E.

A general version of inequalities of type (1.1) and (1.2) for a compact $E \subset \mathbb{R}^N$ was proved in [B2, B3]. Similar inequalities were rediscovered later by Totik [T1, T2] but only for $N = 1$.

2. Cauchy–Poisson transform and extremal function. Let us recall the definition of the Cauchy–Poisson transform (see e.g. [St, StW]).

2.1. Definition. Let $\mathbb{H}_+$ and $\mathbb{H}_-$ be the upper half-plane and the lower half-plane in $\mathbb{C}$, respectively. We shall denote by $\mathcal{P}u$ the *Cauchy–Poisson transform* of a Borel function $u : \mathbb{R} \to \mathbb{R}$, $u(t) = O(|t|^\kappa)$, $\kappa \in (0, 1)$, in $\mathbb{H}_+$:

$$\mathcal{P}u(\zeta) = (\Im\zeta) \frac{1}{\pi} \int_{-\infty}^{\infty} |\zeta - t|^{-2} u(t)\, dt \tag{2.1}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} u(ty + x) \frac{dt}{1 + t^2}, \tag{2.2}$$

where $\zeta = x + iy \in \mathbb{H}_+$.

In particular, $\mathcal{P}u$ is well defined if $u(t)$ has logarithmic growth:

$$u(t) = O(\log(1 + |t|)),$$

or if u is globally Hölder continuous, i.e.

$$|u(t) - u(\tau)| \le \text{const} \cdot |t - \tau|^\kappa$$

with $\kappa \in [0, 1)$ (briefly, $u \in \mathrm{HC}_\kappa(\mathbb{R})$).

We also define $\mathcal{P}u$ in the whole plane $\mathbb{C}$ by

$$\mathcal{P}u(\zeta) = \begin{cases} \mathcal{P}u(-\zeta), & \zeta \in \mathbb{H}_-, \\ u(\zeta), & \zeta \in \mathbb{R}. \end{cases}$$

We have

2.2. Proposition. *If* $u \in \mathrm{HC}_\kappa(\mathbb{R})$ *then* $\mathcal{P}u \in \mathcal{H}(\mathbb{H}_+ \cup \mathbb{H}_-) \cap \mathcal{C}(\mathbb{C})$. (Here $\mathcal{H}(\Omega)$ is the space of harmonic functions on an open set $\Omega \subset \mathbb{C}$.)

Proof. Harmonicity of $\mathcal{P}u$ is a consequence of the equality $\Im\zeta|\zeta - t|^{-2} = \Im(1/(\zeta - t))$ and the mean value criterion.

To prove its continuity fix an $x_0 \in \mathbb{R}$. We can write, for $\zeta = x + iy$,

$$\begin{aligned} |\mathcal{P}u(\zeta) - u(x_0)| &= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} (u(ty + x) - u(x_0)) \frac{dt}{1 + t^2} \right| \\ &\le \frac{1}{\pi} \int_{-\infty}^{\infty} |u(ty + x) - u(x_0)| \frac{dt}{1 + t^2} \\ &\le C|x - x_0|^\kappa + C \frac{1}{\pi} \int_{-\infty}^{\infty} |t|^\kappa \frac{dt}{1 + t^2} |y|^\kappa \\ &\le C_1(|x - x_0|^\kappa + |y|^\kappa). \end{aligned}$$

2.3. Remark. $\mathcal{P}u$ is also continuous on $\mathbb{C}$ if $u \in \mathcal{C}(\mathbb{R})$, since we can then apply the Lebesgue bounded convergence theorem. We can also use the Lebesgue theorem if $|u|$ is bounded by $C(1 + |t|)^\kappa$, $\kappa < 1$, in particular, if u has the logarithmic growth $|u(t)| \le C \log(1 + |t|)$.)

To get our main result we need a theorem that establishes relations between the Zakharyuta–Siciak extremal function V_E in $\mathbb{C}^N$ and its restriction to $\mathbb{R}^N$. Here a central role is played by the Cauchy–Poisson transform.

2.4. Theorem. *If E is a compact set in $\mathbb{R}^N$ then for all $x, v \in \mathbb{R}^N$ and $\zeta \in \mathbb{C}$,*

$$V_E(x + \zeta v) \le \mathcal{P}u(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} V_E(x + (\Re\zeta + t\Im\zeta)v) \frac{dt}{1 + t^2}, \tag{2.3}$$

where $u(t) = V_E(x + tv)$, with equality if $N = 1$. In particular, if $v \in \mathbb{S}^{N-1}$, $\varepsilon > 0$ then

$$V_E(x + i\varepsilon v) \le \frac{1}{\pi} \int_{-\infty}^{\infty} V_E(t\varepsilon v + x) \frac{dt}{1 + t^2} \tag{2.4}$$

and

$$\liminf_{\varepsilon \to 0+} \frac{1}{\varepsilon} V_E(x + i\varepsilon v) \le \frac{1}{\pi} \int_{-\infty}^{\infty} t^{-2} V_E(x + tv)\, dt. \tag{2.5}$$

As an immediate consequence we get

2.5. Corollary. *If E is a compact set in $\mathbb{R}^N$ and $x \in \operatorname{int}(E)$ then for any $v \in \mathbb{S}^{N-1}$,*

$$\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} V_E(x+i\varepsilon v) \le \frac{1}{\pi} \int\limits_{|t|\ge \operatorname{dist}_v(x,\partial E)} t^{-2} V_E(x+tv)\,dt. \tag{2.6}$$

Here $\operatorname{dist}_v(x,\partial E)$ is the distance from x to ∂E in direction v defined in the next section.

Proof of Theorem 2.4. Let us recall that if E is a compact subset of $\mathbb{R}^N$ then

$$V_E(z) = \sup\biggl\{\frac{1}{\deg p}\log|h(p(z))| : p \in \mathbb{R}[x],\ \deg p \ge 1,\ \|p\|_E \le 1\biggr\},$$

where $h(\zeta) = \zeta + \sqrt{\zeta^2-1}$ and $|h(\zeta)| = h(\frac{1}{2}|\zeta+1| + \frac{1}{2}|\zeta-1|)$, $h(t) = t + (t^2-1)^{1/2}$, $t \ge 1$ (see [B2]).

Put

$$u(\zeta) = \frac{1}{\deg p}\log|h(p(x+\zeta v))|, \qquad \zeta \in \mathbb{C}.$$

Then

$$u \in \mathcal{SH}(\mathbb{C}) \cap \mathcal{H}(\mathbb{H}_+ \cup \mathbb{H}_-) \cap \mathcal{C}(\mathbb{C}).$$

Moreover, $u \ge 0$ and $u(z) - \frac{1}{2}\log(1+|\zeta|^2) = O(1)$. This implies that $\mathcal{P}u \in \mathcal{C}(\mathbb{C})$ and the function v defined by

$$v(\zeta) = u(\zeta) - \mathcal{P}u(\zeta), \qquad \zeta \in \mathbb{C},$$

is a bounded continuous function on $\mathbb{C}$ that equals 0 on $\mathbb{R}$. Therefore, applying the maximum principle separately to $\mathbb{H}_+$ and $\mathbb{H}_-$ we get the inequality $v \le 0$ in $\mathbb{C}$, whence

$$\frac{1}{\deg p}\log|h(p(x+\zeta v))| \le \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{1}{\deg p}\log|h(p(x+tv))|\,\frac{dt}{1+t^2},$$

and taking the supremum over p gives (2.3).

The proof of equality in case $N=1$ is similar to that in [B4]: it suffices to consider the case $x=0$ and $y=1$.

Let $E \subset \mathbb{R} \subset \mathbb{C}$ be a compact set that satisfies the HCP *condition*, i.e. there exist constants $M > 0$ and $\kappa \in (0,1]$ such that

$$V_E(z) \le M[\operatorname{dist}(z,E)]^\kappa \qquad \operatorname{dist}(z,E) \le 1.$$

Then in particular $V_E \in \mathcal{C}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C}\setminus E)$ and $V_E(\zeta) - \log(1+|\zeta|) = O(1)$ as $\zeta \to \infty$. Hence, by the argument of the proof of Theorem 2.4, the function

$$v(\zeta) = \mathcal{P}V_E|_{\mathbb{R}}(\zeta) - V_E(\zeta), \qquad \zeta \in \mathbb{C},$$

is nonnegative, whence for $\zeta = x + iy$ we get

$$V_E(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} V_E(ty+x)\,\frac{dt}{1+t^2}.$$

Now, if E is an arbitrary compact subset of $\mathbb{R}$, there exists a sequence of compact sets E_k such that $E_{k+1} \subset E_k$, $E_k \in \mathrm{HCP}$ and $E = \bigcap_{k=1}^{\infty} E_k$. Hence $V_{E_k} \nearrow V_E$, and so, by the Lebesgue monotone convergence theorem,

$$\begin{aligned} V_E(\zeta) \nwarrow V_{E_k}(\zeta) &= \mathcal{P}V_{E_k}|_{\mathbb{R}}(\zeta) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} V_{E_k}(ty+x)\,\frac{dt}{1+t^2} \nearrow \frac{1}{\pi}\int_{-\infty}^{\infty} V_E(ty+x)\,\frac{dt}{1+t^2}. \end{aligned}$$

Let us recall that E is said to be *L-regular at* $x_0 \in E$ if

$$\limsup_{z\to x_0} V_E(z) = 0,$$

that is, V_E is continuous at x_0. From Theorem 2.4 we easily derive

2.6. Corollary. *If E is not a pluripolar subset of $\mathbb{R}^N$ (that is, V_E^* is bounded by* const $+ \log(1+|z|)$ *in $\mathbb{C}^N$) then E is L-regular at $x_0 \in E$ if $V_E|_{\mathbb{R}^N}$ is continuous at x_0.*

To show another application of Theorem 2.4 we need the following simple lemma.

2.7. Lemma. *Put*

$$\Lambda_x(v) = \frac{|v|}{\pi}\int_{\mathbb{R}\setminus E} (x-t)^{-2} V_E(t)\,dt, \qquad x \in \mathrm{int}(E),\ v \in \mathbb{R},$$

and let $0 < \varepsilon < 1$. Then, for $|v| \le (1/\sqrt{\varepsilon(1-\varepsilon)})\mathrm{dist}(x, \mathbb{R}\setminus E)$, one has

$$(1-\varepsilon)\Lambda_x(v) \le \frac{1}{\varepsilon} V_E(x+i\varepsilon v) \le \Lambda_x(v).$$

Proof. If $|v| \le (1/\sqrt{\varepsilon(1-\varepsilon)})\mathrm{dist}(x, \mathbb{R}\setminus E)$ then, for an arbitrary $t \in \mathbb{R}\setminus E$, we have $|v| \le |x-t|/\sqrt{\varepsilon(1-\varepsilon)}$. This inequality is equivalent to $|x+i\varepsilon v|^{-2} \ge (1-\varepsilon)|x-t|^{-2}$ and, by the obvious inequality $|x+i\varepsilon v - t|^{-2} \le |x-t|^2$ and by (2.1), we have

$$(1-\varepsilon)\Lambda_x(v) \le \frac{1}{\varepsilon} V_E(x+i\varepsilon v) = \frac{|v|}{\pi}\int_{\mathbb{R}\setminus E} |x+i\varepsilon v - t|^{-2} u(t)\,dt \le \Lambda_x(v).$$

Now, by pluripotential methods developed in [B3] (see Comparison Lemma 1.12 and Corollary 3.2) one easily obtains the following

2.8. Proposition. *Let E be a compact subset of $\mathbb{R}$ with nonempty interior and let $E_0 = \overline{\mathrm{int}(E)}$ be the “fat” part of E. Then for the equilibrium* *measure* λ_E (see e.g. [Kl] for the definition of this notion in $\mathbb{C}^N$) *the following formula holds*:

$$\lambda_E|_{E_0} = \varphi(x)\,dx,$$

where

$$\varphi(x) = \frac{2}{\pi} \int\limits_{\mathbb{R}\setminus E} |x-t|^{-2} V_E(t)\,dt.$$

3. Szegö type inequality for compact sets in $\mathbb{R}^N$. Let $v \in \mathbb{S}^{N-1}$ and let E be a subset of $\mathbb{R}^N$. If $x_0 \in E$ then the *distance from* x_0 *to* ∂E *in direction* v is defined by

$$\operatorname{dist}_v(x_0, \partial E) = \sup\{r > 0 : x_0 + [-r, r]v \subset E\}.$$

If $\operatorname{dist}(x_0, \partial E)$ denotes the usual distance from $x_0 \in E$ to the boundary of E, that is,

$$\operatorname{dist}(x_0, \partial E) = \inf\{|x - x_0| : x \in \partial E\} = \sup\{r > 0 : \overline{B}(x_0, r) \subset E\}$$

then we have

$$\operatorname{dist}(x_0, \partial E) = \inf_{v \in \mathbb{S}^{N-1}} \operatorname{dist}_v(x_0, \partial E).$$

If $E = [-1,1] \times \{0\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$ and $x_0 = (0,0)$, then for $v = (1,0)$ and $v = (0,1)$ we have $\operatorname{dist}_v(x_0, \partial E) = 1$ and $\operatorname{dist}(x_0, \partial E) = 0$, so the usual distance is in general not comparable with directional distances for n linearly independent vectors.

3.1. Theorem. *Let E be a compact subset of $\mathbb{R}^N$. Let $v \in \mathbb{S}^{N-1}$ and let*

$$E_v := \{x \in E : \operatorname{dist}_v(x, \partial E) > 0\}.$$

Assume that there exist positive constants C_1, C_2 and $\kappa \in (0,1)$ such that

$$V_E(x + tv) \le C_1 \log(1 + |t|), \quad t \in \mathbb{R},\ x \in E_v, \tag{3.1}$$

and

$$V_E(x + tv) \le C_2 |t|^\kappa \quad \text{as } t \in [-1,1],\ x \in E_v. \tag{3.2}$$

Then there exists a positive constant M such that for any $p \in \mathbb{R}[x]$ and any $x \in E_v$,

$$|D_v p(x)| \le M(\deg p)(\operatorname{dist}_v(x, \partial E))^{-(1-\kappa)}(\|p\|_E^2 - p^2(x))^{1/2}. \tag{3.3}$$

Proof. Without loss of generality we can assume that

$$\sup_{x \in E_v} \operatorname{dist}_v(x, \partial E) \le 1.$$

To prove (3.3) we need to find an upper bound of $\liminf_{\varepsilon \to 0+} \varepsilon^{-1} V_E(x + i\varepsilon v)$.

By (2.6) we have

$$
\begin{aligned}
\liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon} V_E(x+i\varepsilon v) &\le \frac{1}{\pi} \int\limits_{|t|\ge \mathrm{dist}_v(x,\partial E)} t^{-2} V_E(x+tv)\,dt \\
&= \frac{1}{\pi}\Big[\int\limits_{1\ge|t|\ge \mathrm{dist}_v(x,\partial E)} + \int\limits_{|t|\ge 1}\Big] t^{-2} V_E(x+tv)\,dt \\
&\le \frac{2C_2}{\pi} \int\limits_{\mathrm{dist}_v(x,\partial E)}^{1} t^{\kappa-2}\,dt + \frac{2C_1}{\pi}\int\limits_1^\infty \log(1+t)t^{-2}\,dt \\
&= \frac{2C_2}{\pi}\,\frac{1}{1-\kappa}\,((\mathrm{dist}_v(x,\partial E))^{-(1-\kappa)} - 1) + C_3 \\
&\le M(\mathrm{dist}_v(x,\partial E))^{-(1-\kappa)},
\end{aligned}
$$

where $M = C_3 + 2C_2/(1-\kappa)\pi$. Hence, by Proposition 1.1 we get inequality (3.3).

Applying Theorem 3.1 for all directions $v \in \mathbb{S}^{n-1}$ gives the main result of the paper:

3.2. Theorem. *If a fat compact E in $\mathbb{R}^N$ satisfies the HCP condition with constants $M > 0$ and $0 < \kappa < 1$, then, for all directions $v \in \mathbb{S}^{n-1}$ and all polynomials $p \in \mathbb{R}[x]$, we have the following Szegö type inequality*:

$$|D_v p(x)| \le A(\deg p)(\mathrm{dist}(x,\partial E))^{-(1-\kappa)}(\|p\|_E^2 - p^2(x))^{1/2}, \quad x \in \mathrm{int}(E),$$

where $A = A(E)$ is a constant.

3.3. Remark. Recall that a compact set E in $\mathbb{R}^N$ is said to be *Markov* if there exist constants $M > 0, m \ge 2$ such that for all polynomials p,

$$(\mathcal{M}) \qquad \|\mathrm{grad}\, p\|_E \le M(\deg p)^m \|p\|_E.$$

By Cauchy's Integral Formula, any HCP compact set in $\mathbb{R}^N$ is Markov and till now, no Markov set which is not an HCP set is known.

It is also known (see [Pl]) that Markov's property is equivalent to the following condition:

$$(\mathcal{P}) \quad \exists C_1, C_2\ \forall p \in \mathcal{P}_k(\mathbb{C}^N) \quad |p(z)| \le C_2\|p\|_E \quad \text{as } \mathrm{dist}(z,E) \le C_1 k^{-m}.$$

It was conjectured in [B2] that an inequality of type (3.3) implies Markov's inequality with exponent $1/\kappa$. We note that this is true in the class of HCP sets.

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