ANNALES
POLONICI MATHEMATICI
109.3 (2013)

Estimation of the Carathéodory distance on pseudoconvex domains of finite type whose boundary has Levi form of corank at most one

by Gregor Herbort (Wuppertal)

Abstract. We study the class of smooth bounded weakly pseudoconvex domains $D \subset \mathbb{C}^n$ whose boundary points are of finite type (in the sense of J. Kohn) and whose Levi form has at most one degenerate eigenvalue at each boundary point, and prove effective estimates on the invariant distance of Carathéodory. This completes the author's investigations on invariant differential metrics of Carathéodory, Bergman, and Kobayashi in the corank one situation and on invariant distances on pseudoconvex finite type domains in dimension two.

1. Introduction. Invariant metrics and distances render valuable services in the study of mapping theory for a long time (see e.g. [Kr, Vor]). Indeed, their boundary behavior is important for the question of whether or not there exists a biholomorphic or proper holomorphic mapping between two given bounded domains in $\mathbb{C}^n$.

The most important of such distance functions were introduced by Carathéodory, Bergman, and Kobayashi. We recall their definitions.

The *Carathéodory distance* on a bounded domain $D \subset \mathbb{C}^n$ is defined by

$$d_D^{\mathrm{Cara}}(A,B) := \{d^P(f(A), f(B)) \mid f : D \to \mathbb{E}, \text{ holomorphic}\},$$

where $\mathbb{E}$ denotes the unit disc in the plane and d^P the Poincaré distance on $\mathbb{E}$, in detail

$$d^P(a,b) := \frac{1}{2}\log\frac{1+\mu(a,b)}{1-\mu(a,b)}, \qquad \mu(a,b) := \left|\frac{a-b}{1-\overline{a}b}\right|.$$

It was introduced by C. Carathéodory in 1926 and was the first known distance function in several complex variables that remains invariant under biholomorphic mappings. The Carathéodory distance of the unit disc agrees with the Poincaré distance.

2010 *Mathematics Subject Classification*: Primary 32F45; Secondary 32T25, 32T45.
Key words and phrases: regular type, plurisubharmonic weights, Cauchy–Riemann equation.

DOI: 10.4064/ap109-3-1

The *Carathéodory pseudodifferential metric* is given by

$$F_D^{\mathrm{Cara}}(z;X) := \sup\{|\langle \partial f(z), X\rangle| \mid f : D \to \mathbb{E}, \text{ holomorphic}, f(z)=0\},$$

where for a vector $X \in \mathbb{C}^n$ and a differentiable function f we denote by $\langle \partial f(z), X\rangle$ the directional derivative

$$\langle \partial f(z), X\rangle := \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z) X_k.$$

The *Kobayashi differential metric* is defined as follows:

$$F_D^{\mathrm{Kob}}(z;X) := \inf\{\alpha^{-1} > 0 \mid \exists f : \mathbb{E} \to D, \text{ holomorphic}, \\ f(0)=z,\ f'(0)=\alpha X\}.$$

Our results that we will state in Section 2 will also involve the *Bergman metric* F_D^{B} which is defined as the Kähler metric with potential $\log K_D(z,z)$, where $K_D : D \times D \to \mathbb{C}$ denotes the Bergman kernel function on D.

In many special cases good estimates have been found for the Carathéodory pseudodifferential metric; see [Cat1, Gra, Her-1, Her-2a]. In the case of the pseudodifferential metrics of Kobayashi and Bergman it is possible to get good and in some cases even precise estimates also for the associated distances d_D^{Kob} and d_D^{B}, respectively, because these are the "integrated forms" of the corresponding differential metrics; see [Bal-Bon, Died-Ohs, Her-3].

By "integrated form" we mean the following: Assume that F_D is one of the pseudodifferential metrics of Bergman or Kobayashi. For two points $A, B \in D$ we define $\mathscr{L}(A,B)$ as the family of all piecewise smooth paths $\gamma : [0,1] \to D$ from A to B. Then we call

$$d_{F_D}(A,B) = \inf\Big\{ \int_0^1 F_D(\gamma(t);\dot\gamma(t))\,dt \;\Big|\; \gamma \in \mathscr{L}(A,B)\Big\}$$

the integrated form of F_D. We have $d_D^{\mathrm{B}} = d_{F_D^{\mathrm{B}}}$, which follows from the definition from Riemannian geometry, and $d_D^{\mathrm{Kob}} = d_{F_D^{\mathrm{Kob}}}$, which was proved by H. L. Royden [Roy].

The case of the Carathéodory distance is more difficult, because the "integrated form" of $\mathrm{Cara}_D(z;X)$ is the inner distance associated to d_D^{Cara} (see [Rei]). In general, however, the inner Carathéodory distance is not equal to the Carathéodory distance. A simple example is given by the annulus in the plane.

In this article we continue the investigations of [Her-3] on the boundary behavior of the above invariant distances on a finite type pseudoconvex domain $D \subset \mathbb{C}^2$. We treat the more general case of bounded pseudoconvex domains with a smooth boundary such that at each boundary point of D the Levi form of the boundary has at most one degenerate eigenvalue.

We establish precise estimates for d_D^{Cara}, d_D^{Kob}, and d_D^{B} analogous to those obtained for the distances of Bergman and Kobayashi in [Her-3].

The paper is organized as follows: In Section 2 we state as a main result the estimation of the distances of Carathéodory, Bergman, and Kobayashi, after we have introduced all the necessary notations (distinguished normal coordinates, and pseudoballs). In Section 3 we establish the "engulfing property" for these pseudoballs (that are constructed in analogy to those from [N-S-W, Cat2]); compare [Cat2, §1]. Section 4 contains the construction of some plurisubharmonic auxiliary functions that will be needed in order to apply a $\overline{\partial}$ solution theorem from Hörmander's L^2-theory for the Cauchy–Riemann operator. In Section 5 we construct, given two points in D, a family of holomorphic auxiliary functions that exhibit a good behavior at those points. By means of these functions we construct in Section 6 an appropriate candidate for the supremum that defines the Carathéodory distance. Finally, in Section 7 we show the corresponding upper estimates for the Kobayashi and Bergman distances.

Remark. It should be mentioned here that also K. Verma, in a joint paper [BMV] with G. Balakumar and P. Mahajan, has independently obtained precise estimates on the Carathéodory and Kobayashi distances on pseudoconvex finite type Levi corank one domains.

2. Statement of the results. Throughout this section we assume that $D \subset\subset \mathbb{C}^n$ is smoothly bounded and pseudoconvex. We choose a defining function $r \in C^\infty(U_0)$, where U_0 is an open neighborhood of ∂D.

Definition 2.1. A boundary point ζ of D is said to be of *finite regular type* if there exists a bound N such that any non-singular holomorphic curve passing through ζ has an order of contact of at most N with ∂D at ζ. The maximal order of contact between such a non-singular curve and ∂D at ζ is called the *regular type* and is denoted by $t(\partial D, \zeta)$.

Remark 2.1. The pseudoconvexity assumption on D implies that $t(\partial D, \zeta)$ is an even integer.

Throughout this paper we will suppose that

(a) each boundary point ζ is of finite type, and
(b) the rank of the Levi form of ∂D at ζ is at least $n-2$.

In this case the Catlin multitype of any $\zeta \in \partial D$ is given by

$$\mathscr{M}(\partial D, \zeta) = (1, 2, \ldots, 2, t(\partial D, \zeta)).$$

If we write $M_w := \{r = r(w)\}$, then the hypersurface M_w is also smooth for all w that lie sufficiently close to ∂D, and further $w \mapsto t(M_w, w)$ is an upper semicontinuous function of w, hence we can find an integer m such that

$$t(M_w, w) \le 2m \tag{2.1}$$

for any w in a suitable neighborhood U_0 of ∂D.

2.1. Normal coordinates and pseudoballs. Let $\zeta \in U_0$. We want to put the defining function r of D near ζ in a "normalized" form so that certain unwelcome pure terms in its Taylor expansion about ζ disappear.

In [Her-2a] the following was proved:

LEMMA 2.1.1. *There exists a radius $R_0 > 0$ and for any $\zeta \in U_0$ a holomorphic mapping $F_\zeta : \mathbb{C}^n \to \mathbb{C}^n$ such that:*

(a) *Over the ball $B(\zeta, 2R_0)$, the function r is normalized to $r = \rho_\zeta \circ F_\zeta$, where*

$$\rho_\zeta(w) = r(\zeta) + \operatorname{Re} w_1 + |w''|^2 + \sum_{j=2}^{2m} P_j(\zeta, w_n) + (\operatorname{Im} w_1) \sum_{j=2}^{2m} Q_j(\zeta, w_n) + 2 \operatorname{Re} \sum_{a=2}^{n-1} w_a g_a(\zeta, w_n) + \mathcal{R}(\zeta, w),$$

and

- (i) *the $P_j(\zeta, \cdot)$ and $Q_j(\zeta, \cdot)$ are real-valued homogeneous polynomials of degree j and do not exhibit pure terms,*
- (ii) *the g_a are complex polynomials without holomorphic terms,*
- (iii) *the remainder term $\mathcal{R}$ can be estimated by*

$$|\mathcal{R}(\zeta, w)| \le C_0\big(|\operatorname{Im} w_1|^3 + |\operatorname{Im} w_1|^2(|w''| + |w_n|) + |\operatorname{Im} w_1|(|w''|^2 + |w_n|^{2m+1}) + |w''|^3 + |w''|^2|w_n| + |w_n|^{2m+1}\big).$$

(b) *The mapping F_ζ can be described as follows: If $n \ge 3$, then*

$$F_\zeta = \widetilde{F}_\zeta \circ \sigma,$$

where σ is a permutation of the coordinates $(z_1, \ldots, z_n)$, and

$$\widetilde{F}_\zeta(z) = \begin{pmatrix} \big((\partial r(\zeta),\, z - \zeta) + f_1(\zeta, z_n - \zeta_n)\big)(1 + f_2(\zeta, z_n - \zeta_n)) \\ A(\zeta) \cdot (z'' - \zeta'') + h(\zeta, z_n - \zeta_n) \\ z_n - \zeta_n \end{pmatrix}, \tag{2.2}$$

where $z'' = (z_2, \ldots, z_{n-1})$, the $(n-2) \times (n-2)$ matrix $A(\zeta)$ is invertible and its determinant is $\ge c > 0$ uniformly in ζ (with some unimportant $c > 0$). The mapping $h(\zeta, \cdot) = (h_2(\zeta, \cdot), \ldots, h_{n-1}(\zeta, \cdot))$

consists of holomorphic polynomials of degree $\le 2m$. *Also the functions* $f_1(\zeta,\cdot)$, $f_2(\zeta,\cdot)$ *are holomorphic polynomials of degree* $\le 2m$. *They all vanish at* 0.

In the case that $n = 2$, *we have*

$$\widetilde{F}_\zeta(z) = \begin{pmatrix} \big((\partial r(\zeta),\, z-\zeta) + f_1(\zeta, z_2-\zeta_2)\big)(1 + f_2(\zeta, z_2-\zeta_2)) \\ z_2-\zeta_2 \end{pmatrix}, \tag{2.3}$$

where f_1 *and* f_2 *have a meaning analogous to the case* $n \ge 3$.

(c) *With a suitable constant* $L_0 > 0$ *we have*

$$\frac{1}{L_0}|z-\zeta| \le |F_\zeta(z)| \le L_0|z-\zeta|. \tag{2.4}$$

2.2. A pseudodistance and the main result. Let U_0 be as in (2.1). According to the normal form ρ_ζ of the local defining function at ζ we define the radii:

DEFINITION 2.2. Let $\zeta \in U_0$ and $\delta > 0$. Then we put

$$\tau_n(\zeta,\delta) := \min_{2\le l\le 2m}\left(\frac{\delta}{\|P_l(\zeta,\cdot)\|}\right)^{1/l},$$

where $\|P_l(\zeta,\cdot)\|$ denotes the sum of the absolute values of the coefficients of $P_l(\zeta,\cdot)$.

Note that this is well-defined, because if $P_2(\zeta,\cdot) = \cdots = P_{2m}(\zeta,\cdot) = 0$, then $t(\partial D,\zeta) > 2m$ contrary to our choice of the number $2m$. With some constant $\gamma_0 > 0$ we have the estimate

$$\tau_n(\zeta,\delta) \ge \gamma_0\sqrt{\delta} \quad \text{for all } \zeta \in U_0 \text{ and } \delta > 0. \tag{2.5}$$

Next we define

$$\tau_1(\zeta,\delta) := \delta, \qquad \tau_2(\zeta,\delta) = \cdots = \tau_{n-1}(\zeta,\delta) = \sqrt{\delta},$$

the polydiscs

$$R_\delta(\zeta) := \{w = (w_1,\dots,w_n) \in \mathbb{C}^n \mid |w_k| < \tau_k(\zeta,\delta) \text{ for } k = 1,\dots,n\},$$

and finally the "pseudoballs"

$$Q_\delta(\zeta) = \{z \in B(\zeta, 2R_0) \mid F_\zeta(z) \in R_\delta(\zeta)\}.$$

For $A, B \in D$ we define $M(A,B) := \{\delta > 0 \mid A \in Q_\delta(B)\}$ and let

$$d'(A,B) := \begin{cases} \inf M(A,B) & \text{if } B \in U_0 \text{ and } M(A,B) \ne \emptyset, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Finally we introduce the "pseudodistance" function

$$d(A,B) := \min\{d'(A,B), |A-B|\}. \tag{2.7}$$

Then we can state our main result:

Main Theorem 2.1. *Assume that $D \subset\subset \mathbb{C}^n$ is a smoothly bounded pseudoconvex domain such that all its boundary points are of regular type $\le 2m$ and the Levi form of ∂D at each point $\zeta \in \partial D$ has at least $n-2$ positive eigenvalues. Denote by d^* any of the invariant distance functions of Carathéodory, Bergman, or Kobayashi. Then, with some unimportant constant $C_* > 0$,*

$$C_* \varrho(A,B) \le d^*(A,B) \le \frac{1}{C_*}\varrho(A,B) \tag{2.8}$$

for all $A, B \in D$, where

$$\varrho(A,B) := \begin{cases} \varrho_B(A) + \varrho_A(B) & \text{if } A, B \in U_0, \\ \varrho_A(B) & \text{if } A \in U_0,\ B \notin U_0, \\ \varrho_B(A) & \text{if } B \in U_0,\ A \notin U_0, \\ \log(1+|A-B|) & \text{if } A, B \notin U_0, \end{cases}$$

and

$$\varrho_x(A) := \log\left(1 + \frac{d(A,x)}{\delta_D(x)} + \frac{|[F_x(A)]''|}{\sqrt{\delta_D(x)}} + \frac{|[F_x(A)]_n|}{\tau_n(x,\delta_D(x))}\right) \quad \text{for } x \in U_0.$$

It is well-known that $d_D^{\mathrm{Cara}} \le d_D^{\mathrm{B}}$, $d_D^{\mathrm{Cara}} \le d_D^{\mathrm{Kob}}$, and from [Her-2a] we know that F_D^{B} and F_D^{Kob} have equivalent growth at ∂D. This implies $\widehat{C}^{-1} d_D^{\mathrm{B}} \le d_D^{\mathrm{Kob}} \le \widehat{C} d_D^{\mathrm{B}}$ with some constant $\widehat{C}$.

Therefore, we have to estimate $d_D^{\mathrm{Cara}}(A,B)$ from below and $d_D^{\mathrm{Kob}}(A,B)$ from above in terms of $\varrho(A,B)$, for $A, B \in D$.

3. Crucial properties of the pseudoballs and pseudodistance. The definitions of the pseudoballs $Q_\delta(\zeta)$ resemble those from [Cho-1], but the mapping F_ζ is not the same, and some of the important properties of $Q_\delta(\zeta)$ cannot be obtained by citing the corresponding lemmas there. In particular a certain important property, which we call the "engulfing property" (see Lemma 3.3.1 below), is not discussed in [Cho-1].

We first want to clarify the following question: Assume that $\delta > 0$ is small and $\zeta_1, \zeta_2 \in U_0$ are such that $\zeta_1 \in Q_\delta(\zeta_2)$. How does the radius $\tau_n(\zeta_1,\delta)$ compare with $\tau_n(\zeta_2,\delta)$? The answer is given in Lemma 3.2.3 below. It looks similar to [Cho-1, Cor. 2.8], and its proof is based on ideas analogous to those in [Cho-1, p. 808].

3.1. Special tangent vector fields. Let r be the defining function we fixed at the beginning. For $j = 1, \ldots, n$ we denote by r_{z_j} the derivative $r_{z_j} = \frac{\partial r}{\partial z_j}$; the derivatives $r_{\bar z_j}$ are defined accordingly. For a vector field

$X = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j} + \sum_{j=1}^n b_j \frac{\partial}{\partial \bar{z}_j}$ we write

$$\partial r(X) := \sum_{j=1}^n r_{z_j} a_j, \qquad \overline{\partial} r(X) = \sum_{j=1}^n r_{\bar{z}_j} b_j.$$

The normal field N to the level sets of r is given by

$$N := \sum_{k=1}^n r_{\bar{z}_k} \frac{\partial}{\partial z_k}. \tag{3.1}$$

For vector fields X, Y on some open set $U \subset \mathbb{C}^n$ we denote by $[X, Y]$ the Lie bracket of X and Y.

After shrinking the neighborhood $U_0 \supset \partial D$ we have $|\nabla r| \geq c_0 > 0$ with some constant c_0.

In particular, we can cover ∂D by open sets $U_1, \ldots, U_n \subset U_0$ such that

$$|r_{z_k}| \geq c_0/n \quad \text{ on } \bar{U}_k$$

for $k = 1, \ldots, n$. The tangent fields

$${}^k L'_j := \frac{\partial}{\partial z_j} - \frac{r_{z_j}}{r_{z_k}} \frac{\partial}{\partial z_k}, \qquad j \neq k,$$

are defined over U_k.

Lemma 3.1.1. *For $n = 2$ define ${}^1L_* := {}^1L'_2$ and ${}^2L_* := {}^2L'_1$. For $n \geq 3$ there exists a constant $c_1 > 0$ with the following property: For any $k \in \{1, \ldots, n\}$ we can rearrange the ${}^kL'_1, \ldots, {}^kL'_{k-1}, {}^kL'_{k+1}, \ldots, {}^kL'_n$ (which corresponds to renumbering the coordinates $\{z_j\}_{j \neq k}$) in such a way that for the resulting list, which will be denoted by ${}^kL_2, \ldots, {}^kL_n$, each eigenvalue of the matrix $({}^k\mathcal{L}_{a\bar{b}})_{a,b=2}^{n-1}$ is not smaller than c_1 at each point of $\overline{U}_k$. Put*

$${}^k\mathcal{L}_{a\bar{b}} := \partial\overline{\partial} r([{}^kL_a, \overline{{}^kL_b}]).$$

If we denote by ${}^k\mathcal{L}^{a\bar{\nu}}$ the entries of the inverse of $({}^kL_{a\bar{b}})_{a,b=2}^{n-1}$, and define

$${}^ks_\nu := -\sum_{a=2}^{n-1} {}^k\mathcal{L}_{n\bar{a}} \, {}^k\mathcal{L}^{a\bar{\nu}}, \quad \nu = 2, \ldots, n-1,$$

then the vector field

$${}^kL_* := {}^kL_n + \sum_{\nu=2}^{n-1} {}^ks_\nu \, {}^kL_\nu$$

satisfies $\partial r([{}^kL_j, \overline{{}^kL_*}]) = 0$ for $j = 2, \ldots, n-1$. In particular, $({}^kL_2, \ldots, {}^kL_{n-1}, {}^kL_*)$ are the special vector fields of a boundary system in the sense of [Cat1].

Proof. The Levi form of ∂D is semipositive at each boundary point, and our assumption on eigenvalues implies that, with some positive constant c_1,

$$\min_{\overline{U}_k} \sum_{\nu \neq k} \det\big((\partial\overline{\partial} r([{}^kL'_a, \overline{{}^kL'_b}]))_{a,b\neq\nu}\big) \geq n^2 c_1.$$

Therefore we can make an arrangement as claimed in the lemma. The properties of the vector field ${}^kL_*$ follow by direct computation. ∎

For each $k \in \{1,\dots,n\}$ we introduce on U_k a system of continuous functions ${}^kC_l$, $l \geq 2$, as follows: With the notations of Lemma 3.1.1 we let

$$(3.2)\qquad {}^kC_l := \max_{a,b\geq 1,\, a+b=l} \big|({}^kL_*)^{a-1}(\overline{{}^kL_*})^{b-1})(\partial\overline{\partial} r([{}^kL_*, \overline{{}^kL_*}])\big|.$$

REMARK 3.1. (a) Let $\zeta \in U_k$. By [BloGra] the smallest integer l such that ${}^kC_l(\zeta) > 0$ is equal to the regular type $t(M_\zeta, \zeta)$

(b) Further, there exists (after possibly shrinking the $U_1,\dots,U_n$) a constant $c_2 > 0$ such that for any $k = 1,\dots,n$ we have

$$\max_{2\leq l\leq 2m} {}^kC_l \geq c_2$$

throughout U_k. In particular, for any $\delta > 0$ and $\zeta \in U_k$, the number

$$(3.3)\qquad {}^k\eta(\zeta,\delta) := \min_{2\leq l\leq 2m}\left(\frac{\delta}{{}^kC_l(\zeta)}\right)^{1/l}$$

is well-defined. Moreover, with some unimportant constant $c_3 > 0$,

$$(3.4)\qquad c_3\sqrt{\delta} \leq {}^k\eta(\zeta,\delta) \leq \frac{1}{c_3}\delta^{1/2m}.$$

REMARK 3.2. The radii ${}^k\eta(\zeta,\delta)$ certainly depend on the choice we made for the local boundary system. But if we choose another one, then for the resulting radii ${}^k\eta(\zeta,\delta)'$ the ratio ${}^k\eta(\zeta,\delta)/{}^k\eta(\zeta,\delta)'$ is bounded from above and from below by uniform positive constants.

We want to compare the radius ${}^k\eta(\zeta,\delta)$ with $\tau_n(\zeta,\delta)$ for $\zeta \in U_k$. This is done in several steps.

Let on U_k a boundary system $(L_2,\dots,L_{n-1},L_*)$ be given as in Lemma 3.1.1 (we drop the superscript k). Then we define the *Levi determinants*

$$\varLambda := \det(\mathcal{L}_{a\bar b})_{a,b=2}^n \quad\text{and}\quad \varLambda' := \det(\mathcal{L}_{a\bar b})_{a,b=2}^{n-1}.$$

By standard linear algebra we obtain

$$(3.5)\qquad \varLambda = \varLambda' \cdot \partial r([L_*, \overline{L_*}]).$$

3.2. Comparison of the radii $\eta(\zeta,\cdot)$ and $\tau_n(\zeta;\cdot)$. We want to compare the quantities ${L_*}^{a-1}{\overline{L}_*}^{b-1}\partial r([L_*,\overline{L_*}])$ and $\frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial\bar w_n^b} \circ F_\zeta$. This is done in the next two lemmas. Because of (3.5) it suffices to work with $L_*^{a-1}\overline{L}_*^{b-1}\varLambda$ instead.

LEMMA 3.2.1. *There exists a constant $C_1 > 0$ such that for small enough $\delta > 0$, any $\zeta \in U_k$, and any multi-index $\alpha \neq 0$ the estimate*

$$\left|\frac{\partial^\alpha \rho_\zeta}{\partial w^\alpha}\right| \le C_1 \frac{\delta}{\delta^{\alpha_1}\tau_2(\zeta,\delta)^{\alpha_2}\cdot\ldots\cdot\tau_n(\zeta,\delta)^{\alpha_n}} \tag{3.6}$$

holds in the polydisc $R_\delta(\zeta)$.

Proof. This is obtained by a Taylor series argument in analogy to [Cho-1, Prop. 2.3]. ∎

LEMMA 3.2.2. *There exists a constant $C_2 > 0$ such that the following holds: Let $\zeta \in U_k$ and let $L_2,\ldots,L_{n-1},L_*$ denote a boundary system near ζ as in Lemma 3.1.1. Then*

$$\left|L_*^{a-1}\overline{L}_*^{b-1}\varLambda - \varLambda'\cdot\frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a\partial\bar w_n^b}\circ F_\zeta\right| \le C_2\theta\cdot\frac{\delta}{\tau_n(\zeta,\delta)^{a+b}} \tag{3.7}$$

on $Q_{\theta\delta}(\zeta)$, for $0<\theta<1$ and $\delta>0$, and any positive integers a, b.

Proof.

STEP 1. We transform everything into the normal coordinates induced by F_ζ. To do this we let

$$\begin{aligned}\widehat{\mathcal L}_{\nu\bar\mu} := (\rho_\zeta)_{w_\nu\bar w_\mu} &- \frac{1}{|(\rho_\zeta)_{w_1}|^2}\big((\rho_\zeta)_{w_\nu\bar w_1}(\rho_\zeta)_{w_1}(\rho_\zeta)_{\bar w_\mu} + (\rho_\zeta)_{w_1\bar w_\mu}(\rho_\zeta)_{\bar w_1}(\rho_\zeta)_{\bar w_\nu}\big)\\ &+\frac{(\rho_\zeta)_{w_1\bar w_1}(\rho_\zeta)_{w_\mu}(\rho_\zeta)_{\bar w_\nu}}{|(\rho_\zeta)_{w_1}|^2}.\end{aligned}$$

Then, by the chain rule, for $a,b = 2,\ldots,n$ we have

$$\mathcal L_{a\bar b} = \sum_{\nu,\mu=2}^{n}\frac{\partial(F_\zeta)_\nu}{\partial z_a}\widehat{\mathcal L}_{\nu\bar\mu}\circ F_\zeta\overline{\frac{\partial(F_\zeta)_\mu}{\partial z_b}},$$

which gives

$$\varLambda = \widehat\varLambda\circ F_\zeta|\det A(\zeta)|^2,$$

where we put $\widehat\varLambda := \det(\widehat{\mathcal L}_{\nu\bar\mu})_{\nu,\mu=2}^n$.

The vector field L_* transforms under F_ζ into a vector field $\widehat L_*$, and

$$L_*^{a-1}\overline{L}_*^{b-1}\varLambda = |\det A(\zeta)|^2(\widehat L_*^{a-1}\overline{\widehat L_*}^{b-1}\widehat\varLambda)\circ F_\zeta.$$

Now note that for $a,b = 2,\ldots,n-1$ we even have

$$\mathcal L_{a\bar b} = \sum_{\nu,\mu=2}^{n-1}\frac{\partial(F_\zeta)_\nu}{\partial z_a}\widehat{\mathcal L}_{\nu\bar\mu}\circ F_\zeta\overline{\frac{\partial(F_\zeta)_\mu}{\partial z_b}},$$

due to the special form of F_ζ. In particular this implies

$$\varLambda' = |\det A(\zeta)|^2\,\widehat\varLambda'\circ F_\zeta$$

and finally

$$\Big|L_*^{a-1}\overline{L}_*^{b-1}\Lambda - \Lambda' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b} \circ F_\zeta\Big|$$
$$= |\det A(\zeta)|^2 \Big|\widehat{L}_*^{a-1}\overline{\widehat{L}_*}^{b-1}\widehat{\Lambda} - \widehat{\Lambda}' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b}\Big| \circ F_\zeta$$
$$\le C_2' \Big|\widehat{L}_*^{a-1}\overline{\widehat{L}_*}^{b-1}\widehat{\Lambda} - \widehat{\Lambda}' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b}\Big| \circ F_\zeta$$

with some universal constant $C_2' > 0$.

STEP 2. We have to estimate $\big|\widehat{L}_*^{a-1}\overline{\widehat{L}_*}^{b-1}\widehat{\Lambda} - \widehat{\Lambda}' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b}\big|$ on the polydisc $R_{\theta\,\delta}(\zeta)$ and can do this by means of [Her-2b, Lemma 5.2].

For $p \ge 1$ we let M_p' denote the set of all derivatives $\partial^{\nu+\mu}\rho_\zeta/\partial w_n^\nu \partial \bar{w}_n^\mu$, where $\nu + \mu \le p$, and M_p'' the set of all products

$$\frac{\partial^{\nu'+\mu'+1}\rho_\zeta}{\partial w_j^\alpha \partial \bar{w}_j^\beta\, \partial w_n^{\nu'} \partial \bar{w}_n^{\mu'}} \cdot \frac{\partial^{\nu''+\mu''+1}\rho_\zeta}{\partial w_s^\gamma \partial \bar{w}_s^\delta\, \partial w_n^{\nu''} \partial \bar{w}_n^{\mu''}},$$

where $\alpha + \beta = \gamma + \delta = 1$, $2 \le j, s \le n-1$, and $\nu' + \nu'' + \mu' + \mu'' \le p$. Finally we let $M_p := M_p' \cup M_{p+1}''$ and denote by S_p the set of all functions of the form $|(\rho_\zeta)_{w_1}|^{-2N}$ times a polynomial in the derivatives of ρ_ζ of order $\le p$. Then, by [Her-2b, Lemma 5.2],

$$\widehat{L}_*^{a-1}\overline{\widehat{L}_*}^{b-1}\widehat{\Lambda} - \widehat{\Lambda}' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b} \in S_{a+b}M_{a+b-1},$$

where the right-hand side is the set of all sums of products fg with $f \in S_{a+b}$ and $g \in M_{a+b-1}$. But there exists a constant $C_{a+b}'' > 0$ such that for all $\theta \in (0,1)$ and $\delta > 0$,

$$\sup_{R_{\theta\delta}(\zeta)} |g| \le C_{a+b}'' \frac{\theta\delta}{\tau_n(\zeta,\delta)^{a+b}}$$

for $g \in M_{a+b-1}$, and therefore

$$\sup_{R_{\theta\delta}(\zeta)} \Big|\widehat{L}_*^{a-1}\overline{\widehat{L}_*}^{b-1}\widehat{\Lambda} - \widehat{\Lambda}' \cdot \frac{\partial^{a+b}\rho_\zeta}{\partial w_n^a \partial \bar{w}_n^b}\Big| \le C_{a+b}'' \frac{\theta\delta}{\tau_n(\zeta,\delta)^{a+b}}. \ \blacksquare$$

LEMMA 3.2.3. *There exist constants $C_e > 1, \delta_0 > 0$ such that if $\zeta_1, \zeta_2 \in U_k$ and $\zeta_1 \in Q_\delta(\zeta_2)$, then*

$$\eta(\zeta_1,\delta) \le C_e \tau_n(\zeta_2,\delta) \le C_e^2 \eta(\zeta_1,\delta) \tag{3.8}$$

whenever $0 < \delta < \delta_0$.

Proof. We know that $F_{\zeta_2}(\zeta_1) \in R_\delta(\zeta_2)$. From (3.7) (with $\zeta = \zeta_2$) and from (3.6) we see that

$$|L_*^{a-1}\,\overline{L}_*^{b-1}\,\Lambda(\zeta_1)| \le C'\left|\frac{\partial^{a+b}\rho_{\zeta_2}}{\partial w_n^a \partial \bar{w}_n^b}(F_{\zeta_2}(\zeta_1))\right| + C'\frac{\delta}{\tau_n(\zeta_2,\delta)^{a+b}} \le C''\frac{\delta}{\tau_n(\zeta_2,\delta)^{a+b}}$$

for any integers $a, b \ge 1$. Together with the Leibniz rule and (3.5) this gives

$$\begin{aligned} |L_*^{a-1}\,\overline{L}_*^{b-1}\,\partial r([L_*,\overline{L_*}])(\zeta_1)| &= \frac{1}{\Lambda'}|L_*^{a-1}\,\overline{L}_*^{b-1}\,\Lambda(\zeta_1)| + E_{ab} \\ &\le C''\frac{\delta}{\tau_n(\zeta_2,\delta)^{a+b}} + E_{ab}, \end{aligned}$$

where E_{ab} is a sum of terms of the form

$$(\text{positive continuous function}) \cdot |L_*^{p-1}\,\overline{L}_*^{q-1}\,\Lambda(\zeta_1)|$$

where $p, q \ge 1$ are integers such that $p + q < a + b$. Therefore

$$E_{ab} \le C''\frac{\delta}{\tau_n(\zeta_2,\delta)^{a+b}}.$$

The functions $C_l(\zeta_1)$ defined in (3.2) can now be estimated by

$$C_l(\zeta_1) \le C'''\frac{\delta}{\tau_n(\zeta_2,\delta)^l},$$

hence $\tau_n(\zeta_2,\delta) \le (C'''\delta/C_l(\zeta_1))^{1/l}$. This implies

$$\tau_n(\zeta_2,\delta) \le C_3\eta(\zeta_1,\delta).$$

Next, in order to estimate $\eta(\zeta_1,\delta)$ from above by means of $\tau_n(\zeta_2,\delta)$, we use (3.7) again. First we fix $l \ge 2$ such that $\tau_n(\zeta_2,\delta) = (\delta/\|P_l(\zeta_2,\cdot)\|)^{1/l}$. Then we can find integers $a, b \ge 1$ such that $a + b = l$ and

$$\left|\frac{\partial^l\,\rho_{\zeta_2}(0)}{\partial w_n^a \partial \bar{w}_n^b}\right| \ge \gamma_m\|P_l(\zeta_2,\cdot)\| = \gamma_m\frac{\delta}{\tau_n(\zeta_2,\delta)^l}$$

with some unimportant constant $\gamma_m > 0$. We choose a small $\theta > 0$. If now $\zeta_1 \in Q_{\theta\delta}(\zeta_2)$, then by (3.7) we obtain

$$\begin{aligned} |L_*^{a-1}\,\overline{L}_*^{b-1}\,\Lambda(\zeta_1)| &\ge \Lambda'(\zeta_1)\left|\frac{\partial^{a+b}\rho_{\zeta_2}}{\partial w_n^a \partial \bar{w}_n^b}(F_{\zeta_2}(\zeta_1))\right| - C_2\theta\frac{\delta}{\tau_n(\zeta_2,\delta)^l} \\ &\ge \Lambda'(\zeta_1)\left[\left|\frac{\partial^{a+b}\rho_{\zeta_2}}{\partial w_n^a \partial \bar{w}_n^b}(0)\right| - \left|\frac{\partial^{a+b}\rho_{\zeta_2}}{\partial w_n^a \partial \bar{w}_n^b}(F_{\zeta_2}(\zeta_1)) - \frac{\partial^{a+b}\rho_{\zeta_2}}{\partial w_n^a \partial \bar{w}_n^b}(0)\right|\right] - C_2\theta\frac{\delta}{\tau_n(\zeta_2,\delta)^l}. \end{aligned}$$

The second term on the right is $\le C_3\theta\delta/\tau_n(\zeta_2,\delta)^l$ and the first is $\ge \gamma_m\Lambda'(\zeta_1)\delta/\tau_n(\zeta_2,\delta)^l$. This gives

$$|L_*^{a-1}\overline{L}_*^{b-1}\,\Lambda(\zeta_1)| \ge \gamma_m\Lambda'(\zeta_1)\left(1 - \frac{C_4}{\gamma_m\Lambda'(\zeta_1)}\theta\right)\frac{\delta}{\tau_n(\zeta_2,\delta)^l}.$$

On the left-hand side of this estimate we can replace $L_*^{a-1}\overline{L}_*^{b-1}\Lambda(\zeta_1)$ with $\Lambda'(\zeta_1)\cdot L_*^{a-1}\overline{L}_*^{b-1}\,\partial r([L_*,\overline{L_*}])(\zeta_1)$, which causes an error that can be controlled by $\widehat{C}\delta/\tau_n(\zeta_2,\delta)^{l-1}$.

For sufficiently small θ and after shrinking δ_0 we obtain

$$|L_*^{a-1}\overline{L}_*^{b-1}\,\partial r([L_*,\overline{L_*}])(\zeta_1)| \geq C_5\frac{\delta}{\tau_n(\zeta_2,\delta)^l}$$

with some unimportant $C_5 > 0$. This proves

$$C_l(\zeta_1) \geq C_5\frac{\delta}{\tau_n(\zeta_2,\delta)^l}$$

and finally

$$\eta(\zeta_1,\delta) \leq \left(\frac{\delta}{C_l(\zeta_1)}\right)^{1/l} \leq \frac{1}{C_5}\tau_n(\zeta_2,\delta)$$

provided that $\zeta_1 \in Q_{\theta\delta}(\zeta_2)$.

Now we take $\delta' = \delta/\theta$ and choose $\zeta_1 \in Q_\delta(\zeta_2) = Q_{\theta\delta'}(\zeta_2)$. Then, by what we proved so far, we get

$$\eta(\zeta_1,\delta) \leq \eta(\zeta_1,\delta') \leq \frac{1}{C_5}\tau_n(\zeta_2,\delta') \leq C_e\tau_n(\zeta_2,\delta) \leq C_e^2\eta(\zeta_1,\delta)$$

with $C_e := \frac{1+C_3}{C_5}\theta^{-1/2}$. ∎

Now we are able to describe how, given $\zeta \in U_k$, the radii $\tau_n(\zeta',\delta)$ behave if ζ' varies within $Q_\delta(\zeta)$.

Corollary 3.2.4.

(a) *For any* $\zeta \in U_k$ *and* $0 < \delta < \delta_0$ *we have*

$$\begin{gathered}\tau_n(\zeta,\delta) \leq C_e\eta(\zeta,\delta), \qquad \eta(\zeta,\delta) \leq C_e\tau_n(\zeta,\delta),\\ \frac{c_3}{C_e}\sqrt{\delta} \leq \tau_n(\zeta,\delta) \leq \frac{C_e}{c_3}\delta^{1/(2m)},\end{gathered} \tag{3.9}$$

where c_3 *is as in* (3.4).

(b) *If* $\zeta_1 \in Q_\delta(\zeta_2)$, *then*

$$\tau_n(\zeta_1,\delta) \leq C_e^2\tau_n(\zeta_2,\delta), \qquad \tau_n(\zeta_2,\delta) \leq C_e^2\tau_n(\zeta_1,\delta). \tag{3.10}$$

Proof. (a) In the preceding lemma just take $\zeta = \zeta_1 = \zeta_2$.

(b) We again use the above lemma and part (a) and find

$$\tau_n(\zeta_1,\delta) \leq C_e\eta(\zeta_1,\delta) \leq C_e^2\tau_n(\zeta_2,\delta), \quad \tau_n(\zeta_2,\delta) \leq C_e\eta(\zeta_1,\delta) \leq C_e^2\tau_n(\zeta_1,\delta). \ ∎$$

3.3. Comparison of pseudoballs. We next prove a property of pseudoballs that we call the "engulfing property".

LEMMA 3.3.1. *After enlarging the constant C_e from Lemma 3.2.3 we can achieve the following: Suppose that $\zeta_1, \zeta_2 \in U_k$ and $\zeta_1 \in Q_\delta(\zeta_2)$ for $0 < \delta < \delta_0$. Then*

(a) $Q_\delta(\zeta_1) \subset Q_{C_e\delta}(\zeta_2)$,
(b) $\zeta_2 \in Q_{C_e\delta}(\zeta_1)$,
(c) $Q_\delta(\zeta_2) \subset Q_{C_e^2\delta}(\zeta_1)$.

We will show this by applying the Schwarz lemma. For positive numbers T, δ we let

$$G_\delta := \{s \in \mathbb{C} \mid \operatorname{Re} s < T\delta + T|\operatorname{Im} s|\}.$$

LEMMA 3.3.2. *Let $\zeta \in U_k$ and $0 < \delta < \delta_0$. Then for every mapping $h : R_\delta(\zeta) \to G_\delta$ we have*

$$|h(t)| \leq \frac{2T}{1 - 2^{-2m}}(|h(0)| + \delta)$$

whenever $t \in R_{\delta/2}(\zeta)$.

Proof. We let $h_2 := h - T\delta$. This mapping has values in the slit plane $\{re^{i\alpha} \mid 0 < \alpha < 2\pi\}$, where a branch ϕ of the square root exists that takes -1 into i. Next let $h_1 := \phi \circ h_2$. Then the function

$$h_3 := \frac{h_1 - h_1(0)}{h_1 - \overline{h_1(0)}}$$

maps $R_\delta(\zeta)$ into the unit disc with $h_3(0) = 0$. The Schwarz lemma yields

$$|h_3(t)| \leq \psi_0(t, \delta) := \max\left\{\frac{|t_1|}{\delta}, \frac{|t''|}{\sqrt{\delta}}, \frac{|t_n|}{\tau_n(\zeta, \delta)}\right\}.$$

This implies after some computation

$$\left|h_1(t) - \frac{h_1(0) - \overline{h_1(0)}\psi_0(t,\delta)^2}{1 - \psi_0(t,\delta)^2}\right| \leq 2\psi_0(t,\delta)\,\frac{\operatorname{Im} h_1(0)}{1 - \psi_0(t,\delta)^2}$$

and finally

$$|h_1(t)| \leq \frac{|h_1(0)|(1 + \psi_0(t,\delta)^2) + 2\psi_0(t,\delta)\operatorname{Im} h_1(0)}{1 - \psi_0(t,\delta)^2} \leq \frac{1 + \psi_0(t,\delta)}{1 - \psi_0(t,\delta)}|h_1(0)|.$$

But $|h_1(0)| = \sqrt{|h_2(0)|} \leq \sqrt{|h(0)| + T\delta}$, hence

$$|h_1(t)| \leq \frac{1 + 2^{-2m}}{1 - 2^{-2m}}\sqrt{|h(0)| + T\delta}$$

because of $\tau_n(\zeta, \delta/2) \leq 2^{-2m}\tau_n(\zeta, \delta)$. This implies

$$|h(t)| \leq |h_2(t)| + T\delta = |h_1(t)|^2 + T\delta \leq \frac{2T}{1 - 2^{-2m}}(|h(0)| + \delta). \ \blacksquare$$

Proof of Lemma 3.3.1. (a) We assume that $\zeta_1 \in Q_\delta(\zeta_2)$ and $\zeta_1, \zeta_2 \in U_k$. Let $F := F_{\zeta_2} \circ F_{\zeta_1}^{-1}$. We want to show that, after enlarging C_e if necessary, we have $F(R_\delta(\zeta_1)) \subset R_{C_e\delta}(\zeta_2)$. This is equivalent to $Q_\delta(\zeta_1) \subset Q_{C_e\delta}(\zeta_2)$.

The key observation is $\rho_{\zeta_2} \circ F = \rho_{\zeta_1}$. On $R_\delta(\zeta_1)$ we have $\rho_{\zeta_1} \le T_1\delta$ with some unimportant $T_1 > 0$. This gives, for $x \in R_\delta(\zeta_1)$ by means of Lemma 2.1.1(c),

$$\begin{aligned}(3.11) \qquad T_1\delta \ge \rho_{\zeta_1}(x) &= \rho_{\zeta_2} \circ F(x)\\ &\ge \operatorname{Re}(F_1(x)) + |F_2(x)|^2 + \cdots + |F_{n-1}(x)|^2\\ &\quad - \sum_{j=2}^{2m} \|P_j(\zeta_2,\cdot)\|\,|F_n(x)|^j - |\operatorname{Im} F_1(x)| \sum_{j=2}^{2m} \|Q_j(\zeta_2,\cdot)\|\,|F_n(x)|^j\\ &\quad - 2\sum_{a=2}^{n-1} |F_a(x)|\,|g_a(\zeta_2, F_n(x))| - |\mathcal{R}(\zeta_2, F(x))|,\end{aligned}$$

where the remainder is estimated by

$$\begin{aligned}|\mathcal{R}(\zeta_2, F(x))| \le T_2\Big(&|\operatorname{Im} F_1(x)| + |F_n(x)| \sum_{j=2}^{n-1} |F_j(x)|^2\\ &+ |F_n(x)|^{2m+1} + |F_2(x)|^3 + \cdots + |F_{n-1}(x)|^3\Big).\end{aligned}$$

From [Her-2a, Lemma 3.2] we know that

$$|g_a(\zeta_2, F_n(x))| \le T_3|F_n(x)|\Big(\sum_{j=2}^{2m} \|P_j(\zeta_2,\cdot)\|\,|F_n(x)|^j\Big)^{1/2}, \qquad a = 2,\dots,n-1.$$

But as $\zeta_1 \in Q_\delta(\zeta_2)$, or equivalently $F_{\zeta_2}(\zeta_1) \in R_\delta(\zeta_2)$, we have (observe that we are assuming $x \in R_\delta(\zeta_1)$, in particular $|x| \le \delta^{1/(2m)}$)

$$|F_n(x)| = |x_n + (\zeta_1 - \zeta_2)_n| \le \tau_n(\zeta_1,\delta) + \tau_n(\zeta_2,\delta) \le (1 + C_e^2)\tau_n(\zeta_2,\delta).$$

Further,

$$|F_2(x)|^3 + \cdots + |F_{n-1}(x)|^3 \le \delta^{1/(2m)}(|F_2(x)|^2 + \cdots + |F_{n-1}(x)|^2),$$

in particular

$$\begin{aligned}&|\mathcal{R}(\zeta_2, F(x))|\\ &\qquad \le T_2\big(|\operatorname{Im} F_1(x)| + \delta^{1/(2m)}(|F_2(x)|^2 + \cdots + |F_{n-1}(x)|^2) + \delta^{1+1/(2m)}\big).\end{aligned}$$

Substituting this into (3.11) we eventually find

$$\begin{aligned}(3.12) \quad T_1\delta \ge \operatorname{Re}(F_1(x)) &+ (1 - \widehat{C}\delta^{1/(2m)})\big(|F_2(x)|^2 + \cdots + |F_{n-1}(x)|^2\big)\\ &-\widehat{C}\delta - T_3|\operatorname{Im} F_1(x)|.\end{aligned}$$

So we can apply Lemma 3.3.2 to F_1. Since $F(0) = F_{\zeta_2}(\zeta_1) \in R_\delta(\zeta_2)$, hence $|F_1(0)| \le T_5\delta$, we obtain $|F_1(x)| \le T_6\delta$ on $R_{\delta/2}(\zeta_1)$. Then, using (3.12) we get

$$|F_2(x)|^2 + \dots + |F_{n-1}(x)|^2 \le T_7\delta \quad \text{ for } x \in R_{\delta/2}(\zeta_1).$$

This gives claim (a) if we enlarge C_e.

(b) Now we define $\widehat{F} := F_{\zeta_1} \circ F_{\zeta_2}^{-1}$. We want to show that $\widehat{F}(R_\delta(\zeta_2)) \subset R_{C_e\delta}(\zeta_1)$. Once we have proved this we see that

$$Q_\delta(\zeta_2) \subset F_{\zeta_1}^{-1}\big(\widehat{F}(R_\delta(\zeta_2))\big) \subset F_{\zeta_1}^{-1}(R_{C_e\delta}(\zeta_1)) = Q_{C_e\delta}(\zeta_1).$$

We let $y := F_{\zeta_2}(\zeta_1)$. Then $y \in R_\delta(\zeta_2)$, and we choose a holomorphic automorphism ϕ_y of $R_\delta(\zeta_2)$ with $\phi_y(0) = y$ and $\phi_y(y) = 0$. Now we use $\rho_{\zeta_1} \circ \widehat{F} = \rho_{\zeta_2}$, in particular

$$\rho_{\zeta_2} \circ \phi_y = \rho_{\zeta_1} \circ \widehat{F} \circ \phi_y.$$

We repeat the arguments from the proof of (a) for $\widehat{F} \circ \phi_y$ in place of F and $\rho_{\zeta_2} \circ \phi_y$ in place of ρ_{ζ_1}. So we obtain, noting that $\widehat{F} \circ \phi_y(0) = 0$ (hence Lemma 3.3.2 applies to $h := \widehat{F}_1$),

$$\widehat{F}(R_\delta(\zeta_2)) \subset R_{C_e\delta}(\zeta_1),$$

provided we enlarge C_e (which is possible uniformly in δ and the ζ's). So we obtain (b) from $F_{\zeta_1}(\zeta_2) = \widehat{F}(0) \in R_{C_e\delta}(\zeta_1)$, which means that $\zeta_2 \in Q_{C_e\delta}(\zeta_1)$.

(c) By (b) we have $\zeta_2 \in Q_{C_e\delta}(\zeta_1)$, hence by (a), with the roles of ζ_1 and ζ_2 interchanged,

$$Q_\delta(\zeta_2) \subset Q_{C_e\delta}(\zeta_2) \subset Q_{C_e^2\delta}(\zeta_1). \ \blacksquare$$

3.4. Properties of the pseudodistance. We next study suitable substitutes for the symmetry and the triangular inequality of the pseudodistance.

Let δ_0 denote the number that appeared in the preceding subsection. For any $t > 0$ we denote by S_t the strip

$$S_t := \{|r| < t\}.$$

We assume that δ_0 is so small that for any $z \in S_{\delta_0}$ its orthogonal projection $z^* \in \partial D$ onto ∂D is uniquely defined.

LEMMA 3.4.1. *There exists a constant $\widetilde{C}_0 > 0$ such that (after shrinking δ_0):*

(a) $S_{\widetilde{C}_0\delta} \subset \bigcup_{\zeta\in\partial D} Q_\delta(\zeta)$ *for* $0 < \delta < \delta_0$.

(b) $d'(z, z^*) \le \widetilde{C}_0^{-1}\delta_D(z)$ *for each* $z \in S_{\delta_0}$.

Proof. (a) For any $z \in S_{\delta_0}$ we have $z = z^* - \delta_D(z)\frac{\nabla r}{|\nabla r|}(z^*)$, and hence

$$\big|r(z) + \delta_D(z)|\nabla r(z^*)|\big| \le \widetilde{C}_1\delta_D(z)^2$$

with some constant $\widetilde{C}_1 > 0$. After shrinking δ_0 we get

$$\widetilde{C}_2\delta_D(z) \le |r(z)| \le \widetilde{C}_2^{-1}\delta_D(z)$$

on S_{δ_0}. This implies in conjunction with (2.4) that

$$|F_{z^*}(z)| \le L_0\delta_D(z) \le \frac{L_0}{\widetilde{C}_2}\,|r(z)|$$

for $z \in S_{\delta_0}$. Let $\widetilde{C}_0 := \frac{\widetilde{C}_2}{2L_0(1+\gamma_0^{-1})}$. Then, using (2.5), we find

$$|[F_{z^*}(z)]_\nu| \le \frac{L_0}{\widetilde{C}_2}\widetilde{C}_0\,\delta \le \tau_\nu(z^*,\delta), \qquad 1 \le \nu \le n,$$

for $z \in S_{\widetilde{C}_0\delta}$. This gives the claim.

(b) If $\widetilde{C}_2$ has the same meaning as in (a), then $z \in S_{\delta_0}$ belongs to $S_{\widetilde{C}_0\delta}$ if we choose $\delta := (\widetilde{C}_0\widetilde{C}_2)^{-1}\delta_D(z)$. Hence, by the arguments for (a) we see that $z \in Q_\delta(z^*)$. This implies

$$d'(z,z^*) \le \delta = (\widetilde{C}_0\widetilde{C}_2)^{-1}\delta_D(z). \blacksquare$$

Lemma 3.4.2. *There exists a constant $\widehat{C}_1 > 0$, depending only on δ, C_e, and the diameter R_D of D, such that:*

(a) *For $A, B \in D$,*

$$d(B,A) \le \widehat{C}_1\, d(A,B).$$

(b) *Whenever $d'(A,B) < \infty$, then*

$$d'(A,B) \le \widehat{C}_1 d(A,B).$$

(c) *The triangular inequality holds in the form*

$$d(A,B) \le \widehat{C}_1(d(A,C) + d(B,C)) \quad \textit{for } A,B,C \in \overline{D}.$$

(d) *For $A \in D$ and $B \in S_{\delta_0}$,*

$$d(A,B) \le \widehat{C}_1(d(A,B^*) + \delta_D(B)).$$

(e) *Let R_0 be as in Lemma 2.1.1. If $A, Q \in U_0$ and $A \in B(Q,R_0)$, then*

$$\frac{1}{3}\Big(|[F_Q(A)]_1| + |[F_Q(A)]''|^2 + \sum_{l=2}^{2m} \|P_l(\zeta,\cdot)\|\,|[F_Q(A)]_n|^l\Big)$$

$$\le d'(A,Q) \le 2\Big(|[F_Q(A)]_1| + |[F_Q(A)]''|^2 + \sum_{l=2}^{2m} \|P_l(\zeta,\cdot)\|\,|[F_Q(A)]_n|^l\Big).$$

Proof. (a) We adapt the proof of [Her-3, Lemma 3.1]. If $d(A,B) = |A-B|$, we simply have $d(B,A) \le |A-B| = d(A,B)$. Hence we assume that $d(A,B) = d'(A,B)$ and consider two cases. If $d'(A,B) > \delta_0/2$, then

$$d(B,A) \le |A-B| \le R_D \le \frac{2R_D}{\delta_0} d'(A,B) = \frac{2R_D}{\delta_0} d(A,B)$$

If $d'(A,B) \le \delta_0/2$, we choose $\delta \in \big(d'(A,B), \frac{3}{2}d'(A,B)\big)$. Then $A \in Q_\delta(B)$. By Lemma 3.3.1 we obtain $B \in Q_{C_e\delta}(A)$, hence

$$d(B,A) \le C_e\delta \le \tfrac{3}{2}C_e d'(A,B) = \tfrac{3}{2}C_e d(A,B)$$

(b) Assume that $d'(A,B) < \infty$. We must show that $d'(A,B) \le \widehat{C}|A-B|$ with some constant $\widehat{C} > 0$. We start with the observation that $A \notin Q_{d'(A,B)/2}(B)$. If now $|[F_A(B)]_1| > \frac{1}{2}d'(A,B)$, we get $2L_0|A-B| \ge 2|[F_A(B)]_1| > d'(A,B)$.

If $|[F_A(B)]_l| > \sqrt{\frac{1}{2}d'(A,B)}$ for some $l \in \{2,\dots,n-1\}$, then

$$2L_0^2 R_D\,|A-B| \ge 2L_0^2|A-B|^2 \ge 2|F_A(B)|^2 > d'(A,B).$$

If finally $|[F_A(B)]_n| > \tau_n\big(B, \frac{1}{2}d'(A,B)\big)$, we obtain (using (2.5))

$$L_0|A-B| \ge |[F_A(B)]_n| \ge \gamma_0\big(\tfrac{1}{2}d'(A,B)\big)^{1/2},$$

hence

$$d'(A,B) \le \frac{2L_0^2}{\gamma_0^2}|A-B|^2 \le \frac{2L_0^2 R_D}{\gamma_0^2}|A-B|.$$

This implies $d'(A,B) \le 2(c_1L_0)^2R_D^2|A-B|$, and hence the assertion.

(c) The triangular inequality is proved in analogy to [Her-3, Lemma 3.1].

(d) Follows from (c) and Lemma 3.4.1(b).

(e) We write for short

$$T := \Big(|[F_Q(A)]_1| + |[F_Q(A)]''|^2 + \sum_{l=2}^{2m} \|P_l(\zeta,\cdot)\|\,|[F_Q(A)]_n|^l\Big).$$

Certainly $F_Q(A) \in R_{2T}(Q)$. By definition of $d'(A,Q)$ this yields $d'(A,Q) \le 2T$. On the other hand we can estimate from below each δ for which $F_Q(A) \in R_\delta(Q)$, namely $|[F_Q(A)]_1| < \delta$, $|[F_Q(A)]''| \le \sqrt{\delta}$, and $|[F_Q(A)]_n| < \tau_n(Q,\delta)$. The latter is equivalent to

$$\sum_{l=2}^{2m} \|P_l(\zeta,\cdot)\|\,|[F_Q(A)]_n|^l < \delta.$$

Summing these inequalities gives $d'(A,Q) \ge \frac{1}{3}T$. ∎

3.5. Covering a δ-collar around the boundary by a special mesh of pseudoballs Q_δ. We will need an important application of Lemma 3.3.1, namely, we have to be able to cover a thin collar around ∂D of width δ by a finite number of $Q_\delta(\zeta)$'s in such a way that any x in this layer is contained in a finite number of $Q_\delta(\zeta)$'s, and this number does not depend on δ (cf.

[Cat2, Lemma 3.3]). This will be essential for the proof of Lemma 4.3 in the next section.

More precisely:

Lemma 3.5.1. *There exists a number $\mathcal{N}_0$ with the following property: Given $\delta_0 > \delta > 0$ we can find a set $\{\zeta^{(\nu)} \mid \nu \in T_\delta\} \subset \partial D$, where T_δ is a finite index set, such that:*

(a) $S_{\widetilde{C}_0\delta} \subset \bigcup_{\nu \in T_\delta} Q_\delta(\zeta^{(\nu)})$.

(b) *For any $x \in S_{\widetilde{C}_0\delta}$ the set*

$$A_x := \{\nu \in T_\delta \mid Q_\delta(x) \cap Q_\delta(\zeta^{(\nu)}) \neq \emptyset\}$$

has at most $\mathcal{N}_0$ elements. (*Here $\widetilde{C}_0$ is the constant from Lemma* 3.4.1.)

Proof. We start with a finite covering $(Q_\delta(y^{(k)}))_{k\in T'_\delta}$ of ∂D and select a subcovering with the desired property. Let $\zeta^{(1)} = y^{(1)}$. If $\partial D \subset Q_\delta(\zeta^{(1)})$ we are done. Otherwise we have a point $y^{(i_2)} \in \partial D \setminus Q_\delta(\zeta^{(1)})$. Let $\zeta^{(2)} = y^{(i_2)}$. Again, if $\partial D \subset Q_\delta(\zeta^{(1)}) \cup Q_\delta(\zeta^{(2)})$, we choose $T_\delta = \{1, 2\}$. Inductively assume that for $\nu \geq 3$ we have found $\zeta^{(1)}, \ldots, \zeta^{(\nu-1)} \in \{y^{(i)} \mid i \in T'_\delta\}$ such that $\zeta^{(k)} \notin \bigcup_{s=1}^{k-1} Q_\delta(\zeta^{(s)})$ for any $k = 2, \ldots, \nu-1$. Then $\partial D \subset \bigcup_{s=1}^{\nu-1} Q_\delta(\zeta^{(s)})$ or there exists an $i_\nu \in T'_\delta$ such that $\zeta^{(\nu)} := y^{(i_\nu)} \notin \bigcup_{s=1}^{\nu-1} Q_\delta(\zeta^{(s)})$. This defines a sequence $(\zeta^{(\nu)})_{\nu\in T_\delta}$ with T_δ finite such that

$$\zeta^{(\nu)} \notin \bigcup_{s=1}^{\nu-1} Q_\delta(\zeta^{(s)}). \tag{3.13}$$

Our claim is that for any $x \in \partial D$ the cardinality of A_x is bounded uniformly in δ. Let C_e denote the constant from Lemma 3.3.1. We claim that

$$Q_{C_e^{-2}\delta}(\zeta^{(\nu)}) \cap Q_{C_e^{-2}\delta}(\zeta^{(l)}) = \emptyset$$

for $\nu \neq l$. To see this assume that $\nu > l$ and that there exists $y \in Q_{C_e^{-2}\delta}(\zeta^{(\nu)}) \cap Q_{C_e^{-2}\delta}(\zeta^{(l)})$. Then by 3.3.1 we obtain $\zeta^{(\nu)} \in Q_{C_e^{-1}\delta}(y) \subset Q_\delta(\zeta^{(l)})$, which contradicts (3.13).

We next estimate the volume of $Q_{C_e^{-2}\delta}(\zeta^{(\nu)})$ from below whenever $\nu \in A_x$, namely

$$\mathrm{Vol}(Q_{C_e^{-2}\delta}(\zeta^{(\nu)})) \geq c_5 C_e^{-2n} \delta^n \tau_n(\zeta^{(\nu)}, \delta)^2, \tag{3.14}$$

$c_5 > 0$ being an unimportant constant. For any $z \in Q_\delta(\zeta^{(\nu)}) \cap Q_\delta(x)$ we get

$$\tau_n(z, \delta) \leq C_e^2 \tau_n(\zeta^{(\nu)}, \delta),$$

and, since $x \in Q_{C_e\delta}(z)$, it follows that

$$\tau_n(x, \delta) \leq C_e^2 \tau_n(z, C_e\delta) \leq C_e^3 \tau_n(z, \delta) \leq C_e^5 \tau_n(\zeta^{(\nu)}, \delta).$$

In particular by means of (3.14) we obtain

$$\mathrm{Vol}(Q_{C_e^{-2}\delta}(\zeta^{(\nu)})) \geq c_5 C_e^{-10-2n}\delta^n \tau_n(x,\delta)^2. \tag{3.15}$$

At the same time, $\zeta^{(\nu)} \in Q_{C_e\delta}(z) \subset Q_{C_e^2\delta}(x)$, hence $Q_\delta(\zeta^{(\nu)}) \subset Q_{C_e^3\delta}(x)$ for any $\nu \in A_x$. Thus

$$\bigcup_{\nu\in A_x} Q_{C_e^{-2}\delta}(\zeta^{(\nu)}) \subset \bigcup_{\nu\in A_x} Q_\delta(\zeta^{(\nu)}) \subset Q_{C_e^3\delta}(x).$$

Let $\sharp$ denote cardinality. In conjunction with (3.15) the above yields

$$(\sharp\, A_x) c_5 C_e^{-10-2n}\delta^n\tau_n(x,\delta)^2 \leq \mathrm{Vol}\Big(\bigcup_{\nu\in A_x} Q_{C_e^{-2}\delta}(\zeta^{(\nu)})\Big) \leq c_6\delta^n C_e^{3n+2}\tau_n(x,\delta)^2.$$

This implies $\sharp\, A_x \leq c_6 C_e^{5n+12}/c_5$ independently of δ. ■

4. Plurisubharmonic weight functions. In the next step we construct for a pseudoball $Q_\delta(\zeta)$ (with small $\delta > 0$ and ζ close to ∂D) a negative smooth plurisubharmonic function $\phi_{\zeta,\delta}$ whose Levi form "fits well" the geometric form of $Q_\delta(\zeta)$, and is bounded from below by a uniform constant.

For a precise statement of this we prepare some

Notation. Let U_0 and $U_1,\ldots,U_n \subset U_0$ denote the open sets we have fixed before the statement of Lemma 3.1.1. We choose open sets $\widetilde{U}_k \subset\subset U_k$, for $k = 1,\ldots,n$, such that $\partial D \subset \widetilde{U}_1 \cup \cdots \cup \widetilde{U}_n$. On each U_k we fix a set of vector fields ${}^kL_1,\ldots,{}^kL_{n-1},{}^kL_*$ of a boundary system. Let the normal field N be defined as in (3.1). Over U_k, each $X \in \mathbb{C}^n$ has a unique representation in the form

$$X = b_1 N + \sum_{\nu=2}^{n-1} b_\nu\, {}^kL_\nu + b_n\, {}^kL_*$$

with smooth coefficients $b_1,\ldots,b_n$ on U_k.

We may assume that δ_0 has been chosen so small that

(a) $S_{\delta_0} \subset \widetilde{U}_1 \cup \cdots \cup \widetilde{U}_n$,

(b) $Q_\delta(\zeta) \subset\subset U_k$ for any $\zeta \in S_{\widetilde{C}_0\delta} \cap \widetilde{U}_k$, $\delta < \delta_0$, $k = 1,\ldots,n$.

Lemma 4.1. *There exists a constant K_0 such that, after possibly shrinking δ_0, for any $0 < \delta < \delta_0$ and any $\zeta \in S_{\widetilde{C}_0\delta}$ we can find a smooth plurisubharmonic function $\psi_{\zeta,\delta}$ on $\widetilde{D}_{\zeta,\delta/K_0} := \{\rho_\zeta < \delta/K_0\}$ with the following properties:*

(i) $-\frac{5}{12}K_0^{-2/3} \leq \psi_{\zeta,\delta} < 0$ *on* $\widetilde{D}_{\zeta,\delta/K_0}$.

(ii) *On* $R_{\delta/K_0^{5/6}}(\zeta) \cap D_{\zeta,\delta/K_0}$ *the Levi form* $\mathscr{L}_{\psi_{\zeta,\delta}}$ *of* $\psi_{\zeta,\delta}$ *satisfies*

$$
\mathscr{L}_{\psi_{\zeta,\delta}}(w,X) \geq \begin{cases} \dfrac{1}{K_0}\left(\dfrac{|X_1|^2}{\delta^2} + \dfrac{|X''|^2}{\delta} + \dfrac{|X_n|^2}{\tau_n(\zeta,\delta)^2}\right) & \textit{for } n \geq 3, \\ \dfrac{1}{K_0}\left(\dfrac{|X_1|^2}{\delta^2} + \dfrac{|X_2|^2}{\tau_2(\zeta,\delta)^2}\right) & \textit{for } n = 2, \end{cases} \tag{4.1}
$$

for $X \in \mathbb{C}^n$.

Proof. For $n = 2$ this is contained in [Cat2], so assume $n \geq 3$. Let $\delta < \delta_0$. We apply [Her-2a, Theorem 4] for any $\zeta \in S_{\widetilde{C}_0\delta}$. There exists a universal radius $R_2 > 0$ (independent of δ and ζ), and on the disc $\Delta(0, 3R_2)$ a subharmonic function $w_n \mapsto \widetilde{P}(\zeta, w_n)$ such that the plurisubharmonic function

$$\widehat{\varphi}_\zeta(w) := \mathrm{Re}(w_1 + M w_1^2) + \tfrac{1}{2}|w''|^2 + \widetilde{P}(\zeta, w_n)$$

satisfies, with suitably chosen $M, K > 0$, the estimate

$$2\rho_\zeta(w) - M|w_1|^2 - K\mathscr{B}(\zeta, w') \leq \widehat{\varphi}_\zeta(w) \leq \frac{1}{2}\rho_\zeta(w) - \frac{1}{M}|w_1|^2 - \frac{1}{K}\mathscr{B}(\zeta, w')$$

on the ball $B(0, 2R_2)$, where we write

$$\mathscr{B}(\zeta, w') := |w''|^2 + \sum_{j=2}^{2m} \|P_j(\zeta, \cdot)\|\, |w_n|^j. \tag{4.2}$$

Let $K_0 > 1$ be a constant, to be chosen later. From the pseudoconvexity of D, and hence of $\{\rho_\zeta < 0\}$, it follows that also the function

$$\varphi_{\zeta,\delta,0} := \log \frac{(1 + K_0^{-1})\delta}{(1 + K_0^{-1})\delta - \rho_\zeta} + A|w|^2$$

is plurisubharmonic on $\{\rho_\zeta < K_0^{-1}\delta\}$, once A is sufficiently large.

We define

$$
\begin{aligned}
\widetilde{\varphi}_{\zeta,\delta}(w) := & -\frac{1}{K_0} + \frac{\widehat{\varphi}_\zeta(w)}{\delta} \\
& + \sum_{j=2}^{2m} \frac{1}{\delta^{2/j}}\left(-(-\widehat{\varphi}_\zeta(w) + \delta)^{2/j} + K_*^{-1}\|P_j(\zeta,\cdot)\|^{2/j}|w_n|^2\right)
\end{aligned} \tag{4.3}
$$

with a constant K_* to be chosen later. On $\{\rho_\zeta < \delta/K_0\} \cap B(0, 2R_2)$ we have

$$\widetilde{\varphi}_{\zeta,\delta}(w) < -K_*^{-1}\delta^{-2/j}\mathscr{B}(w')^{2/j} \quad \text{for any } j = 2, \ldots, 2m,$$

and further the function

$$\widetilde{\varphi}_{\zeta,\delta}(w) - \frac{1}{\sqrt{K_*}}\left(\frac{|w''|^2}{\delta} + \frac{|w_n|^2}{\tau_n(\zeta,\delta)^2}\right)$$

is plurisubharmonic on $R_{3\delta/(2K_*)}(\zeta)$.

The Levi form of $\varphi_{\zeta,\delta,0}$ satisfies

$$\mathscr{L}_{\varphi_{\zeta,\delta,0}} \geq -\mathscr{L}_{\log(1+K_0^{-1})\delta-\rho_\zeta} + A\mathscr{L}_{|w|^2} \geq \frac{1}{2}\,\frac{\partial\rho_\zeta\overline{\partial\rho_\zeta}}{((1+K_0^{-1})\delta-\rho_\zeta)^2} \geq \frac{8}{K_*}\,\frac{\partial\rho_\zeta\overline{\partial\rho_\zeta}}{\delta^2}$$

on $R_{3\delta/(2K_*)}(\zeta)$ if $K_* \gg 1$. But on this set

$$\partial\rho_\zeta\overline{\partial\rho_\zeta} \geq \frac{1}{8}|dw_1|^2 - M_1\delta\mathscr{L}_{|w''|^2} - M_1\left(\frac{\delta}{\tau_n(\zeta,\delta)}\right)^2|dw_n|^2$$

with an unimportant constant M_1. Hence the function

$$\widetilde{\psi}_{\zeta,\delta}(w) := \varphi_{\zeta,\delta,0}(w) + \widetilde{\varphi}_{\zeta,\delta}(w) - \frac{1}{K_*}\left(\frac{|w_1|^2}{\delta^2} + \frac{|w''|^2}{\delta} + \frac{|w_n|^2}{\tau_n(\zeta,\delta)^2}\right)$$

is plurisubharmonic on $R_{5\delta/(4K_*)}(\zeta)$.

We have $\widehat{\psi}_{\zeta,\delta}(0) = \widetilde{\psi}_{\zeta,\delta}(0) = 0$ and

$$\widetilde{\psi}_{\zeta,\delta}(w) < \log(1+2K_0^{-1}) + A|w|^2 - \frac{|w_1|^2}{\delta^2} - \frac{1}{K_*}\sum_{j=2}^{2m}\left(\frac{\mathscr{B}(w')}{\delta}\right)^{2/j}$$

on $\widetilde{D}_{\zeta,\delta/K_0}$. This implies that on $\partial R_{5\delta/(4K_*)}(\zeta)\cap\widetilde{D}_{\zeta,\,\delta/K_0}$, with another unimportant constant $\widetilde{A}$,

$$\widetilde{\psi}_{\zeta,\delta} < \frac{2}{K_0} + \widetilde{A}\delta^{1/m} - \frac{2}{K_*^2} < -\frac{1}{2K_*^2} = -\frac{1}{2K_0^{2/3}}, \tag{4.4}$$

if we choose $K_0 := K_*^3$ and $\delta_0 \leq (4\widetilde{A}K_*)^{-3m}$.

Furthermore, there exists a constant $L_2 > 0$, independent of δ and K_*, such that $\widetilde{\psi}_{\zeta,\delta} > -L_2/T$ on $R_{\delta/T}(\zeta)$ for any $T > 1$. This shows that $\widetilde{\psi}_{\zeta,\delta} > -1/(3K_*^2)$ on $R_{\delta/(3L_2K_*^2)}(\zeta)$. We enlarge K_* (and hence K_0) so that $K_* > 9L_2^2$. Then $R_{\delta/K_*^{5/2}}(\zeta) \subset R_{\delta/(3L_2K_*^2)}(\zeta)$.

After a standard regularization procedure we can assume that $\widetilde{\psi}_{\zeta,\delta}$ is smooth. (Note that $\widetilde{P}(\zeta,\cdot)$ is only continuous!)

Next we choose a convex function $\varkappa : \mathbb{R}\to\mathbb{R}$ such that $\varkappa(x) = -5/(12K_*^2)$ for $x \leq -1/(2K_*^2)$ and $\varkappa(x) = x$ for $x > -1/(3K_*^2)$. Then the function $\psi_{\zeta,\delta} := \varkappa\circ\widetilde{\psi}_{\zeta,\delta}$ is plurisubharmonic on $\{\rho_\zeta < \delta/K_0\}$ and has the desired properties. ■

We push the weight functions $\psi_{\zeta,\delta}$ forward to the original domain $\{r < \delta/K_0\}$:

Lemma 4.2. *There exists a constant $C_* > 0$ such that for each $\zeta \in S_{\widetilde{C}_0\delta}$ and $0 < \delta < \delta_0$ we can find a plurisubharmonic function $\phi_{\zeta,\delta} < 0$ on $\{r < \delta/K_0\}$ such that:*

(i) $-\frac{5}{12}K_0^{-2/3} \leq \phi_{\zeta,\delta} < 0$.

(ii) *The Levi form of $\phi_{\zeta,\delta}$ satisfies*

$$\mathscr{L}_{\phi_{\zeta,\delta}}(z; b_1 N + \sum_{k=2}^{n-1} b_k L_k + b_n L_*) \geq \begin{cases} C_*\Big(\dfrac{|b_1|^2}{\delta^2} + \dfrac{\sum_{j=2}^{n-1}|b_j|^2}{\delta} + \dfrac{|b_n|^2}{\eta(\zeta,\delta)^2}\Big) & \text{if } n \geq 3, \\ C_*\Big(\dfrac{|b_1|^2}{\delta^2} + \dfrac{|b_2|^2}{\eta(\zeta,\delta)^2}\Big) & \text{if } n = 2 \end{cases}$$

on $Q_{\delta/K_0^{5/6}}(\zeta)$, where $(L_2,\dots,L_{n-1},L_*)$ is a boundary system as at the beginning of this section.

Proof. Again we argue only for $n \geq 3$. We put $\phi_{\zeta,\delta} := \psi_{\zeta,\delta} \circ F_\zeta$. This function is plurisubharmonic on $\{r < \delta/K_0\} \cap B(\zeta, R_0)$. But because of (2.4) in conjunction with (4.4) we have $\widetilde{\psi}_{\zeta,\delta} \circ F_\zeta \leq -L_4 \delta^{-1/2m}$, so that for sufficiently small δ we can extend $\widetilde{\psi}_{\zeta,\delta} \circ F_\zeta$ by setting

$$\approx\!\!\psi_{\zeta,\delta} := \begin{cases} \max\{\widetilde{\psi}_{\zeta,\delta} \circ F_\zeta, -5/(12K_0^{2/3})\} & \text{on } \{r < \delta/K_0\} \cap B(\zeta, R_0), \\ -5/(12K_0^{2/3}) & \text{on } \{r < \delta/K_0\} \setminus B(\zeta, R_0). \end{cases}$$

If $\varkappa$ is chosen as in the proof of the preceding lemma, we choose $\phi_{\zeta,\delta}$ as a regularization of $\varkappa \circ \approx\!\!\psi_{\zeta,\delta}$.

This function is defined and plurisubharmonic on $\{r < \delta/K_0\}$, and it satifies (i). To see that the estimate (ii) on the Levi form of $\phi_{\zeta,\delta}$ holds on $Q_{\delta/K_0^{5/6}}(\zeta)$ we may assume that the boundary system $L_2,\dots,L_{n-1},L_*$ is defined on $B(\zeta, 2R_0)$. If we write $X = b_1 N + \sum_{p=2}^{n-1} b_p L_p + b_n L_*$, then

$$\begin{aligned} &b_1 = \frac{1}{|\nabla r|}\langle \partial r, X\rangle, \\ (4.5)\qquad &b_k = X_k + b_1 \cdot (r_{\bar z_n} s_k - r_{\bar z_k}) - X_n \cdot s_k, \qquad k = 2,\dots,n-1, \\ &b_n = X_n - b_1 \cdot r_{\bar z_n}. \end{aligned}$$

On $F_\zeta^{-1}(\{\psi_{\zeta,\delta} > -1/(3K_0^{2/3})\})$ we obtain

$$\begin{aligned} \mathscr{L}_{\phi_{\zeta,\delta}} &\geq \varkappa' \circ \widetilde{\psi}_{\zeta,\delta} \circ F_\zeta \cdot \mathscr{L}_{\psi_{\zeta,\delta}}(F_\zeta(z), F_\zeta'(z)X) \geq \mathscr{L}_{\psi_{\zeta,\delta}}(F_\zeta(z), F_\zeta'(z)X) \\ &\geq \frac{1}{2K_*}\bigg(\frac{\big|\big(\partial\rho_\zeta(F_\zeta(z)), F_\zeta'(z)X\big)\big|^2}{\delta^2} + \frac{1}{\delta}\sum_{p=2}^{n-1} |\langle \partial (F_\zeta)_p(z), X\rangle|^2 + \frac{|X_n|^2}{\tau_n(\zeta,\delta)^2}\bigg) \\ &\geq \frac{1}{2K_*}\bigg(\frac{\big|\big(\partial\rho_\zeta(F_\zeta(z)), F_\zeta'(z)X\big)\big|^2}{\delta^2} + \frac{1}{K_*\delta}\sum_{p=2}^{n-1} |\langle \partial (F_\zeta)_p(z), X\rangle|^2 + \frac{|X_n|^2}{\tau_n(\zeta,\delta)^2}\bigg). \end{aligned}$$

Now we have $\langle \partial\rho_\zeta(F_\zeta(z), F'_\zeta(z)X\rangle = \langle \partial r(z), X\rangle$. Next we observe that

$$\begin{aligned}\langle \partial [F_\zeta]_p(z), X\rangle &= \sum_{\nu=2}^{n} \frac{\partial [F_\zeta]_p}{\partial z_\nu}(z) X_\nu \\ &= \sum_{\nu=2}^{n-1} \frac{\partial [F_\zeta]_p}{\partial z_\nu}(z) b_\nu - b_1 r_{\bar z_n} \sum_{\nu=2}^{n-1} \frac{\partial [F_\zeta]_p}{\partial z_\nu}(z) s_\nu + \sigma_p \circ F_\zeta \cdot (b_n + b_1\, r_{\bar z_n}),\end{aligned}$$

where the functions σ_p are the coefficients in the representation

$$\widehat{L}_* = \widehat{L}_n + \sum_{p=2}^{n-1} \sigma_p \widehat{L}_p$$

and $\widehat{L}_k$ are the vector fields into which the L_k transform under F_ζ. The σ_p are given by

$$\sigma_p \circ F_\zeta = \sum_{\nu=2}^{n-1} \frac{\partial [F_\zeta]_p}{\partial z_\nu}(z) s_\nu + \frac{\partial [F_\zeta]_p}{\partial z_n}(z). \tag{4.6}$$

By computation one can check that

$$\sigma_p = -\sum_{l=2}^{n-1} \widehat{\mathcal{L}}^{p\bar l} \widehat{\mathcal{L}}_{n\bar l},$$

where the $\mathcal{L}_{n\bar l}$ and $\mathcal{L}^{p\bar l}$ are defined as in the first step in the proof of Lemma 3.2.2. Now by Lemma 3.2.1 we get, on $R_\delta(\zeta)$,

$$|\sigma_p| \le \sqrt{\delta}/\tau_n(\zeta,\delta).$$

This implies ($M_2 > 1$ being independent of K_*) that

$$\begin{aligned}\frac{1}{K_*\delta} |\langle \partial [F_\zeta]_p(z), X\rangle|^2 &\ge \frac{1}{K_*\delta} \Big| \sum_{\nu=2}^{n-1} \frac{\partial [F_\zeta]_p}{\partial z_\nu}(z) b_\nu \Big|^2 - \frac{M_2 |b_1|^2}{K_*\delta} - \frac{|b_n|^2 |\sigma_p|^2}{K_*\delta} \\ &\ge \frac{c'}{K_*\delta} \sum_{\nu=2}^{n-1} |b_\nu|^2 - \frac{M_2}{K_*\delta} |b_1|^2 - \frac{1}{K_*} \frac{|b_n|^2}{\tau_n(\zeta,\delta)^2} \\ &\ge \frac{c'}{K_*\delta} \sum_{\nu=2}^{n-1} |b_\nu|^2 - \frac{M_2}{K_*\delta} |b_1|^2 - \frac{C_e^2}{K_*} \frac{|b_n|^2}{\eta(\zeta,\delta)^2}\end{aligned}$$

with some constant $c' > 0$, because the matrix $A(z) := \big(\frac{\partial F_p}{\partial z_\nu}(z)\big)_{p,\nu=2}^{n-1}$ is invertible, and

$$|A(z)\cdot Y| \approx |Y| \tag{4.7}$$

on $R_\delta(\zeta)$. Plugging all this into the lower estimate on the Levi form of $\phi_{\zeta,\delta}$ we obtain the desired lower bound on $\mathscr{L}_{\phi_{\zeta,\delta}}$. Note that, with some unimportant constant $M > 0$, one has

$$\frac{|X_n|^2}{\tau_n(\zeta,\delta)^2} \geq \frac{|b_n|^2}{\tau_n(\zeta,\delta)^2} - 2\frac{|b_1|^2|r_{z_n}|^2}{\tau_n(\zeta,\delta)^2} \geq \frac{|b_n|^2}{C_e^2\eta_n(\zeta,\delta)^2} - M\frac{|\langle\partial r, X\rangle|^2}{\delta}. \blacksquare$$

In what follows we will need another two weight functions that can be constructed in analogy to the corresponding ones in [Cat2, Cho-1, Cho-2].

Lemma 4.3. *For a sufficiently large constant $K_0 > 0$ and small enough $\delta_0 > 0$ we can find a family $(\lambda_\delta)_{0<\delta<\delta_0}$ of plurisubharmonic functions such that $0 \leq \lambda_\delta \leq 1$ and, on U_k,*

$$\mathscr{L}_{\lambda_\delta}\Big(z; b_1 N + \sum_{p=2}^{n-1} b_p L_p + b_n L_*\Big) \geq \begin{cases} \dfrac{1}{K_0}\left(\dfrac{|b_1|^2}{\delta^2} + \dfrac{|b_2|^2 + \cdots + |b_{n-1}|^2}{\delta} + \dfrac{|b_n|^2}{\tau_n(z,\delta)^2}\right) & \text{if } n \geq 3, \\ \dfrac{1}{K_0}\left(\dfrac{|b_1|^2}{\delta^2} + \dfrac{|b_2|^2}{\tau_2(z,\delta)^2}\right) & \text{if } n = 2, \end{cases}$$

where $(L_2,\ldots,L_*)$ denote the vector fields of a boundary system as in Lemma 3.1.1.

Proof. Let K_0 and δ_0 be as in the preceding lemma. We apply Lemma 3.5.1 and put

$$\lambda_\delta := \frac{12K_0^{2/3}}{5\mathcal{N}_0}\sum_{\nu\in T_\delta}\left(\phi_{\zeta^{(\nu)},\delta} + \frac{5}{12K_0^{2/3}}\right).$$

This function has values between 0 and 1 and its Levi form satisfies the desired estimates. $\blacksquare$

We define for small $t, \varepsilon_0 > 0$ the function

$$\mathscr{J}_{\zeta,t}(\widetilde{z}) := (t^2 + |\widetilde{z}_1|^2 + \mathscr{B}(\zeta,\widetilde{z}')^2)^{1/2}$$

with $\mathscr{B}$ as in (4.2), and for a small radius $0 < R_1 < R_0$ the open set

$$U_{\zeta,t} := \{\rho_\zeta < \varepsilon_0 t\} \cup \{|\widetilde{z}| < R_1 \mid \rho_\zeta(\widetilde{z}) < \varepsilon_0\mathscr{J}_{\zeta,t}(\widetilde{z})\}.$$

The functions λ_δ serve as bricks in the construction of the following family of plurisubharmonic functions:

Lemma 4.4. *Let $n \geq 2$. Then, for suitable $R_1 \leq R_0/(2L_0^3)$ and $t_0, \varepsilon_0 > 0$, we can find for $0 < t \leq t_0$ a smooth plurisubharmonic function $E_{\zeta,t}$ on the domain $U_{\zeta,t}$ such that, with a universal constant $L_1 > 1$:*

(i) $-L_1\mathscr{J}_{\zeta,t} \leq E_{\zeta,t} \leq -(1/L_1)\mathscr{J}_{\zeta,t}$.

(ii) *The Levi form of $E_{\zeta,t}$ satisfies*

$$\mathscr{L}_{E_{\zeta,t}}(\widetilde{z}, Y) \geq \frac{1}{L_1}\left(\frac{|Y_1|^2}{\mathscr{J}_{\zeta,t}(\widetilde{z})^2} + \frac{|Y_2|^2 + \cdots + |Y_{n-1}|^2}{\mathscr{J}_{\zeta,t}(\widetilde{z})} + \frac{|Y_n|^2}{\tau_n(\zeta,\mathscr{J}_{\zeta,t}(\widetilde{z}))^2}\right).$$

(iii) *The domain*
$$D_t^\zeta := \{\widetilde{z} \in B(0,R_1) \mid \rho_\zeta(\widetilde{z}) < t \text{ or } \rho_\zeta(\widetilde{z}) < -\varepsilon_0 E_{\zeta,t}(\widetilde{z})\}$$
is pseudoconvex.

Proof. For $n = 2$ this is [Cat2, Prop. 4.1], and for $n \geq 3$ we can copy the proof of [Cho-2, Prop. 2.2]. ∎

REMARK 4.5. (a) After shrinking δ_0 once again we can achieve that for each $\delta \in (0, \delta_0)$ we can stick together the domains $F_\zeta^{-1}(D_{\delta/K_0}^\zeta)$ and $\{r < t_0\}$ whenever $\zeta \in S_\delta$. For details the reader may consult [Cat2, Sec. 5] and [Cho-2, Prop. 2.7]. During this procedure the parameters t_0 and ε_0 must possibly be shrunk. We will denote the resulting domain by $D_{\zeta,\delta}$. By the preceding lemma we obtain the following: There exists $\theta_0 > 0$ such that for any $z \in D_{\zeta,\delta}$ and $0 < \theta \leq \theta_0$,
$$\begin{cases} F_\zeta^{-1}(P_{\widetilde{z}}^\theta) \subset D_{\zeta,\delta} & \text{if } \widetilde{z} := F_\zeta(z) \in D_{\delta/K_0}^\zeta, \\ \Delta_n(z, c_0 t_0) \subset D_{\zeta,\delta} & \text{otherwise,} \end{cases}$$
where for $\theta \in (0, \theta_0]$ we denote by $P_{\widetilde{z}}^\theta$ the polydisc
$$P_{\widetilde{z}}^\theta := \Delta(\widetilde{z}_1, \theta \mathscr{J}_{\zeta,\delta}(\widetilde{z})) \times \Delta_{n-2}\Big(\widetilde{z}'', \sqrt{\theta \mathscr{J}_{\zeta,\delta}(\widetilde{z})}\Big) \times \Delta(\widetilde{z}_n, \tau_n(\zeta, \theta \mathscr{J}_{\zeta,\delta}(\widetilde{z})).$$

(b) Using Lemma 4.4(i) we see that we can choose $s_1 \in (0,1)$ independently of δ, ζ in such a way that the function
$$\widehat{W}_{\zeta,\delta}(z) := \begin{cases} \max\{E_{\zeta,\delta/K_0} \circ F_\zeta, -s_1\} & \text{on } F_\zeta^{-1}(D_\zeta^{\delta/K_0}), \\ -s_1 & \text{on } D_{\zeta,\delta} \setminus F_\zeta^{-1}(D_\zeta^{\delta/K_0}), \end{cases}$$
becomes plurisubharmonic on $D_{\zeta,\delta}$ whenever $\delta < t_0$.

5. Holomorphic auxiliary functions. We can proceed in a similar manner to the construction of peak functions in [DieHer]. These were of the form $(1-c)\sum_{m=0}^\infty c^m F_m$, where the "bricks" F_m had to satisfy certain conditions. In the next lemma we want to construct the functions F_m that are suitable for the problem at hand.

LEMMA 5.1. *There exist constants $\widehat{C}, k_0, \mathscr{N}_1 > 0$ and for any $\zeta \in \partial D$ and $A \in D$ a family $(F_{A,\zeta,k})_{0<k<k_0} \subset \mathscr{O}(D)$ with the following properties:*

(i) *For any $k \in (0, k_0)$ we have $F_{A,\zeta,k} \in \mathscr{O}(D_{\zeta,k})$, $F_{A,\zeta,k}(\zeta) = 1$.*

(ii) *Let $0 < k < k_0$ and $\zeta \in \partial D$. Then:*

(a) *If $A \notin B(\zeta, R_1)$, then $F_{A,\zeta,k}(A) = 0$.*

(b) *If $A \in B(\zeta, R_1)$, then*
$$F_{A,\zeta,k}(A) = \begin{cases} 1 & \text{if } V_{\zeta,k}(A) \leq 1, \\ 0 & \text{if } V_{\zeta,k}(A) > 1, \end{cases}$$
where $V_{\zeta,k}(z) := |[F_\zeta(z)]_1|^2/k^2 + (1/k)\mathscr{B}(\zeta, [F_\zeta(z)]')$.

(iii) *For any* $k \in (0, k_0)$ *we have* $\|F_{A,\zeta,k}\|_{H^\infty(D)} \le \widehat{C}$.

(iv) *For* $z \in D \cap B(\zeta, R_0)$,

$$|F_{A,\zeta,k}(z) - 1| \le \widehat{C}\biggl(\frac{|[F_\zeta(z)]_1|}{k} + \frac{|[F_\zeta(z)]''|}{\sqrt{k}} + \frac{|[F_\zeta(z)]_n|}{\tau_n(\zeta, k)}\biggr).$$

(v) *For any* $z \in D$, $z \notin Q_{\mathscr{N}_1 k}(\zeta)$ *and* $p \le k$,

$$|F_{A,\zeta,p}(z)| \le \widehat{C}p/k.$$

We will prove this for $n \ge 3$. (The case $n = 2$ goes in a completely analogous way.) We will apply the $\overline{\partial}$-technique developed by [Hör] with suitable plurisubharmonic weight functions on the pseudoconvex domain $D_{\zeta,k}$.

Before proceeding to the details we need some preparations.

Let U_0 and δ_0 be as before. For $0 < t < \delta_0$ we define

$$\begin{aligned} V'_{\zeta,t}(w) &:= \frac{|w_1|^2}{t^2} + \frac{1}{t}\mathscr{B}(\zeta, w'), & V_{\zeta,t} &:= V'_{\zeta,t} \circ F_\zeta, \\ V''_{\zeta,t}(w) &:= \frac{|w_1|^2}{t^2} + \frac{|w''|^2}{t} + \frac{|w_n|^2}{\tau_n(\zeta,t)^2}, & \widehat{V}_{\zeta,t} &:= V''_{\zeta,t} \circ F_\zeta. \end{aligned} \tag{5.1}$$

Then

$$V_{\zeta,t} \le (2m+1)\max\{\widehat{V}_{\zeta,t}, \widehat{V}^m_{\zeta,t}\}, \qquad \widehat{V}_{\zeta,t} \le (2m+1)\max\{V_{\zeta,t}, V^{1/m}_{\zeta,t}\}. \tag{5.2}$$

We further define, for $A \in B(\zeta, R_0)$, the functions

$$\begin{aligned} \widetilde{V}_{A,\zeta,t}(w) &:= \frac{|w_1 - [F_\zeta(A)]_1|^2}{t^2} \\ &\qquad + \frac{|w'' - [F_\zeta(A)]''|^2}{t} + \frac{|w_n - [F_\zeta(A)]_n|^2}{\tau_n(\zeta,t)^2}, \\ V_{A,\zeta,t} &:= \widetilde{V}_{A,\zeta,t} \circ F_\zeta. \end{aligned} \tag{5.3}$$

We have

$$V_{A,\zeta,t}(z) \ge \max\bigl\{\tfrac{1}{2}\widehat{V}_{\zeta,t}(z) - 2\widehat{V}_{\zeta,t}(A), \tfrac{1}{2}\widehat{V}_{\zeta,t}(A) - 2\widehat{V}_{\zeta,t}(z)\bigr\}. \tag{5.4}$$

The $\overline{\partial}$-equation. Let $\chi : \mathbb{R} \to [0,1]$ denote a smooth function with $\chi(x) = 1$ for $x \le 2$ and $\chi(x) = 0$ for $x \ge 3$. With $s \ge 1/(2 + 4(2m+1))^2$ to be chosen later, we write

$$\chi_1(z) := \begin{cases} \chi(sV_{\zeta,k}(z)) & \text{if } |z - \zeta| < R_1, \\ 0 & \text{if } |z - \zeta| \ge R_1. \end{cases}$$

Then there exists $k_0 > 0$ such that χ_1 is smooth on $D_{\zeta,k}$ for any $k \le k_0$. In fact, if $|z - \zeta| = R_1/2$, then $|F_\zeta(z)| \ge R_1/(2L_0)$ and using (3.4) and Lemma 3.2.3 we get $V_{\zeta,k}(z) \ge \gamma \cdot k^{-1/(2m)}$ with an unimportant constant γ, and finally $sV_{\zeta,k}(z) \geq \frac{\gamma}{(2+4(2m+1))^2 k^{1/(2m)}} \geq 3$ if we choose

$$k \leq k_0 := \left(\frac{\gamma}{3(2+4(2m+1))}\right)^{4m}.$$

This proves the smoothness of χ_1 on $D_{\zeta,k}$.

We intend to solve the equation

$$\overline{\partial} u_k = \overline{\partial}\chi_1 \tag{5.5}$$

by means of Hörmander's theory of the $\overline{\partial}$-equation with plurisubharmonic weights. The desired functions will then be given by

$$F_{A,\zeta,k} = \chi_1 - u_k.$$

To ensure that the $F_{A,\zeta,k}$ behave in the desired manner, we must choose the plurisubharmonic weight functions carefully. We want them to exhibit a logarithmic pole at ζ and A. We proceed as follows.

5.1. Choosing plurisubharmonic weights. Let $\lambda \in \mathscr{C}^\infty(\mathbb{R})$ be an increasing function such that $\lambda(x) = x$ for $x \leq 1/2$ and $\lambda(x) = 2/3$ for $x \geq 3/4$.

We can choose $s_0 > 3(192m(2m+1))^m$ such that for any $\zeta \in \partial D$ and $k \in (0, t_0)$ we have

$$\{s_0 V_{\zeta,k} \leq 1\} \subset Q_{k/K_0^{5/6}}(\zeta) \cap \{r < k/K_0\}. \tag{5.6}$$

Furthermore, by Lemma 4.4(i),

$$E_{\zeta,k/K_0}(w) \geq -L_1 \mathscr{J}_{\zeta,k/K_0}(w) \geq -L_8 k \tag{5.7}$$

for $w \in R_{k/K_0}(\zeta)$, with some unimportant $L_8 > 0$.

Hence we can find a small $k_0 \ll 1$ such that

$$E_{\zeta,k/K_0} \geq -L_8 k_0 > -s_1 \tag{5.8}$$

on $\{s_0 V'_{\zeta,k} \leq 1\}$ whenever $0 < k \leq k_0$. (For the definition of s_1, see Remark 4.5(b).)

Lemma 5.1.1.

(i) *Assume that* $|A - \zeta| \geq R_0/(2L_0^3)$. *Define*

$$w_{A,\zeta}(z) = \log \lambda(s_0 V_{\zeta,k}(z)) + \log \frac{|z-A|^2}{R_0^2}$$

and choose $s = 3$. *Then*

$$w_{A,\zeta} \geq \lambda(2s_0/3) - 2\log 4 - 6\log L_0 \qquad \textit{on } \operatorname{supp}(\overline{\partial}\chi_1).$$

(ii) *Assume that* $|A - \zeta| < R_0/(2L_0^3)$ *and* $V_{\zeta,k}(A) \leq 1$ *and let*

$$w_{A,\zeta}(z) := \log \lambda(s_0 V_{\zeta,k}(z)) + \log \lambda(V_{A,\zeta,k}(z)).$$

If $s=(2+4(2m+1))^{-2}$, *then*

$$w_{A,\zeta} \geq \log\lambda(2s_0/s) + \log\frac{2}{3} \quad \textit{on } \operatorname{supp}(\overline{\partial}\chi_1).$$

(iii) *Assume that* $|A-\zeta| < R_0/(2L_0^3)$ *and* $V_{\zeta,k}(A) > 1$. *There is a constant* $v_* \in (0,1)$, *depending only on* R_0, *such that* $\min_{\partial B(\zeta,7R_0/8)} V_{A,\zeta,k} \geq v_*$. *Define* $\omega_k := \mathscr{J}_{\zeta,k}(F_\zeta(A))$ *and choose*

$$w_{A,\zeta} := \log\lambda(s_0 V_{\zeta,k}) + \log\lambda\left(\frac{3}{4v_*}V_{A,\zeta,\omega_k}\right).$$

(*For the definition of* $V_{A,\zeta,t}$ *see* (5.3).) *Let* $s := 3\cdot(192m(2m+1))^m$. *Then*

$$w_{A,\zeta} \geq \log\lambda(2s_0/s) - \log(128m) \quad \textit{on } \operatorname{supp}(\overline{\partial}\chi_1).$$

Proof. (i) We first note

$$|z-A| \geq |A-\zeta| - |z-\zeta| \geq \frac{R_0}{2L_0^3} - |z-\zeta| \geq \frac{R_0}{2L_0^3} - L_0|F_\zeta(z)|$$

and, for $z \in \operatorname{supp}(\overline{\partial}\chi_1)$, using $V_{\zeta,k}(z) \leq 1$,

$$|[F_\zeta(z)]_1| \leq k, \quad |[F_\zeta(z)]''| \leq \sqrt{k}, \quad |[F_\zeta(z)]_n| \leq \tau_n(\zeta,k) \leq c_1 k^{1/(2m)}. \tag{5.9}$$

Combining this with Lemma 3.2.3 and (3.4) we obtain

$$L_0|F_\zeta(z)| \leq L_0\widehat{c}_1 k^{1/(2m)}$$

with an unimportant constant $\widehat{c}_1$ and hence $R_0/(2L_0^3) - |z-\zeta| \geq R_0/(2L_0^3) - L_0|F_\zeta(z)| \geq R_0/(4L_0^3)$ if $k \leq k_0 \ll 1$. This proves

$$w_{A,\zeta} \geq \lambda(2\widetilde{s}/3) - 2\log 4 - 6\log L_0 \quad \text{on } \operatorname{supp}(\overline{\partial}\chi_1).$$

(ii) We have to estimate $V_{A,\zeta,k}$ from below on $\operatorname{supp}(\overline{\partial}\chi_1)$.

By (5.4) and the second estimate from (5.2), we see that

$$V_{A,\zeta,k}(z) \geq \tfrac{1}{2}\widehat{V}_{\zeta,k}(z) - 2\widehat{V}_{\zeta,k}(A) \geq \tfrac{1}{2}\widehat{V}_{\zeta,k}(z) - 2(2m+1).$$

For $z \in \operatorname{supp}(\overline{\partial}\chi_1)$ we have, using the first estimate from (5.2),

$$\max\{\widehat{V}_{\zeta,k}(z), V_{\zeta,k}^m(z)\} \geq \frac{1}{2m+1}V_{\zeta,k}(z) \geq \frac{2}{(2m+1)s} \geq \frac{1}{\sqrt{s}}$$

since $s < (2m+1)^{-2}$. This and the above estimate imply

$$V_{A,\zeta,k}(z) \geq \frac{1}{2\sqrt{s}} - 2(2m+1) = 1$$

by our choice of s. From this we get the desired estimate.

(iii) For $z \in \partial B(\zeta, 7R_0/8)$ we obtain

$$|F_\zeta(z) - F_\zeta(A)| \geq L_0^{-1}|z-\zeta| - L_0|\zeta - A| \geq \frac{3}{8L_0}R_0 =: R_3.$$

On the other hand $\omega_k^2 \le k_0^2 + \widehat{L}_6^2 R_0^2 \le L_7^2 R_0^2$, with unimportant constants L_6, L_7. This leads to

$$V_{A,\zeta,\omega_k}(z) \ge \min_{|w|\ge R_3}\left(\frac{|w_1|^2}{(L_7R_0)^2} + \frac{|w''|^2}{L_7R_0} + \frac{|w_n|^2}{\tau_n(\zeta, L_7R_0)^2}\right),$$

which implies the existence of v_*.

For any $y \ne 0$ we can show (by elementary estimates) that $V_{\zeta,\mathscr{I}_{\zeta,k}(y)}(y) \ge 1/(24m)$ provided that $V'_{\zeta,k}(y) \ge 1$. We exploit this for $y := F_\zeta(A)$. Since $V'_{\zeta,k}(F_\zeta(A)) = V_{\zeta,k}(A) \ge 1$ we get

$$\widehat{V}_{\zeta,\omega_k}(A) = V''_{\zeta,\omega_k}(F_\zeta(A)) \ge \frac{1}{24m},$$

so that for $z \in \operatorname{supp}(\overline{\partial}\chi_1)$,

$$\begin{aligned}\frac{3}{4v_*}V_{A,\zeta,\omega_k}(z) &\ge \frac{3}{4}V_{A,\zeta,\omega_k}(z) \ge \frac{3}{8}\widehat{V}_{\zeta,\omega_k}(A) - \frac{3}{2}\widehat{V}_{\zeta,\omega_k}(z)\\ &\ge \frac{3}{8}\widehat{V}_{\zeta,\omega_k}(A) - \frac{3}{2}(2m+1)\max\{\widehat{V}_{\zeta,\omega_k}(z), V^{1/m}_{\zeta,\omega_k}(z)\}\\ &\ge \frac{3}{8}\widehat{V}_{\zeta,\omega_k}(A) - \frac{3}{2}(2m+1)\max\{\widehat{V}_{\zeta,k}(z), V^{1/m}_{\zeta,k}(z)\}\\ &\ge \frac{1}{64m} - \frac{3}{2}(2m+1)\left(\frac{3}{s}\right)^{1/m} = \frac{1}{128m}\end{aligned}$$

by the choice of s. ∎

Remark 5.1.2. The function $w_{A,\zeta}$ can be viewed as a function on $D_{\zeta,k}$.

Proof. In case (i) of Lemma 5.1.1, we see that $w_{A,\zeta}$ is defined on $B(\zeta, R_0)$, but on $\partial B(\zeta, R_0)$ the function $s_0 V_{\zeta,k}$ has values $\ge s_0 L_5 k^{-1/(2m)} \ge 1$ if k is sufficiently small (here, L_5 is an unimportant constant). Hence we can define $\lambda \circ V_{\zeta,k} := \log(2/3)$ outside $B(\zeta, R_0)$.

In case (ii), we use

$$V_{A,\zeta,k}(z) \ge \tfrac{1}{2}\widehat{V}_{\zeta,k}(z) - 2\widehat{V}_{\zeta,k}(A) \ge \tfrac{1}{2}\widehat{V}_{\zeta,k}(z) - 2 = \tfrac{1}{2}V''_{\zeta,k}(F_\zeta(z)) - 2.$$

On $\partial B(\zeta, R)$ we have $|F_\zeta(z)| \ge R_0/L_0$, and, as in case (i), we find that $V_{A,\zeta,k}(z) \ge 1$ on $\partial B(\zeta, R)$ if k is small enough.

Case (iii) is similar to case (ii). ∎

The above functions $w_{A,\zeta}$ are not yet plurisubharmonic, but we can overcome this by

Lemma 5.1.3. *One can choose a constant M_1 uniformly in ζ and $k \le k_0$ so large that the function*

$$\widehat{w}_{A,\zeta} := M_1\widehat{W}_{\zeta,k} + w_{A,\zeta} + M_1|z|^2$$

is plurisubharmonic throughout $D_{\zeta,k}$.

Proof. For a set M let ξ_M denote the characteristic function of M.

In case (i) of Lemma 5.1.1,

$$w_{A,\zeta} = \log\lambda(s_0 V_{\zeta,k}(z)) + \log\frac{|z-A|^2}{R_0^2}.$$

The function $\log\lambda(s_0 V_{\zeta,k}) = \log\lambda(s_0 V'_{\zeta,k}) \circ F_\zeta$ is not plurisubharmonic. We consider the Levi form of $\log\lambda(s_0 V'_{\zeta,k})$. By computation we find

$$\begin{aligned}
\mathscr{L}_{\log\lambda(s_0 V'_{\zeta,k})} &= s_0(\log\lambda)'(s_0 V'_{\zeta,k})\mathscr{L}_{V'_{\zeta,k}} + s_0^2(\log\,\lambda)''(s_0 V'_{\zeta,k})\,\partial V'_{\zeta,k}\overline{\partial V'_{\zeta,k}}\\
&\ge s_0^2(\log\,\lambda)''(s_0 V'_{\zeta,k})\,\partial V'_{\zeta,k}\overline{\partial V'_{\zeta,k}}\\
&\ge -K_1\cdot\xi_{\{1/2\le s_0 V'_{\zeta,k}\le 3/4\}} s_0^2\,\partial V'_{\zeta,k}\overline{\partial V'_{\zeta,k}} \quad \text{with } K_1 := \max_{[1/2,3/4]}|(\log\,\lambda)''|\\
&\ge -K_1\cdot\xi_{\{1/2\le s_0 V'_{\zeta,k}\le 3/4\}} s_0\,\mathscr{L}_{V'_{\zeta,k}}.
\end{aligned}$$

But the Levi form of $E_{\zeta,\,k/K_0}$ satisfies

$$\mathscr{L}_{E_{\zeta,\,k/K_0}}(w;Y) \ge \frac{1}{L_1}\left(\frac{|Y_1|^2}{\mathscr{I}_{\zeta,k/K_0}(w)^2} + \frac{\sum_{l=2}^{n-1}|Y_l|^2}{\mathscr{I}_{\zeta,k/K_0}(w)} + \frac{|Y_n|^2}{\tau_n(\zeta,\,\mathscr{I}_{\zeta,k/K_0}(w))^2}\right).$$

On $\{s_0 V'_{\zeta,k} \le 3/4\}$ we even have

$$\mathscr{L}_{E_{\zeta,\,k/K_0}}(w;Y) \ge L_9\left(\frac{|Y_1|^2}{k^2} + \frac{\sum_{l=2}^{n-1}|Y_l|^2}{k} + \frac{|Y_n|^2}{\tau_n(\zeta,k)^2}\right), \tag{5.10}$$

with some unimportant constant $L_9 > 0$. But if $V'_{\zeta,k}(w) \le 1$, we get

$$\frac{1}{k}\sum_{l=2}^{2m}\|P_l(\zeta,\cdot)\|\,|w_n|^{l-2}|dw_n|^2 \le 2m\,\frac{|dw_n|^2}{\tau_n(\zeta,k)^2},$$

and therefore

$$4mK_1 s_0 L_9^{-1} E_{\zeta,\,k/K_0} + \log\lambda(s_0 V'_{\zeta,k})$$

becomes plurisubharmonic. From (5.8) we infer

$$\{s_0 V_{\zeta,k} \le 3/4\} \subset \{\widehat{W}_{\zeta,k} = E_{\zeta,\,k/K_0}\}, \tag{5.11}$$

and the claim follows in case (i).

In case (ii) we have

$$w_{A,\zeta}(z) := \log\lambda(s_0 V_{\zeta,k}(z)) + \log\lambda(V_{A,\zeta,k}(z))$$

and we need to estimate the Levi form of $\log\lambda(V_{A,\zeta,k}) = \log\lambda(\widetilde{V}_{A,\zeta,k}) \circ F_\zeta$. First we obtain (as in case (i))

$$\mathscr{L}_{\log\lambda(\widetilde{V}_{A,\zeta,k})} \ge -K_1\xi_{\{1/2\le V_{A,\zeta,k}\le 3/4\}}\cdot\mathscr{L}_{V''_{\zeta,k}}$$

with K_1 as above. But, after shrinking k_0, we find, with some constant $L_{10} > 0$,

$$\mathscr{I}_{\zeta,k/K_0}(w) \le L_{10}k_0 < s_1/L_1$$

for all w such that $V''_{\zeta,k}(w) \le 1$. Moreover

$$\mathscr{L}_{E_{\zeta,\,k/K_0}} \ge L_{11}\mathscr{L}_{V''_{\zeta,k}}$$

on $\{1/2 \le V_{A,\zeta,k} \le 3/4\}$ with some constant $L_{11} > 0$. Plugging F_ζ into the function $M_1 E_{\zeta,\,k/K_0} + \log\lambda(s_0 V'_{\zeta,k}) + \log\lambda \circ \widetilde{V}_{A,\zeta,k}$, we obtain the assertion, after enlarging M_1 if necessary.

In case (iii),

$$w_{A,\zeta} = \log\lambda(s_0 V'_{\zeta,k}) \circ F_\zeta + \log\lambda\biggl(\frac{3}{4v_*}\widetilde{V}_{A,\zeta,\omega_k} \circ F_\zeta\biggr).$$

The Levi form of the first member is treated as before. To estimate the Levi form of the second term we start with

$$\mathscr{L}_{\log\lambda(\frac{3}{4v_*}\widetilde{V}_{A,\zeta,\omega_k})} \ge -\frac{3}{4v_*}K_1 \cdot \xi_{\{\widetilde{V}_{A,\zeta,\omega_k} \le v_*\}} \cdot \mathscr{L}_{V''_{\zeta,\omega_k}}.$$

Now, on $\{\widetilde{V}_{A,\zeta,\omega_k} \le v_*\}$ we have

$$\mathscr{J}_{\zeta,k/K_0} \le 4^{m+2}(m+1)\omega_k. \tag{5.12}$$

Next we can choose a number t_1 (independently of k,ζ) such that

$$4t_1^2 + 2t_1 + 2\tau_n(\zeta, t_1) < (R_1/10)^2.$$

Two cases can occur:

(1) Suppose that $\omega_k \le \min\bigl\{\frac{s_1}{2L_1 4^{m+2}(m+1)}, t_1\bigr\}$. Then

$$\{\widetilde{V}_{A,\zeta,\omega_k} \le v_*\} \subset D_\zeta^{k/K_0} \cap \{E_{\zeta,k/K_0} \ge -s_1/2\},$$

hence

$$\{V_{A,\zeta,\omega_k} \le v_*\} \subset F_\zeta^{-1}(D_\zeta^{k/K_0}) \cap \{\widehat{W}_{\zeta,k} = E_{\zeta,k/K_0} \circ F_\zeta\}.$$

From Lemma 4.4(ii) and (5.12) we see that for a large enough M_1 we can achieve that $M_1\widehat{W}_{\zeta,k} + w_{A,\zeta}$ becomes plurisubharmonic on $D_{\zeta,k}$.

(2) Suppose that $\omega_k \ge \min\bigl\{\frac{s_1}{2L_1 4^{m+2}(m+1)}, t_1\bigr\}$. Then

$$\mathscr{L}_{\log\lambda(\frac{3}{4v_*}\widetilde{V}_{A,\zeta,\omega_k})} \ge -\frac{3}{4v_*}K_1 \cdot \xi_{\{\widetilde{V}_{A,\zeta,\omega_k} \le v_*\}} \cdot \mathscr{L}_{|w|^2}$$

and

$$\mathscr{L}_{\log\lambda(\frac{3}{4v_*}V_{A,\zeta,\omega_k})} \ge -\frac{3}{4v_*}K_1 \cdot \xi_{\{V_{A,\zeta,\omega_k} \le v_*\}} \cdot \mathscr{L}_{|F_\zeta|^2} \ge -L_{10}\mathscr{L}_{|z|^2},$$

with some universal constant L_{10}, as follows from the definition of F_ζ. This gives the claim, after another enlargement of M_1. ∎

With a view to (v) of Lemma 5.1 we introduce one more weight function.

Lemma 5.1.4. *For any $0 < k \le k_0$ let*

$$\widehat{U}_{\zeta,k} := \log\frac{k}{k - \widehat{W}_{\zeta,k}}.$$

Then $\widehat{U}_{\zeta,k}$ *is plurisubharmonic on* $D_{\zeta,k}$. *If* L_1 *is the constant from Lemma 4.4, then, given two numbers* $p \le k \le k_0$, *we have*

$$\widehat{U}_{\zeta,p}(x) \le \log L_1 + \log(k_0/s_1) + \log(p/k)$$

whenever $x \in D_{\zeta,p}$ *but* $x \notin Q_k(\zeta)$.

Proof. Let $x \in D_{\zeta,p} \setminus Q_k(\zeta)$. Then we consider two cases:

(a) If $\widehat{W}_{\zeta,p}(x) = -s_1$, then

$$\widehat{U}_{\zeta,p}(x) = \log \frac{p}{p+s_1} \le \log \frac{p}{k} + \log \frac{k_0}{p+s_1} < \log \frac{k_0}{s_1} + \log \frac{p}{k}.$$

(b) Assume that $\widehat{W}_{\zeta,p}(x) = E_{\zeta,p/K_0}(F_\zeta(x))$. We use $\mathscr{I}_{\zeta,p/K_0}(F_\zeta(x)) \ge (p/K_0)^2 + k^2$, because $F_\zeta(x) \notin R_k(\zeta)$, which implies

$$E_{\zeta,p/K_0}(F_\zeta(x)) \le -L_1^{-1}\sqrt{(p/K_0)^2 + k^2},$$

and hence

$$\widehat{U}_{\zeta,p}(x) \le \log \frac{p}{p + L_1^{-1}\sqrt{(p/K_0)^2+k^2}} \le \log L_1 + \log(p/k). \ \blacksquare$$

Now we define the desired plurisubharmonic weight function on $D_{\zeta,k}$:

$$\Phi_k := 2n\widehat{w}_{A,\zeta} + 2\widehat{U}_{\zeta,k}. \tag{5.13}$$

5.2. Proof of Lemma 5.1. Applying [Her-2a, Theorem 4], we find a smooth solution u_k to

$$\overline{\partial} u_k = v = \overline{\partial}\chi_1$$

such that

$$\int_{D_{\zeta,k}} |u_k|^2 e^{-\Phi_k}\, d^{2n}z \le K_3 \int_{D_{\zeta,k}} |\overline{\partial}\chi_1|^2_{\partial\overline{\partial}\Phi_k} e^{-\Phi_k}\, d^{2n}z, \tag{5.14}$$

where, for a $(0,1)$-form $v = v_1 d\bar z_1 + \cdots + v_n d\bar z_n$, we denote by $|v|^2_{\partial\overline{\partial}\Phi_k}$ the square of the length of v measured in the hermitian metric $ds^2 := \sum_{a,b=1}^n \frac{\partial^2\Phi_k}{\partial z_1\partial\bar z_b} dz_a\, d\bar z_b$. If $(H^{a\bar b})_{a,b=1}^n$ denotes the inverse of the coefficient matrix $\big(\frac{\partial^2\Phi_k}{\partial z_1\partial\bar z_b}\big)_{a,b=1}^n$, then $|v|^2_{\partial\overline{\partial}\Phi_k} = \sum_{a,b=1}^n H^{a\bar b} v_a \overline{v}_b$.

We estimate the right-hand side of (5.14) as follows.

Lemma 5.2.1. *With some unimportant constant* $K_4 > 0$ *one has*

$$\int_{D_{\zeta,k}} |\overline{\partial}\chi_1|^2_{\partial\overline{\partial}\Phi_k} e^{-\Phi_k} d^{2n}z \le K_4 \operatorname{Vol}(Q_{3k/s}(\zeta)). \tag{5.15}$$

Proof. First of all,

$$\overline{\partial}\chi_1 = s\chi'(sV_{\zeta,k})\overline{\partial}V_{\zeta,k} = s\chi'(sV_\zeta)\cdot F_\zeta^*\overline{\partial}V'_{\zeta,k},$$

hence

$$|\overline{\partial}\chi_1|^2_{\partial\overline{\partial}\Phi_k} \le s^2(\max\chi')^2 \cdot \xi_{\{sV_{\zeta,k}\le 3/4\}}|F_\zeta^*\overline{\partial}V'_{\zeta,k}|^2_{\partial\overline{\partial}\Phi_k}.$$

But

$$\partial V'_{\zeta,k}\overline{\partial V'_{\zeta,k}} \le V'_{\zeta,k}\mathscr{L}_{V'_{\zeta,k}} \le \frac{1}{s}\mathscr{L}_{V'_{\zeta,k}}$$

on $\{sV'_{\zeta,k}\le 3/4\}$. On the other hand, using (5.10), we get

$$\mathscr{L}_{E_{\zeta,k/K_0}} \ge \frac{L_9}{2m}\mathscr{L}_{V'_{\zeta,k}} \ge \frac{L_9 s}{2m}\partial V'_{\zeta,k}\overline{\partial V'_{\zeta,k}}$$

and hence

$$\mathscr{L}_{E_{\zeta,k/K_0}\circ F_\zeta} \ge \frac{L_9 s}{2m}F_\zeta^*\partial V'_{\zeta,k}\overline{F_\zeta^*\partial V'_{\zeta,k}},$$

which gives

$$|\overline{\partial}\chi_1|^2_{\partial\overline{\partial}\Phi_k} \le \frac{2ms}{K_5L_9}(\max\chi')^2 \cdot \xi_{\{sV_{\zeta,k}\le 3/4\}} \tag{5.16}$$

if we recall (see (5.8) and (5.11)) that $\operatorname{supp}(\overline{\partial}\chi_1) \subset \{sV_{\zeta,k} \le 3/4\} \subset \{s_0V_{\zeta,k}\le 3/4\} \subset \{\widehat{W}_{\zeta,k} = E_{\zeta,k/K_0}\}$, and therefore,

$$\mathscr{L}_{\Phi_k} \ge K_5\mathscr{L}_{E_{\zeta,k/K_0}\circ F_\zeta} \ge \frac{K_5L_9s}{2m}F_\zeta^*\partial V'_{\zeta,k}\overline{F_\zeta^*\partial V'_{\zeta,k}}$$

with some constant $K_5>0$.

We estimate Φ_k from below on $\operatorname{supp}(\overline{\partial}\chi_1)$ by

$$\begin{aligned}\Phi_k &= 2n\widehat{w}_{A,\zeta} + \widehat{U}_{\zeta,k} = 2n(M_1\widehat{W}_{\zeta,k} + w_{A,\zeta} + M_1|z|^2) + \widehat{U}_{\zeta,k}\\ &\ge -2n\,M_1s_1 + 2nw_{A,\zeta} + \widehat{U}_{\zeta,k}\\ &\ge -K_6 + \widehat{U}_{\zeta,k} \qquad \text{by Lemma 5.1.1.}\end{aligned}$$

But on $\operatorname{supp}(\overline{\partial}\chi_1)$ we have $\mathscr{J}_{\zeta,k/K_0}\circ F_\zeta \le L_1K_7k$ with some constant $K_7 > 0$, and hence

$$\widehat{U}_{\zeta,k} = \log\frac{k}{k-E_{\zeta,k/K_0}} \ge \log\frac{k}{k+L_1^{-1}\mathscr{J}_{\zeta,k/K_0}\circ F_\zeta} \ge \log\frac{1}{1+K_7}.$$

This altogether shows that $e^{-\Phi_k}\le K_8$ on $\operatorname{supp}(\overline{\partial}\chi_1)$, and in particular

$$\begin{aligned}&\int\limits_{D_{\zeta,k}} |\overline{\partial}\chi_1|^2_{\partial\overline{\partial}\Phi_k}e^{-\Phi_k}\,d^{2n}z\\ &\qquad\le \frac{2msK_8}{K_5L_9}(\max\chi')^2\operatorname{Vol}(\operatorname{supp}(\overline{\partial}\chi_1)) \le K_4\operatorname{Vol}(Q_{3/s}(\zeta))\end{aligned}$$

with an unimportant constant K_4, as desired. ∎

Lemma 5.2.2. *The function $F_{A,\zeta,k}$ satisfies* (i) *and* (ii) *of Lemma* 5.1.

Proof. (i) Certainly, by our construction, $F_{A,\zeta,k}$ is holomorphic on $D_{\zeta,k}$. Since $e^{-\Phi_k}$ is not locally integrable at ζ and A, the function u_k must have zeros at ζ and A. Hence

$$F_{A,\zeta,k}(\zeta)=\chi_1(\zeta)=\chi(0)=1.$$

(ii) By our construction of the $\overline{\partial}$-data we have, with R_1 as in Lemma 4.4,

$$F_{A,\zeta,k}(A)=\chi(sV_{\zeta,k}(A))=\begin{cases}0 & \text{if } |A-\zeta|\ge R_1,\\ 1 & \text{if } |A-\zeta|\le R_1,\ V_{\zeta,k}(A)\le 1,\\ 0 & \text{if } |A-\zeta|\le R_1,\ V_{\zeta,k}(A)>1.\end{cases}\ \blacksquare$$

Lemma 5.2.3. *The $F_{A,\zeta,k}$ satisfy* (iii) *of Lemma* 5.1.

Proof. We fix $z\in D$ and consider two cases:

(1) Suppose that $z\notin F_\zeta^{-1}(D^\zeta_{k/K_0})$. Since the polydisc $P_z:=\Delta_n(z,c_0t_0)$ is contained in $D_{\zeta,k}$, by the mean value inequality we get

$$\begin{aligned}(5.17)\quad |F_{A,\zeta,k}(z)|^2 &\le \frac{1}{\mathrm{Vol}(P_z)}\int\limits_{P_z}|F_{A,\zeta,k}(x)|^2\,d^{2n}x\\ &\le 2+(c_0t_0)^{-2n}\int\limits_{P_z}|u_k(x)|^2\,d^{2n}x\\ &\le 2+(c_0t_0)^{-2n}\,\mathrm{Vol}(\{V_{\zeta,k}\le 3/s\})\exp\Big(\max_{P_z}\Phi_k\Big)\le C^*,\end{aligned}$$

with some constant C^*, uniformly in z,A,ζ,k.

(2) Assume that $z\in F_\zeta^{-1}(D^\zeta_{k/K_0})$. Now we set $\widetilde{z}:=F_\zeta(z)$ and denote by $P_{\widetilde{z}}$ the polydisc around $\widetilde{z}$ with polyradius $(\sigma,\sqrt{\sigma},\ldots,\sqrt{\sigma},\tau_n(\zeta,\sigma))$, where $\sigma:=\theta_0\mathscr{J}_{\zeta,k}(\widetilde{z})$. Then again $P_{\widetilde{z}}\subset D^\zeta_k$, and by the mean value inequality,

$$\begin{aligned}(5.18)\qquad |F_{A,\zeta,k}(z)|^2 &= |F_{A,\zeta,k}\circ F_\zeta^{-1}(\widetilde{z})|^2\\ &\le \frac{1}{\sigma^n\tau_n(\zeta,\sigma)^2}\int\limits_{P_{\widetilde{z}}}|F_{A,\zeta,k}\circ F_\zeta^{-1}(y)|^2\,d^{2n}y\\ &\le \frac{C'}{\sigma^n\tau_n(\zeta,\sigma)^2}\int\limits_{F_\zeta^{-1}(P_{\widetilde{z}})}|F_{A,\zeta,k}(x)|^2\,d^{2n}x\\ &\le \frac{C''}{\sigma^n\tau_n(\zeta,\sigma)^2}\Big(\mathrm{Vol}(F_\zeta^{-1}(P_{\widetilde{z}}))+\int\limits_{F_\zeta^{-1}(P_{\widetilde{z}})}|u_k(x)|^2\,d^{2n}x\Big)\\ &\le \frac{C'''}{\sigma^n\tau_n(\zeta,\sigma)^2}(\sigma^n\tau_n(\zeta,\sigma)^2+k^n\tau_n(\zeta,k)^2)\le\widetilde{C},\end{aligned}$$

where $\widetilde{C}$ does not depend on A,z,ζ,k, since $\sigma\ge\theta_0k$. Hence in each case $|F_{A,\zeta,k}(z)|\le\widehat{C}$ with some unimportant constant $\widehat{C}$. ∎

By means of the Schwarz lemma we prove (iv):

Lemma 5.2.4. *The functions $F_{A,\zeta,k}$ satisfy* (iv) *of Lemma* 5.1.

Proof. We let $\widehat{s} := \min\left\{\frac{1}{6ms}, \frac{1}{\sqrt{3s}}, 1\right\}$. Then first of all we have

$$Q_{\widehat{s}\,C_e^{-1}k}(\zeta) \subset Q_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta) \subset \{V_{\zeta,k} < 2/s\}.$$

Hence u_k is holomorphic on $Q_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta)$, and in particular $u_k \circ F_\zeta^{-1}$ must be holomorphic on $R_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta)$. Further, we find $t > 0$, depending only on $\widehat{s}$ and C_e, such that for each $x \in R_{\widehat{s}C_e^{-1}k}(\zeta)$ the polydisc about x with polyradius $(tk, t\sqrt{k}, t\tau_n(\zeta, k))$ is contained in $R_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta)$.

Therefore we can apply the mean value inequality to find

$$\begin{aligned}
|u_k \circ F_\zeta^{-1}(x)|^2 &\le \frac{1}{t^{2n}k^n\tau_n(\zeta,k)^2} \int\limits_{R_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta)} |u_k \circ F_\zeta^{-1}(y)|^2\, d^{2n}y \\
&\le I := C'\frac{1}{k^n\tau_n(\zeta,k)^2} \int\limits_{Q_{(\widehat{s}\,C_e^{-1})^{1/(2m)}k}(\zeta)} |u_k(\widehat{x})|^2\, d^{2n}\widehat{x},
\end{aligned}$$

since the Jacobian determinant of F_ζ is bounded away from zero independently of k, z, ζ. From (5.14) and (5.15) and the fact that $\Phi_k < 0$ we obtain $I \le C''$ uniformly in k, z, ζ. Thus

$$\max_{R_{\widehat{s}\,C_e^{-1}\,k}(\zeta)} |u_k \circ F_\zeta^{-1}| \le \sqrt{C''}.$$

This, in conjunction with $u_k \circ F_\zeta^{-1}(0) = 0$ and the Schwarz lemma, implies, for all $w \in R_{\widehat{s}\,C_e^{-1}k}(\zeta)$,

$$|u_k \circ F_\zeta^{-1}(w)| \le \frac{C_e\sqrt{C''}}{\widehat{s}} \max\left\{\frac{|w_1|}{k}, \frac{|w''|}{\sqrt{k}}, \frac{|w_n|}{\tau_n(\zeta,k)}\right\}. \tag{5.19}$$

Now let $z \in D$ be arbitrary. We consider two cases:

(1) Suppose that $z \in Q_{\widehat{C}_e^{-1}k}(\zeta)$. Then $w := F_\zeta(z) \in R_{\widehat{C}_e^{-1}k}(\zeta)$, and hence

$$|u_k(z)| = |u_k \circ F_\zeta^{-1}(w)| \le \max\left\{\frac{|[F_\zeta](z)_1|}{k}, \frac{|[F_\zeta(z)]''|}{\sqrt{k}}, \frac{|[F_\zeta(z)]_n|}{\tau_n(\zeta,k)}\right\},$$

from which (iv) of Lemma 5.1 follows.

(2) Suppose that $z \notin Q_{\widehat{C}_e^{-1}k}(\zeta)$. Then

$$\frac{|[F_\zeta(z)]_1|^2}{k^2} + \frac{|[F_\zeta(z)]''|^2}{k} + \frac{|[F_\zeta(z)]_n|^2}{\tau_n(\zeta,k)^2} \ge \widehat{s}C_e^{-2},$$

from which, in conjunction with (iii), the claim follows. ∎

Finally we establish the last property (v), which will later allow us to construct a holomorphic "peaking function" at ζ.

LEMMA 5.2.5. *For the family of functions $F_{A,\zeta,k}$ statement* (v) *of Lemma* 5.1 *holds.*

Proof. Assume that $0 < k \le k_0$, and $\mathscr{N}_1 := 1 + 3(2 + 4(2m+1))^2$. Let $0 < p \le k$ be given.

(A) We first handle $z \in B(\zeta, R_0) \cap D \setminus Q_{\mathscr{N}_1 k}(\zeta)$. Let us estimate $|F_{A,\zeta,p}(z)|$ from above. Put $\widetilde{z} = F_\zeta(z)$ and denote by $\widetilde{P}_{\widetilde{z},k}$ the polydisc around $\widetilde{z}$ with polyradius $(\theta\sigma, \sqrt{\theta\sigma}, \ldots, \sqrt{\theta\sigma}, \tau_n(\zeta, \theta\sigma))$, where $\sigma := \mathscr{J}_{\zeta,k}(\widetilde{z})$ and the number θ will be chosen later. Finally let $P_{z,k} := F_\zeta^{-1}(\widetilde{P}_{\widetilde{z},k})$. We want to show that

$$P_{z,k} \cap \operatorname{supp} \chi(sV_{\zeta,p}) = \emptyset. \tag{5.20}$$

Assume that there exists $w \in P_{z,k} \cap \operatorname{supp}\, \chi(sV_{\zeta,p})$. Then $V_{\zeta,p}(w) \le 3/s$ and $F_\zeta(w) \in \widetilde{P}_{\widetilde{z},k}$, hence

$$|[F_\zeta(w)]_1 - [F_\zeta(z)]_1| < \theta \mathscr{J}_{\zeta,k}(\widetilde{z})$$

and

$$|[F_\zeta(z)]_1| \le |[F_\zeta(w)]_1| + \theta \mathscr{J}_{\zeta,k}(\widetilde{z}) \le \frac{3}{s}\, p + \theta\sigma. \tag{5.21}$$

In a similar way we obtain

$$|[F_\zeta(z)]''| \le \frac{3}{s}\sqrt{p} + \sqrt{\theta\sigma}, \tag{5.22}$$

$$|[F_\zeta(z)]_n| \le \frac{3}{s}\tau_n(\zeta, p) + \tau_n(\zeta, \theta\sigma). \tag{5.23}$$

This yields

$$\sigma \le k + |[F_\zeta(z)]_1| + |[F_\zeta(z)]''|^2 + \sum_{l=2}^{2m} \|P(\zeta,\cdot)\|\, |[F_\zeta(z)]_n|^l \le k + T_1(p + \theta^{1/m}\sigma)$$

with $T_1 := 4^{m+2} m (1 + 3/s)^{2m}$. So choose $\theta := \min\{\theta_0, (2T_1)^{-m}\}$ to obtain $\sigma \le 2T_1 p$. Plugging this into (5.21) through (5.23) we find

$$F_\zeta(z) \in R_{(1+3/s)k}(\zeta) \subset R_{\mathscr{N}_1 k}(\zeta),$$

contrary to our assumption on z.

In particular, $F_{A,\zeta,p} = u_k$ on $P_{z,k}$ and $F_{A,\zeta,p} \circ F_\zeta^{-1} = u_p \circ F_\zeta^{-1}$ on $\widetilde{P}_{\widetilde{z},p}$ for $p \le k$. The mean value property gives

$$\begin{aligned}
|F_{A,\zeta,p}(z)|^2 &= |F_{A,\zeta,p} \circ F_\zeta^{-1}(\widetilde{z})|^2 = \frac{1}{\operatorname{Vol}(\widetilde{P}_{\widetilde{z},p})} \int_{\widetilde{P}_{\widetilde{z},p}} |F_{A,\zeta,p} \circ F_\zeta^{-1}(y)|^2\, d^{2n}y \\
&= \frac{1}{\operatorname{Vol}(\widetilde{P}_{\widetilde{z},p})} \int_{\widetilde{P}_{\widetilde{z},p}} |u_p \circ F_\zeta^{-1}(y)|^2 d^{2n}y \le \frac{\widetilde{C}}{\operatorname{Vol}(\widetilde{P}_{\widetilde{z},p})} \int_{P_{z,p}} |u_p(x)|^2 d^{2n}x \\
&\le \widetilde{C}_1 \exp\Big(\max_{P_{z,p}} \varPhi_p\Big) \le \widetilde{C}_2 \exp\Big(2 \max_{P_{z,p}} \widehat{U}_{\zeta,p}\Big).
\end{aligned}$$

But by Lemma 5.1.4 we have $2\max_{P_{z,p}} \widehat{U}_{\zeta,p} \le 2\log\big(L_1 \frac{k_0}{s}\cdot\frac{p}{k}\big)$, because for $p \le k$ one has $P_{z,p} \cap Q_{\mathscr{N}_1 k/2}(\zeta) = \emptyset$, which can be shown similarly to (5.20). This implies $\widetilde{C}_2 \exp(2\max_{P_{z,p}} \widehat{U}_{\zeta,p}) \le \widetilde{C}_3(p/k)^2$ with some $\widetilde{C}_3 > 0$.

(B) It remains to estimate $|F_{A,\zeta,p}(z)|$ for $z \in D \setminus B(\zeta, R_0)$. But then the polydisc $\Delta_n(z, c_0t_0) \subset D_{\zeta,k/K_0}$ introduced in Remark 4.5 is contained in $D_{\zeta,k/K_0}$ and we apply the mean value inequality. Since $\chi_1 = 0$ on $\Delta_n(z_0, c_0t_0)$, we have

$$\begin{aligned} |F_{A,\zeta,p}(z)|^2 = |u_p(z)|^2 &\le \frac{1}{\mathrm{Vol}(\Delta_n(z, c_0t_0))} \int\limits_{\Delta_n(z,c_0t_0)} |u_p(x)|^2\, d^{2n}x \\ &\le K_9 \exp\Big(\max_{\Delta_n(z,c_0t_0)} e^{2\Phi_p}\Big) \quad \text{by (5.14), (5.15)} \\ &\le K_{10}(p/k)^2, \end{aligned}$$

with unimportant constants $K_9, K_{10} > 0$, because $\Delta_n(z, c_0t_0) \cap Q_k(\zeta) = \emptyset$.

This completes the proof of Lemma 5.2.5 and hence of Lemma 5.1. ∎

6. Estimation of the Carathéodory distance from below. We recall that at the beginning of Sec. 3.4 we denoted, for a point $z \in S_{\delta_0}$, the orthogonal projection of z by z^*. For small enough δ_0 this is well-defined.

We fix $A, B \in D$ with $\delta_D(A), \delta_D(B) \le \delta_0$. Let $\zeta := B^*$. Then we study the properties of the function

$$F_{A,B} := (1-c)\sum_{l=0}^{\infty} c^l F_{A,B^*,d^{-l-1}}$$

for

$$\frac{1}{2} < c < 1 \quad \text{and} \quad d > \left(\frac{nL_0C_e}{c_3\delta_0}\right)^{m^2} + 4^m(1+3\mathscr{N}_1) + \frac{4(n+1+\gamma_1)\mathscr{N}_1}{\gamma_1}$$

with

$$\gamma_1 := \frac{c_3}{L_0C_e},$$

where c_3 and C_e are the constants from (3.9).

Our first step is now

LEMMA 6.1. *The number c can be chosen uniformly in A and B in such a way that:*

(a) *The series that defines $F_{A,B}$ converges locally uniformly at each $z_0 \in D$, and in particular $F_{A,B}$ defines a holomorphic function on D.*
(b) *$|F_{A,B}| < 1$ on D.*

Proof. Let $z_0 \in D$. We consider the sets

$$U_l^* := D \cap B(B^*, R_0/2) \cap \{V_{B^*,d^{-l-1}} \le 1\}.$$

Certainly $U_{l+1}^* \subset U_l^*$ for any $l \ge 0$. We proceed in two steps.

STEP 1. Assume that $z_0 \in U_1^*$. Since

$$V_{B^*,d^{-l-1}}(z) = |[F_{B^*}(z)]_1|^2 d^{2(l+1)} + d^{l+1}\mathscr{B}(B^*, [F_{B^*}(z)]')$$

for any $z \in D \cap B(B^*, R_0/2)$, there exist only finitely many l for which $V_{B^*,d^{-l-1}}(z) < 1$. Hence the number

$$m_{z_0} := \max\{\mu \mid z_0 \in U_\mu^*\}$$

is well-defined. Since further $z_0 \notin U^*_{m_{z_0}+1}$ there exists an open neighborhood $W \ni z_0$ such that $z \in U^*_{m_{z_0}-1} \setminus U^*_{m_{z_0}+1}$ for any $z \in W$. In particular,

$$m_{z_0} - 1 \le m_z \le m_{z_0} + 1$$

for all $z \in W$. We next want to show that the series

$$T(z) := (1-c) \sum_{l=m_{z_0}+2}^{\infty} c^l F_{A,B^*,d^{-l-1}}(z)$$

converges uniformly on W. For all $z \in W$ we have $V_{B^*,d^{-m_{z_0}-2}}(z) > 1$, in particular $z \notin Q_{4^{-m}\,d^{-m_{z_0}-2}}(B^*)$.

We note that for $p := d^{-l-1}$ and $k := 4^{-m}\,d^{-m_{z_0}-2}\mathscr{N}_1^{-1}$ we have

$$p/k = 4^m d^{m_{z_0}+2}\mathscr{N}_1 d^{-l-1} = 4^m \mathscr{N}_1 d^{-1} < 1$$

whenever $l \ge m_{z_0} + 2$, so we can apply Lemma 5.1(v) to obtain

$$\sup_{z\in W} |F_{A,B^*,d^{-l-1}}(z)| \le p/k = 4^m \mathscr{N}_1 \widehat{C} d^{m_{z_0}+2} d^{-l-1}. \tag{6.1}$$

This proves the desired local uniform convergence of $F_{A,B}$ on $D\cap B(B^*, R_0)$.

With a view to (b), for the points of $D \cap B(B^*, R_0)$ we plug (6.1) into the series T and find, for $z \in W$,

$$\begin{aligned}
|T(z)| &\le (1-c) \sum_{l=m_{z_0}+2}^{\infty} c^l |F_{A,B^*,d^{-l-1}}(z)| \\
&\le 4^m \mathscr{N}_1 \widehat{C}(1-c) d^{m_{z_0}+1} \sum_{l=m_{z_0}+2}^{\infty} \left(\frac{c}{d}\right)^l \\
&= 4^m \mathscr{N}_1 \widehat{C}(1-c) d^{m_{z_0}+1} \left(\frac{c}{d}\right)^{m_{z_0}+2} \cdot \frac{d}{d-c} \le \frac{4^m \mathscr{N}_1 \widehat{C}}{d-1}(1-c)\cdot c^{m_{z_0}+2}.
\end{aligned}$$

To prove (b) on $D \cap B(B^*, R_0/2)$ we have to consider the sum

$$T_1(z) := (1-c) \sum_{l=0}^{m_{z_0}+1} c^l F_{A,B^*,d^{-l-1}}(z).$$

This can be done using Lemma 5.1(iv), namely

$$\begin{aligned} T_1(z) &= (1-c)\sum_{l=0}^{m_{z_0}+1} c^l + (1-c)\sum_{l=0}^{m_{z_0}+1} c^l (F_{A,B^*,d^{-l-1}}(z)-1) \\ &= 1-c^{m_{z_0}+2} + (1-c)\sum_{l=0}^{m_{z_0}+1} c^l (F_{A,B^*,d^{-l-1}}(z)-1) \end{aligned}$$

and

$$\begin{aligned} &\sum_{l=0}^{m_{z_0}+1} c^l |F_{A,B^*,d^{-l-1}}(z)-1| \\ &\qquad \le \widehat{C}\sum_{l=0}^{m_{z_0}+1} c^l\biggl(\frac{|[F_\zeta(z)]_1|}{d^{-l-1}} + \frac{|[F_\zeta(z)]''|}{\sqrt{d^{-l-1}}} + \frac{|[F_\zeta(z)]_n|}{\tau_n(\zeta, d^{-l-1})}\biggr) \\ &\qquad = \widehat{C}(S_1(c,d)|[F_\zeta(z)]_1| + S_2(c,d)|[F_\zeta(z)]''| + S_3(c,d)[F_\zeta(z)]_n|)), \end{aligned}$$

where

$$\begin{aligned} S_1(c,d) &= \sum_{l=0}^{m_{z_0}+1} d(cd)^l = d\frac{(cd)^{m_{z_0}+2}-1}{cd-1} \le \frac{d}{cd-1}(cd)^{m_{z_0}+2}, \\ S_2(c,d) &= \sum_{l=0}^{m_{z_0}+1} \sqrt{d}\,(c\sqrt{d})^l \le \frac{\sqrt{d}}{c\sqrt{d}-1}(c\sqrt{d})^{m_{z_0}+2}, \\ S_3(c,d) &= \sum_{l=0}^{m_{z_0}+1} \frac{c^l}{\tau_n(\zeta, d^{-l-1})}. \end{aligned}$$

Now we note that

$$\begin{aligned} \tau_n(\zeta, d^{-l-1}) &= \min_{2\le k\le 2m}\biggl(\frac{d^{-l-1}}{\|P_k(\zeta,\cdot)\|}\biggr)^{1/k} \ge d^{\frac{m_{z_0}-l}{2m}} \min_{2\le k\le 2m}\biggl(\frac{d^{-m_{z_0}-1}}{\|P_k(\zeta,\cdot)\|}\biggr)^{1/k} \\ &\ge d^{\frac{m_{z_0}-l}{2m}}\,\tau_n(\zeta, d^{-m_{z_0}-1}), \end{aligned}$$

which implies that

$$\begin{aligned} S_3(c,d) &\le \frac{d^{-m_{z_0}/(2m)}}{\tau_n(\zeta, d^{-m_{z_0}-1})}\sum_{l=0}^{m_{z_0}+1} (cd^{1/(2m)})^l \\ &\le \frac{d^{-m_{z_0}/(2m)}}{\tau_n(\zeta, d^{-m_{z_0}-1})}\,\frac{(cd^{1/(2m)})^{m_{z_0}+2}}{cd^{1/(2m)}-1} = \frac{d^{1/m}}{cd^{1/(2m)}-1}\,\frac{c^{m_{z_0}+2}}{\tau_n(\zeta, d^{-m_{z_0}-1})}. \end{aligned}$$

But by definition of m_{z_0} we have

$$S_1(c,d)|[F_\zeta(z)]_1| \le \frac{d^2}{cd-1}c^{m_{z_0}+2}\,\frac{|[F_\zeta(z)]_1|}{d^{-m_{z_0}-1}} \le \frac{2d^2}{d-2}c^{m_{z_0}+2},$$

$$S_2(c,d)|[F_\zeta(z)]''| \le \frac{d}{c\sqrt{d}-1} c^{m_{z_0}+2} \frac{|[F_\zeta(z)]''|}{d^{-(m_{z_0}+1)/2}} \le n\frac{2\sqrt{d}}{\sqrt{d}-2} c^{m_{z_0}+2},$$
$$S_3(c,d)|[F_\zeta(z)]_n| \le \frac{d^{1/m}}{cd^{1/(2m)}-1} \frac{|[F_\zeta(z)]_n|}{\tau_n(\zeta, d^{-m_{z_0}-1})} c^{m_{z_0}+2} \le \frac{2d^{1/m}}{d^{1/(2m)}-2} c^{m_{z_0}+2}.$$

Plugging this into the expression for $T_1(z)$ we get, for $z \in W$,

$$|T_1(z)| \le 1 - c^{m_{z_0}+2} + 2(n+2)\widehat{C}\frac{d^2}{d^{1/(2m)}-2}(1-c)c^{m_{z_0}+2},$$

and altogether

$$\begin{aligned} (6.2)\qquad & |F_{A,B^*}(z) - (1 - c^{m_{z_0}+2})| \\ & \le (1-c)\left(\frac{2(n+2)\widehat{C}d^2}{d^{1/(2m)}-2} + \frac{4^m \mathscr{N}_1 \widehat{C}}{d-1}\right) c^{m_{z_0}+2} \le \frac{1}{2} c^{m_{z_0}+2}, \end{aligned}$$

if we choose c so close to 1 that

$$(6.3)\qquad (1-c)\left(\frac{2(n+2)\widehat{C}d^2}{d^{1/(2m)}-2} + \frac{4^m \mathscr{N}_1 \widehat{C}}{d-1}\right) < \frac{1}{2}.$$

Thus we obtain properties (a) and (b) of F_{A,B^*} within $D \cap B(B^*, R_0/2)$.

Step 2. Assume that $z_0 \notin U_1$. Then $|z_0 - B^*| \ge \frac{1}{2}R_0$ or $|z_0 - B^*| < \frac{1}{2}R_0$ and simultaneously $V_{B^*,d^{-2}}(z_0) > 1$. In each case we have

$$|z_0 - B^*| > \gamma_1 \cdot \frac{1}{d},$$

hence we can find an open neighborhood W of z_0 such that

$$|z - B^*| > \gamma_1 \cdot \frac{1}{2d}$$

on W. If we choose

$$(6.4)\qquad k := \frac{\gamma_1}{4\mathscr{N}_1(n+1+\gamma_1)} \cdot \frac{1}{d},$$

we see that $d^{-2} < k$ and $W \cap Q_{\mathscr{N}_1 k}(B^*) = \emptyset$.

Again we apply Lemma 5.1(v), where we put $p = d^{-l-1}$ for $l \ge 1$ and k is as in (6.4). This gives, for $l \ge 1$,

$$|F_{A,B^*,d^{-l-1}}(z)| \le \frac{4\widehat{C}\mathscr{N}_1(n+1+\gamma_1)}{d^l}$$

on W. So we obtain the uniform convergence of the series $F_{A,B}$ on W, hence its holomorphy, and

$$\begin{aligned}|F_{A,B}(z)| &\le (1-c)\sum_{l=0}^{\infty} c^l |F_{A,B^*,d^{-l-1}}(z)| \\ &\le (1-c)\Big(\widehat{C} + 4\widehat{C}\mathscr{N}_1(n+1+\gamma_1)\sum_{l=1}^{\infty}\Big(\frac{c}{d}\Big)^l\Big) \\ &\le 2(1-c)\widehat{C}(1+8\widehat{C}\mathscr{N}_1(n+1+\gamma_1)).\end{aligned}$$

If c is chosen close enough to 1 that (6.3) is also satisfied, we obtain $|F_{A,B}| < 1$. ■

Next we begin establishing the lower estimate for the Carathéodory distance stated in (2.8).

Lemma 6.2. *There exists a constant $K > 0$ such that*

$$d_D^{\mathrm{Cara}}(A,B) \ge \frac{1}{K}\varrho(A,B)$$

for any $A, B \in D$ with $d(A,B) \ge K\delta_D(B)$.

Proof. Let $\varrho(A,B)$, $\varrho_B(A)$, and $\varrho_A(B)$ be defined as in Theorem 2.1.

(A) We may replace the open neighborhood U_0 of ∂D by $U_0' := D \cap S_{\delta_0}$, where δ_0 is as described before the statement of Lemma 6.1.

If $A, B \in U_0'$ we will prove

$$d_D^{\mathrm{Cara}}(A,B) \ge \frac{2}{K}\varrho_B(A).$$

By symmetry of $d_D^{\mathrm{Cara}}(A,B)$ we will obtain

$$d_D^{\mathrm{Cara}}(A,B) \ge \frac{1}{K}\varrho(A,B).$$

If $A \notin U_0', B \in U_0'$, we will likewise show that $d_D^{\mathrm{Cara}}(A,B) \ge \frac{2}{K}\varrho_B(A)$.

The case $B \notin U_0', A \in U_0'$ is treated similarly. The case $A, B \notin U_0'$ is trivial, since we only need to apply the Schwarz lemma to D and a large enough ball $\widehat{B}$ that contains D.

(B) The hard part of the proof is where A or B (or both) are allowed to lie arbitrarily close to ∂D.

(B.1) Assume that $A, B \in U_0'$. We define $F_{A,B}$ as in the preceding lemma and put

$$f_{A,B} := \frac{F_{A,B} - F_{A,B}(A)}{1 - \overline{F_{A,B}(A)}F_{A,B}}.$$

This function is holomorphic on D and has values in the unit disc. We get

$$
\begin{aligned}
(6.5)\quad d_D^{\mathrm{Cara}}(A,B) &\ge \frac{1}{2}\log\left(1+\frac{2|f_{A,B}(B)|}{1-|f_{A,B}(B)|}\right)\\
&\ge \frac{1}{2}\log\left(1+\frac{2|f_{A,B}(B)|^2}{1-|f_{A,B}(B)|^2}\right)\\
&= \frac{1}{2}\log\left(1+\frac{2|F_{A,B}(A)-F_{A,B}(B)|^2}{(1-|F_{A,B}(A)|^2)(1-|F_{A,B}(B)|^2)}\right).
\end{aligned}
$$

Let c_3 and C_e be as in (3.9). We can assume that

$$
\delta_D(B) \le 4^{-\frac{\log d}{2\log(1/c)}}\frac{c_3}{6mL_0C_e}, \tag{6.6}
$$

otherwise the estimate of the lemma is obtained from the Schwarz lemma for the Carathéodory distance, applied to D and the ball $B(A,R_D)$.

As in the proof of the preceding lemma, we let

$$
U_l^* := D\cap B(B^*,R_0/2)\cap\{V_{B^*,d^{-l-1}}\le 1\},\qquad l\ge 0.
$$

If $B\notin U_0^*$, we have $V_{B^*,d^{-1}}(B)>1$, which implies

$$
\delta_D(B)\ge \frac{1}{L_0d}\,\frac{1}{1+C_e/c_3},
$$

so there is nothing to do, as explained before.

Assume that $B\in U_0^*$. Then $m_B:=\max\{\mu\mid B\in U_\mu^*\}$ is well-defined, and from $B\notin U_{m_B+1}^*$ we obtain

$$
(d^{m_B+2}|[F_{B^*}(B)]_1|)^2+d^{m_B+2}\Big(|[F_{B^*}(B)]''|^2+\sum_{l=2}^{2m}\|P_l\|\,|[F_{B^*}(B)]_n|^l\Big)\ge 1.
$$

In conjunction with (2.4) and (3.9) this implies

$$
m_B+2\ge\frac{1}{\log d}\left(2\log\frac{c_3}{6nC_eL_0}+\log\frac{1}{\delta_D(B)}\right), \tag{6.7}
$$

which gives, combined with (6.6),

$$
c^{m_B+2}\le\left(\frac{6mC_e}{c_3}\right)^{\frac{2\log(1/c)}{\log d}}\delta_D(B)^{\frac{\log(1/c)}{\log d}}. \tag{6.8}
$$

From (6.2) we obtain

$$
|F_{A,B}(B)-(1-c^{m_B+2})|\le\tfrac{1}{2}c^{m_B+2},
$$

hence, after a little computation,

$$
1-|F_{A,B}(B)|^2\le 3c^{m_B+2}\le 3/4.
$$

We plan to plug this into (6.5). For this we have to estimate $1-|F_{A,B}(A)|^2$.

Two cases can occur: If $A \notin U_0^*$ we have $F_{A,B}(A) = 0$, and by (6.5),

$$(6.9)\quad d_D^{\mathrm{Cara}}(A,B) \ge \frac{1}{2}\log\left(1 + \frac{2|F_{A,B}(B)|^2}{1-|F_{A,B}(B)|^2}\right) \ge \frac{1}{2}\log\left(1 + \frac{1}{24}\left(\frac{c_3}{6mC_e}\right)^{\frac{2\log(1/c)}{\log\, d}} \delta_D(B)^{-\frac{\log(1/c)}{\log\, d}}\right).$$

From this the desired estimate for $d_D^{\mathrm{Cara}}(A,B)$ follows.

The alternative case is that $A \in U_0^*$. Now we define

$$m_{A,B^*} := \max\{\mu \mid A \in U_\mu^*\}$$

and derive from

$$(d^{m_{A,B^*}+1}|[F_{B^*}(A)]_1|)^2 + d^{m_{A,B^*}+1}\Big(|[F_{B^*}(A)]''|^2 + \sum_{l=2}^{2m} \|P_l\|\,|[F_{B^*}(A)]_n|^l\Big) \le 1,$$

combined with Lemma 3.4.2(e), that

$$(6.10)\qquad m_{A,B^*} \le \frac{\log(2/d'(A,B))}{\log d}, \qquad c^{m_{A,B^*}} \ge \left(\frac{d'(A,B)}{2}\right)^{\frac{\log(1/c)}{\log d}}.$$

In order to apply (6.5) we note that

$$F_{A,B}(A) = (1-c)\sum_{l\colon A\in U_l^*} c^l = 1 - c^{m_{A,B^*}+1},$$

which yields

$$(6.11)\qquad 1 - |F_{A,B}(A)|^2 = 2c^{m_{A,B^*}+1} - c^{2m_{A,B^*}+2} \le 2c^{m_{A,B^*}+1},$$

and further

$$\begin{aligned}|F_{A,B}(A) - F_{A,B}(B)| &\ge |-c^{m_{A,B^*}+1} + c^{m_B+2}| - \tfrac{1}{2}c^{m_B+2}\\ &\ge \tfrac{1}{2}c^{m_B+2} - c^{m_{A,B^*}+1} = c^{m_{A,B^*}+1}\big|1 - \tfrac{1}{2}c^{m_B - m_{A,B^*}+1}\big|.\end{aligned}$$

This, combined with (6.5), (6.8), (6.11), and (6.10), yields

$$\begin{aligned}d^{\mathrm{Cara}}(A,B) &\ge \frac{1}{2}\log\left(1 + \frac{1}{3}\left(\frac{1}{c}\right)^{m_B - m_{A,B^*}-1}\left(1 - \frac{1}{2}c^{m_B - m_{A,B^*}+1}\right)^2\right)\\ &\ge \frac{1}{2}\log\left(1 + K'\left(1 - \frac{1}{2}K'\left(\frac{\delta_D(B)}{d'(A,B)}\right)^\nu\right)^2\left(\frac{d'(A,B)}{\delta_D(B)}\right)^\nu\right)\\ &\ge \frac{1}{2}\log\left(1 + \frac{K'}{2}\left(\frac{d'(A,B)}{\delta_D(B)}\right)^\nu\right)\\ &\ge \frac{\nu}{2}\log\left(1 + \left(\frac{1}{2}K'\right)^{1/\nu}\frac{d'(A,B)}{\delta_D(B)}\right),\end{aligned}$$

where

$$\nu := \frac{\log(1/c)}{\log\, d}, \qquad K' = \left(\frac{3mC_e}{c_3}\right)^{2\nu}.$$

Hence we choose

$$K := \frac{1}{\nu}\, K'^{\,1/\nu} \tag{6.12}$$

and obtain the estimate

$$d^{\mathrm{Cara}}(A,B) \geq \frac{2}{K}\log\left(1 + \frac{3}{K}\,\frac{d(A,B)}{\delta_D(B)}\right).$$

(B.2) Suppose that $A \notin U_0'$. Then, by our choice of the number d we even have $A \notin U_0^*$, and the arguments apply that led to (6.9).

So it remains to compare $\varrho_B(A)$ with $\log\big(1 + \frac{1}{K}\frac{d(A,B)}{\delta_D(B)}\big)$. For this we observe that by Lemma 3.4.2,

$$\frac{|[F_B(A)]''|^2}{\delta_D(B)} \leq 3\frac{d'(A,B)}{\delta_D(B)},$$

and hence

$$\frac{|[F_B(A)]''|}{\sqrt{\delta_D(B)}} \leq \sqrt{3\frac{d'(A,B)}{\delta_D(B)}} \leq \sqrt{\frac{3}{K}}\,\frac{d'(A,B)}{\delta_D(B)}$$

and

$$\frac{|[F_B(A)]|}{\sqrt{\delta_D(B)}} \leq L_0\frac{|A-B|}{\sqrt{\delta_D(B)}} \leq L_0\sqrt{R_D}\,\frac{|A-B|}{\delta_D(B)},$$

which implies

$$\frac{|[F_B(A)]''|}{\sqrt{\delta_D(B)}} \leq (3 + L_0\sqrt{R_D})\frac{d(A,B)}{\delta_D(B)}.$$

Likewise we obtain

$$\begin{aligned}\frac{|[F_B(A)]_n|}{\tau_n(B,\delta_D(B))} &\leq \frac{\tau_n(B,2d'(A,B))}{\tau_n(B,\delta_D(B))}\\ &= \frac{\tau_n(B,2d'(A,B))}{2d'(A,B)}\,\frac{\delta_D(B)}{\tau_n(B,\delta_D(B))}\,\frac{2d'(A,B)}{\delta_D(B)} \leq \frac{2d'(A,B)}{\delta_D(B)},\end{aligned}$$

since the function $t \mapsto t/\tau(B,t)$ is increasing in t.

As before, it is easy to see that

$$\frac{|[F_B(A)]_n|}{\tau_n(B,\delta_D(B))} \leq L_0\frac{|A-B|}{\tau_n(B,\delta_D(B))} \leq \frac{C_eL_0}{c_3}\,\frac{|A-B|}{\delta_D(B)}.$$

This, in conjunction with (3.8), proves, after enlarging K,

$$\frac{1}{2}\log\left(1 + \frac{1}{K}\,\frac{d(A,B)}{\delta_D(B)}\right) \geq \varrho_B(A). \ \blacksquare$$

We come to the alternative case where A and B are close to each other.

LEMMA 6.3. *If $d(A,B) \le K\delta_D(B)$, then*

$$d_D^{\mathrm{Cara}}(A,B) \ge \varrho_B(A).$$

Proof. From (2.4) and Lemma 3.4.2(e) we deduce $A, B \in Q_{\widehat{K}\delta_D(B)}(B^*)$ if we choose

$$\widehat{K} := (3 + L_0 C_e/c_3)K.$$

We may assume that

$$\delta_D(B) \le \frac{s_1}{\widehat{K} L_1(2+n+2m)} \tag{6.13}$$

where L_1 and s_1 are as in Lemma 4.4 and Remark 4.5.

Our plan is to compare the Carathéodory distances of the domains D and $Q_{\widehat{K}\,\delta_D(B)}(B^*)$ at A and B. In Lemma 4.4 and Remark 4.5 we choose

$$\delta := 2K_0\widehat{K}\,\delta_D(B).$$

Then our key lemma is

LEMMA 6.4. *With a universal constant $\widetilde{L}$ that does not depend on A or B the following holds: Given $f \in H^\infty(Q_{\delta/K_0}(B^*))$ one can find $\widehat{f} \in H^\infty(D_{B^*,\delta})$ such that*

(a) $\widehat{f}(A) = f(A)$, $\widehat{f}(B) = f(B)$, *and*
(b) $\|\widehat{f}\|_\infty \le \widetilde{L}\|f\|_\infty$.

We postpone a sketch of proof of this for a moment.

With this lemma in hand we can give

Proof of Lemma 6.3. From the definition of F_{B^*} it follows that, with a uniform constant $C_{**} > 0$, for all $z \in D$ one has

$$|F_B(z) - F_{B^*}(z)| \le C_{**}\delta_D(B).$$

We apply Lemma 6.4 to

$$f = f_1 := \frac{[F_B]_1}{\delta/K_0}$$

and find

$$\|f_1\|_\infty \le \frac{|[F_{B^*}]_1| + C_{**}\delta_D(B)}{\delta/K_0} \le \frac{|[F_{B^*}]_1| + C_{**}\delta/(K_0\widehat{K})}{\delta/K_0} \le 1 + C_{**}.$$

This yields, in conjunction with Lemma 6.4,

$$d^{\mathrm{Cara}}(A,B) \ge \frac{1}{2}\log\biggl(1 + 2\frac{|f_1(A)|}{\widetilde{L}(1+C_{**})}\biggr) = \frac{1}{2}\log\biggl(1 + \frac{|[F_B]_1(A)|}{\widetilde{L}\widehat{K}(1+C_{**})\delta_D(B)}\biggr).$$

Next, for $2 \le l \le n-1$ we choose

$$f = f_l := \frac{[F_B]_l}{\sqrt{\delta/K_0}}.$$

A similar argument yields

$$d^{\mathrm{Cara}}(A,B) \ge \frac{1}{2}\log\left(1 + 2\frac{|f_l(A)|}{\widetilde{L}(1+C_{**})}\right) = \frac{1}{2}\log\left(1 + \frac{|[F_B]_l(A)|}{\widetilde{L}\widehat{K}(1+C_{**})\sqrt{\delta_D(B)}}\right).$$

Finally let

$$f_n := \frac{[F_B]_n}{\tau_n(B, \delta/K_0)}.$$

Then

$$\|f_n\|_\infty \le \frac{|[F_{B^*}]_n| + C_{**}\delta_D(B)}{\tau_n(B,\delta/K_0)} \le \frac{C_e}{c_3}(1 + C_{**})$$

and as before we deduce

$$d^{\mathrm{Cara}}(A,B) \ge \frac{1}{2}\log\left(1 + \frac{c_3}{C_e\widetilde{L}\widehat{K}(1+C_{**})}\,\frac{|[F_B]_n(A)|}{\tau_n(B,\delta_D(B))}\right)$$

and so

$$d^{\mathrm{Cara}}(A,B) \ge \frac{1}{2}\log\left(1 + C'\left(\frac{|[F_B]_1(A)|}{\delta_D(B)} + \frac{[F_B]''(A)}{\sqrt{\delta_D(B)}} + \frac{|[F_B]_n(A)|}{\tau_n(B,\delta_D(B))}\right)\right)$$

with

$$C' := \frac{c_3}{n\,C_e\widetilde{L}\widehat{K}(1+C_{**})}.$$

Keeping in mind that we are assuming $d(A,B) \le K\delta_D(B)$, we can estimate the right-hand side from below by $\varrho_B(A)$ and see that the asserted lower bound on $d^{\mathrm{Cara}}(A,B)$ is true. ∎

Proof of Lemma 6.4. The proof proceeds very much along the lines of the proof of Lemma 5.1(iii). Therefore we give a sketch here, omitting technical details.

First, fix $f \in H^\infty(Q_{\delta/K_0}(B^*)$. We choose a smooth function $\chi : \mathbb{C}^n \to [0,1]$ such that $\chi = 1$ on $R_{3\delta/(4K_0)}(B^*)$ and $\chi = 0$ on $\mathbb{C}^n \setminus R_{7\delta/(8K_0)}(B^*)$. Then we want to solve the $\overline{\partial}$-equation $\overline{\partial}u = \alpha$ on $D_{B^*,\delta}$, where

$$\alpha := \overline{\partial}(\chi \circ F_{B^*}) \cdot f.$$

The solution u is to vanish at A and B, so we need a suitable plurisubharmonic weight function $\varPhi_{A,B}$ on $D_{B^*,\delta}$.

We set $t := \delta/K_0$ and try the function

$$\widetilde{w}_{A,B} = \log\lambda \circ \widetilde{V}_{A,B^*,t} + \log\lambda \circ \widetilde{V}_{B,B^*,t},$$

where $\lambda \in C^\infty$ is defined as earlier, namely $\lambda(x) = x$ for $x \le 1/2$ and $\lambda(x) = 2/3$ for $x \ge 3/4$. With $K_1 := \max|(\log\lambda)''|$ we obtain

$$\mathscr{L}_{\widetilde{w}_{A,B}} \ge -K_1(\xi_{\{\widetilde{V}_{A,B^*,t}\le 3/4\}} + \xi_{\{\widetilde{V}_{B,B^*,t}\le 3/4\}})\mathscr{L}_{V''_{B^*,t}}.$$

Now

$$\{\widetilde{V}_{A,B^*,t} \le 3/4\} \cup \{\widetilde{V}_{B,B^*,t} \le 3/4\} \subset R_{\widetilde{K}t}(B^*),$$

where $\widetilde{K} := 4 + L_0C_e/(\widehat{K}c_3)$. Because of our choice of δ and the condition (6.13) on $\delta_D(B)$ we find a constant $M_3 > 1$ such that

$$\Phi_{A,B} := 2n\widehat{w}_{A,B} \circ F_{B^*} + M_3 W_{B^*,\delta}$$

is plurisubharmonic on $D_{B^*,\delta}$. Further, $\Phi_{A,B} < 0$ everywhere, and since $\operatorname{supp}(\alpha) \subset D_{B^*,\delta} \setminus Q_{3\delta/(4K_0)}(B^*)$, we have $\Phi_{A,B} \ge -4n\log 2 - M_3 s_1$ on $\operatorname{supp}(\alpha)$.

In analogy to the proof of Lemma 5.2.1 we find a smooth solution u on $D_{B^*,\delta}$ to $\overline{\partial}u = \alpha$ such that

$$\int_{D_{B^*,\delta}} |u|^2 e^{-\Phi_{A,B}}\, d^{2n}z \le \widehat{C}'\|f\|_\infty^2 \operatorname{Vol}(Q_{\delta/K_0}(B^*))$$

with an unimportant constant $\widehat{C}'$. The desired function is now

$$\widehat{f} := \chi \circ F_{B^*} \cdot f - u.$$

Since $e^{-\Phi_{A,B}}$ is not locally integrable at A and at B, we have $u(A) = u(B) = 0$, so the holomorphic function $\widehat{f}$ satisfies (a).

The proof of estimate (b) is based upon the mean value inequality for $|\widehat{f}|$. It goes analogously to the proof of Lemma 5.1(iii).

Let $z \in D$ be fixed. Two cases are possible.

Case (i). Let $z \notin F_{B^*}^{-1}(D_{\delta/K_0}^{B^*})$. Then the polydisc $P_z := \Delta_n(z, c_0t_0)$ (see Remark 4.5) is contained in $D_{B^*,\delta}$, and we obtain

$$\begin{aligned}|\widehat{f}(z)|^2 &\le \frac{1}{\operatorname{Vol}(P_z)} \int_{P_z} |\widehat{f}(x)|^2\, d^{2n}x \\ &\le \frac{2}{\operatorname{Vol}(P_z)}\Big(\operatorname{Vol}(P_z)\|f\|_\infty^2 + 2^{8n}e^{M_3s_1}\int_{P_z} |u(x)|^2 e^{-\Phi_{A,B}}\, d^{2n}x\Big) \le \widetilde{L}_1^2\|f\|_\infty^2\end{aligned}$$

with some constant $\widetilde{L}_1$.

Case (ii). Assume that $z \in F_{B^*}^{-1}(D_{\delta/K_0}^{B^*})$. Again we put $\widetilde{z} := F_{B^*}(z)$ and choose $P_z := F_{B^*}^{-1}(P_{\widetilde{z}})$, where $P_{\widetilde{z}}$ denotes the polydisc around $\widetilde{z}$ with polyradius $(\sigma, \sqrt{\sigma}, \ldots, \sqrt{\sigma}, \tau_n(B^*, \sigma))$, where $\sigma := \theta_0 \mathscr{J}_{B^*,\delta/K_0}(\widetilde{z})$. (For θ_0

see Remark 4.5). Then, as in the proof of Lemma 5.2.3, we have

$$\begin{aligned}
|\widehat{f}(z)|^2 = |\widehat{f}\circ F_{B^*}^{-1}(\widetilde{z})|^2 &\le \frac{1}{\mathrm{Vol}(P_{\widetilde{z}})}\int\limits_{P_{\widetilde{z}}} |\widehat{f}\circ F_{B^*}^{-1}|^2\, d^{2n}x \\
&\le 2\|f\|_\infty^2 + \frac{2}{\mathrm{Vol}(P_{\widetilde{z}})}\int\limits_{P_{\widetilde{z}}} |u\circ F_{B^*}^{-1}(x)|^2\, d^{2n}x \\
&\le 2\|f\|_\infty^2 + \frac{C''}{\mathrm{Vol}(P_{\widetilde{z}})}\int\limits_{F_{B^*}^{-1}(P_{\widetilde{z}})} |u(y)|^2\, d^{2n}y \le \widetilde{L}_2^2\|f\|_\infty^2,
\end{aligned}$$

as before. From this we obtain estimate (b). ∎

7. The upper bound for the Kobayashi distance. We will estimate the Kobayashi distance from above as stated in the main theorem. This will complete the proof of the theorem, as already explained at the end of Section 2.

We start with the following remark: If x, y belong to $T := \{(x,y) \in D\times D \mid \delta_D(x), \delta_D(y) \ge \delta_0\}$, then

$$d_D^{\mathrm{Kob}}(x,y) \le \log(1+\widetilde{C}|x-y|)$$

with some constant $\widetilde{C}$.

Indeed, T is compact. From the continuity of d_D^{Kob} on $D\times D$ it follows that, with a uniform constant $\widetilde{C} > 1$, we have

$$\frac{d_D^{\mathrm{Kob}}(x,y)}{\log(1+\widetilde{C}|x-y|)} \le 1$$

whenever $|x-y| \ge \delta_0/2$ and $\delta_D(x), \delta_D(y) \ge \delta_0$.

If $|x-y| < \delta_0/2$, we apply the Schwarz lemma to $B(y,\delta) \subset D$ to obtain

$$d_D^{\mathrm{Kob}}(x,y) \le d_{B(y,\delta_0)}^{\mathrm{Kob}}(x,y) \le \log\left(1+\frac{2}{\delta_0}|x-y|\right).$$

In each case (since $|x-y| \lesssim d(x,y)$) we obtain the desired upper bound for $d_D^{\mathrm{Kob}}(x,y)$.

Next we establish the upper bound claimed for $d_D^{\mathrm{Kob}}(A,B)$ in the main theorem for $A, B \in D$ with $\delta_D(A), \delta_D(B) \le \delta_0$.

As usual, for $P \in S_{\delta_0}$ we denote by P^* the orthogonal projection of P to the boundary ∂D and by $\nu(P^*)$ the inner unit normal to ∂D at P^*.

From [Her-3, Lemma 7.1] (which carries over to general $n \ge 2$) we obtain

LEMMA 7.1. *Let $P \in D \cap S_{\delta_0}$ and $0 < t < s < \delta_0$. Then*

$$d_D^{\mathrm{Kob}}(P^* - t\nu(P^*), P^* - s\nu(P^*)) \le \frac{1}{2}\log\left(1+\frac{2|t-s|}{t}\right).$$

The construction of the polyhedra $Q_\delta(x)$ for $x \in D \cap S_{\delta_0}$ and $\delta < \delta_0$ allows one to choose $M_0 \gg 1$ such that

$$Q_{2\delta_D(x)/M_0}(x) \subset D \quad \text{for any } x \in D \cap S_{\delta_0}.$$

LEMMA 7.2. *Suppose that $A', B' \in D$ with*

$$\delta_0 > \min\{\delta_D(A'), \delta_D(B')\} \geq M_0\, d(A', B'). \tag{7.1}$$

Then

$$d_D^{\mathrm{Kob}}(A', B') \leq 2M_0 \varrho(A', B').$$

Proof. Assume first that $d(A', B') = d'(A', B')$. Further let $\delta_D(A') \geq \delta_D(B')$ and $\delta_D(B') < \delta_0$. From $A' \in \overline{Q_{d'(A',B')}}(B')$ and (7.1) we obtain $A' \in Q_{\delta_D(B')/M_0}(B')$. Now, by the invariance property of the Kobayashi distance,

$$\begin{aligned} d_D^{\mathrm{Kob}}(A', B') &\leq d^{\mathrm{Kob}}_{Q_{2\delta_D(B')/M_0}(B')}(A', B') = d^{\mathrm{Kob}}_{R_{2\delta_D(B')/M_0}(B')}(F_{B'}(A'), 0) \\ &= \frac{1}{2}\log\left(1 + \frac{2\mu'(A', B')}{1 - \mu'(A', B')}\right), \end{aligned}$$

where

$$\mu'(A', B') = \max\left\{M_0\frac{|[F_{B'}(A')]_1|}{\delta_D(B')}, \frac{\sqrt{M_0}\,|[F_{B'}(A')]''|}{\sqrt{\delta_D(B')}}, \frac{|[F_{B'}(A')]_n|}{\tau(B', 2\delta_D(B')/M_0)}\right\}.$$

But $\mu'(A', B') < 2^{-1/(2m)}$, hence

$$\begin{aligned} &d_D^{\mathrm{Kob}}(A', B') \\ &\leq \frac{M_0}{1 - 2^{-1/(2m)}}\log\left(1 + \max\left\{\frac{|[F_{B'}(A')]_1|}{\delta_D(B')}, \frac{|[F_{B'}(A')]''|}{\sqrt{\delta_D(B')}}, \frac{|[F_{B'}(A')]_n|}{\tau(B', \delta_D(B'))}\right\}\right). \end{aligned}$$

But (using again Lemma 3.4.2(e)) we see that the right-hand side is $\leq 2M_0\, \varrho(A', B')$.

Assume now that $d(A', B') = |A' - B'|$. Then we use

$$D \supset B(B', \delta_D(B')) \supset B(B', \delta_D(B')/M_0) \ni A'.$$

As above we get

$$\begin{aligned} d_D^{\mathrm{Kob}}(A', B') &\leq d^{\mathrm{Kob}}_{B(B',\delta_D(B')/M_0)}(A', B') \leq \frac{1}{2}\log\left(1 + 2\frac{|A' - B'|}{\delta_D(B')}\right) \\ &= \frac{1}{2}\log\left(1 + 2\frac{d(A', B')}{\delta_D(B')}\right) \leq \varrho(A', B') + \varrho(B', A'). \quad \blacksquare \end{aligned}$$

We have to make two more steps in order to remove the condition (7.1).

LEMMA 7.3. *There exist constants $C^*, \widetilde{C} > 0$ such that for any $T > 0$ the point $z - 4C^*T\delta_D(B)\nu(B^*)$ belongs to D whenever $z \in Q_{T\delta_D(B)}(B)$ and B is a point in $D \cap S_{\delta_0}$ with $\delta_D(B) \leq \widetilde{C}/T$.*

Proof. Let $B \in D$ be close to ∂D. Then (recall that r was chosen as a defining function for D) we have

$$r(z - 4C^* T\delta_D(B)\nu(B^*)) = \rho_B\bigl(F_B(z - 4C^* T\delta_D(B)\nu(B^*))\bigr).$$

Now analysis of the Taylor expansions of ρ_B (see Lemma 2.1.1) and of F_B at z gives the claim. ∎

Now let $T := 3\widehat{C}_1^2$, where $\widehat{C}_1$ is the constant from Lemma 3.4.2.

LEMMA 7.4. *Suppose that $A', B' \in D \cap S_{\delta_0}$ with*

$$\frac{1}{M_0}\min\{\delta_D(A'), \delta_D(B')\} \le d(A', B') \le 3\widehat{C}_1^2 \min\{\delta_D(A'), \delta_D(B')\}. \tag{7.2}$$

Then

$$d_D^{\mathrm{Kob}}(A', B') \le 2M_0\varrho(A', B').$$

Proof. We assume that $\delta_D(B') \le \delta_D(A')$ and introduce the auxiliary points

$$A'' := A' - T\,\delta_D(B')\nu(B'^{\,*}), \qquad B'' := B' - T\,\delta_D(B')\nu(B'^{\,*}).$$

We use that $A' \in Q_{T\delta_D(B')}(B')$ and obtain, by means of the Schwarz lemma,

$$\begin{aligned} d_D^{\mathrm{Kob}}(A', B') &\le d_D^{\mathrm{Kob}}(A', A'') + d_D^{\mathrm{Kob}}(A'', B'') + d_D^{\mathrm{Kob}}(B', B'') \\ &\le 2\log(1+2T) + d_D^{\mathrm{Kob}}(A'', B'') \\ &\le 2\log(1+2T) + d^{\mathrm{Kob}}_{Q'_{T\delta_D(B')}(B')}(A'', B'') \\ &= 2\log(1+2T) + d^{\mathrm{Kob}}_{Q_{T\delta_D(B')}(B')}(A', B'), \end{aligned}$$

where $Q'_{T\delta_D(B')}(B') := \{z - 4C^* T\delta_D(B')\nu(B'^{\,*}) \mid z \in Q_{T\delta_D(B')}(B')\} \subset D$.

But now we can repeat the estimation made in the first part of the proof of Lemma 7.2 to bound $d^{\mathrm{Kob}}_{Q_{T\delta_D(B')}(B')}(A', B')$ from above. The lower bound $(1/M_0)\min\{\delta_D(A'), \delta_D(B')\} \le d(A', B')$ allows us also to estimate the quantity $2\log(1+2T)$ in terms of $\varrho(A', B')$. ∎

In a final step we prove the desired upper bound on the Kobayashi distance.

THEOREM 7.5. *For any $A, B \in D \cap S_{\delta_0/2}$,*

$$d_D^{\mathrm{Kob}}(A, B) \le C^*\varrho(A, B).$$

Proof. We assume that $d(A, B) < \delta_0/2$. The claim will follow from the triangular inequality: We put

$$A' = A - d(A, B)\nu(A^*), \qquad B' = B - d(A, B)\nu(B^*).$$

Then

$$\begin{aligned} d(A', B') &\le \widehat{C}_1^2(d(A, A') + d(B, B') + d(A, B)) \\ &\le \widehat{C}_1^2(|A - A'| + |B - B'| + d(A, B)) \\ &\le 3\widehat{C}_1^2 d(A, B) \le 3\widehat{C}_1^2 \min\{\delta_D(A'), \delta_D(B')\} \end{aligned}$$

and

$$\begin{aligned} d_D^{\mathrm{Kob}}(A, B) &\le d_D^{\mathrm{Kob}}(A, A') + d_D^{\mathrm{Kob}}(A', B') + d_D^{\mathrm{Kob}}(B, B') \\ &\le \frac{1}{2}\log\left(1 + 2\frac{d(A, B)}{\delta_D(A)}\right) + \frac{1}{2}\log\left(1 + 2\frac{d(A, B)}{\delta_D(B)}\right) + d_D^{\mathrm{Kob}}(A', B'), \end{aligned}$$

from which, in conjunction with Lemmas 7.2 and 7.4 applied to A', B', and with the estimate $d(A', B') \le 3\widehat{C}_1^2 d(A, B)$, the claim follows. ■

References

[BMV] G. P. Balakumar, P. Mahajan, and K. Verma, *Bounds for invariant distances on pseudoconvex Levi corank one domains and applications*, arXiv:1303.3439.

[Bal-Bon] Z. M. Balogh and M. Bonk, *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. 75 (2000), 504–533.

[BloGra] T. Bloom and I. Graham, *A geometric characterization of points of type m on real submanifolds of $\mathbb{C}^n$*, J. Differential Geom. 12 (1977), 171–182.

[Cat1] D. W. Catlin, *Boundary invariants of pseudoconvex domains*, Ann. of Math. 120 (1984), 529–586.

[Cat2] D. W. Catlin, *Estimate of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. 200 (1989), 429–466.

[Cho-1] S. Cho, *Boundary behavior of the Bergman kernel function on some pseudoconvex domains in $\mathbb{C}^n$*, Trans. Amer. Math. Soc. 345 (1994), 803–817.

[Cho-2] S. Cho, *Estimates of invariant metrics on some pseudoconvex domains in $\mathbb{C}^n$*, J. Korean Math. Soc. 32 (1995), 661–678.

[DieHer] K. Diederich and G. Herbort, *Pseudoconvex domains of semiregular type*, in: Contributions to Complex Analysis and Analytic Geometry, H. Skoda and J.-M. Trépreau (eds.), Aspects Math. E26, Vieweg, Braunschweig, 1994, 127–161.

[Died-Ohs] K. Diederich and T. Ohsawa, *An estimate for the Bergman distance on pseudoconvex domains*, Ann. of Math. 141 (1995), 181–190.

[Gra] I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbf{C}^n$ with smooth boundary*, Trans. Amer. Math. Soc. 207 (1975), 219–240.

[Her-1] G. Herbort, *Invariant metrics and peak functions on pseudoconvex domains of homogeneous finite diagonal type*, Math. Z. 209 (1992), 223–243.

[Her-2a] G. Herbort, *On the invariant differential metrics near pseudoconvex boundary points where the Levi form has corank one*, Nagoya Math. J. 130 (1993), 25–54.

[Her-2b] G. Herbort, *Erratum to my paper: On the invariant differential metrics near pseudoconvex boundary points where the Levi form has corank one*, Nagoya Math. J. 135 (1994), 149–152.

[Her-3] G. Herbort, *Estimation on invariant distances on pseudoconvex domains of finite type in dimension two*, Math. Z. 251 (2005), 673–703.

[Hör] L. Hörmander, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math. 113 (1965), 89–152.

[Kr] S. G. Krantz, *Function Theory of Several Complex Variables*, reprint of the 2nd ed., AMS Chelsea, Providence, RI, 2001.

[N-S-W] A. Nagel, E. Stein, and S. Wainger, *Boundary behavior of functions holomorphic in domains of finite type*, Proc. Nat. Acad. Sci. U.S.A. 78 (1981), 6596–6599.

[Rei] H.-J. Reiffen, *Die differentialgeometrischen Eigenschaften der invarianten Distanzfunktion von Carathéodory*, Schr. Math. Inst. Univ. Münster 26 (1963).

[Roy] H. L. Royden, *Remarks on the Kobayashi metric*, in: Several Complex Variables II, Lecture Notes in Math. 185, Springer, Berlin, 1971, 125–137.

[Vor] N. Vormoor, *Topologische Fortsetzung biholomorpher Funktionen auf dem Rande bei beschränkten streng-pseudokonvexen Gebieten im $\mathbb{C}^n$ mit C^∞-Rand*, Math. Ann. 204 (1973), 239–261.

Gregor Herbort
Fachbereich Mathematik und Naturwissenschaften
Bergische Universität Wuppertal
D-42097 Wuppertal, Germany
E-mail: Gregor.Herbort@math.uni-wuppertal.de

Received 8.3.2012
and in final form 25.4.2013 (2718)