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Boundary blow-up solutions for a cooperative system involving the p-Laplacian

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Abstract. We study necessary and sufficient conditions for the existence of nonnegative boundary blow-up solutions to the cooperative system $\Delta_p u = g(u - \alpha v)$, $\Delta_p v = f(v - \beta u)$ in a smooth bounded domain of $\mathbb{R}^N$, where Δ_p is the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$, f and g are nondecreasing, nonnegative C^1 functions, and α and β are two positive parameters. The asymptotic behavior of solutions near the boundary is obtained and we get a uniqueness result for $p = 2$.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $1 < p < \infty$. We shall consider boundary blow-up solutions to the quasilinear system of the form

$$\begin{cases} \Delta_p u = g(u - \alpha v), & x \in \Omega, \\ \Delta_p v = f(v - \beta u), & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_p is the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, f, g are nondecreasing functions and $\alpha, \beta > 0$.

By a (local weak) *boundary blow-up solution* or *large solution* to (1.1), we mean that $(u, v) \in [W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^1_{\mathrm{loc}}(\Omega)]^2$ and

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = -\int_\Omega g(u - \alpha v)\varphi\, dx, & \forall \varphi \in C_0^\infty(\Omega), \\ \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\phi\, dx = -\int_\Omega f(v - \beta u)\phi\, dx, & \forall \phi \in C_0^\infty(\Omega), \end{cases}$$

and the boundary explosion should be interpreted as follows: For every positive integer k we have $k - u, k - v < 0$ on $\partial\Omega$ in the weak sense, $(\max(k - u, 0), \max(k - v, 0)) \in (W_0^{1,p}(\Omega))^2$.

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Boundary blow-up quasilinear elliptic problems of the form

$$\begin{cases} \Delta_p u = f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

have been the focus of a great number of works, regarding existence and uniqueness of positive solutions together with estimates of their rate of divergence to infinity at $\partial\Omega$; see for example [BM, CCEG, CD, L05, L06, V, Z] for semilinear problems ($p = 2$) and [DL, DG02, DG03, FLS, G09, GW, GW05] for problems with the p-Laplacian. It is known that when

$$\begin{aligned} &\text{either } f \in C^1(\mathbb{R}),\ f'(s) \geq 0 \text{ and } f(s) > 0 \text{ for } s \in \mathbb{R}, \text{ or} \\ &f \in C^1[0,\infty),\ f'(s) \geq 0 \text{ for } s \geq 0,\ f(0) = 0 \text{ and } f(s) > 0 \text{ for } s > 0, \end{aligned} \tag{1.3}$$

the existence of solutions to (1.2) is equivalent to a growth condition on f known as the *Keller–Osserman condition* (see [M]):

$$\int^{\infty} (F(t))^{-1/p}\,dt < \infty, \qquad \text{where} \quad F(t) = \int_0^t f(s)\,ds. \tag{1.4}$$

However, the corresponding problem for quasilinear elliptic systems such as (1.1) has been barely touched on in the literature. It is often studied for more selective classes of nonlinearities f, g and the Laplacian-operator case (i.e. $p = 2$). In [GS], the authors studied boundary blow-up solutions of the system $-\Delta u = \lambda u - u^2 + ruv, -\Delta v = \mu v - v^2 + suv$ with $r, s > 0$ (it falls into the cooperative regime). In [G07, G08, GLS, GR], the authors studied the problem

$$\begin{cases} \Delta_p u = u^a v^b, & x \in \Omega, \\ \Delta_p v = u^c v^e, & x \in \Omega, \\ u = v = \infty, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $a, e > p-1$, and $b, c > 0$; now (1.5) is of competitive type (the former two references studied the case of $p = 2$). Under the condition $(a - p + 1)$ $(e - p + 1) > bc$, the authors proved that the problem (1.5) has positive solutions if and only if $c < a - p + 1$, $b < e - p + 1$, and the positive solution is unique when it exists. In the critical case $(a - p + 1)(e - p + 1) = bc$, infinitely many positive solutions were constructed. Recently, two authors of the present paper studied (1.5) with $b, c < 0$ (see [CZ]; (1.5) is now of cooperative type) and showed the existence and uniqueness of a positive solution, and obtained the exact blow-up rate near the boundary of the solution under the condition $a > p-1$, $e > p-1$ and $(a-p+1)(e-p+1) > bc$.

Dávila et al. [DD] studied (1.1) for $\alpha = 1$, $p = 2$ and got necessary and sufficient conditions for the existence of positive solutions.

Moreover, we mention [AL, D02, LW, L03, LM], in which systems of large solutions were analyzed.

In the present work, we extend some of the results in [DD], such as existence, asymptotic behavior near the boundary and uniqueness of solutions, to the context of the p-Laplacian.

Here is a summary of our main results.

THEOREM 1.1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be nondecreasing, nonnegative C^1 functions such that $f = g = 0$ on $(-\infty, 0]$ and $\alpha, \beta > 0$. The problem* (1.1) *has a nonnegative solution if and only if the following conditions hold:*

- *f and g satisfy the Keller–Osserman condition* (1.4),
- *$\alpha\beta < 1$.*

The asymptotic of solutions to (1.1) is obtained at the price of a technical assumption on the nonlinearities, commonly found in the literature (see e.g. [DG02]). More precisely, let

$$\phi(u) = \int_u^\infty \frac{dt}{(qF(t))^{1/p}}, \tag{1.6}$$

where $F(t) = \int_0^t f(s)\,ds,\ q = p/(p-1)$. We assume in what follows that

$$\liminf_{t\to\infty} \frac{\phi(at)}{\phi(t)} > 1, \qquad \forall a \in (0,1). \tag{1.7}$$

Examples are $f(u) = e^u$ or $f(u) = u^m$, $m > p - 1$. A counter-example is $f(u) = u^{p-1}(\ln(1+u))^{2r}, r > 1$. Moreover, if f satisfies (1.7), the Keller–Osserman condition (1.4) follows.

For $\alpha\beta < \vartheta \le 1$, we let $w_\vartheta > 0$ denote the minimal solution to

$$\begin{cases} \Delta_p w_\vartheta = f\big(\frac{\vartheta-\alpha\beta}{\vartheta} w_\vartheta\big), & x \in \Omega, \\ w_\vartheta = \infty, & x \in \partial\Omega \end{cases} \tag{1.8}$$

(see e.g. [D06, DG02]).

THEOREM 1.2. *Make the same assumptions as in Theorem* 1.1.

(a) *If f satisfies* (1.7) *and is smaller than g at infinity in the sense that*

$$\lim_{t\to\infty} \frac{f(t)}{g(\varepsilon t)} = 0 \quad \textit{for any given } \varepsilon > 0, \tag{1.9}$$

then any nonnegative solution (u, v) of (1.1) *satisfies*

$$\lim_{x\to\partial\Omega} \frac{u}{w_1} = \alpha, \qquad \lim_{x\to\partial\Omega} \frac{v}{w_1} = 1, \tag{1.10}$$

where w_1 is the minimal positive solution to (1.8) *with $\vartheta = 1$.*

(b) *If f satisfies* (1.7) *and is of the order of g at infinity in the sense that for some $\vartheta_0 \in (\beta, 1)$,*

$$\tag{1.11} \begin{aligned} &\liminf_{t\to\infty} \left(\frac{\vartheta}{\alpha}\right)^{p-1} \frac{g((1-\alpha\vartheta)t)}{f((\vartheta-\beta)t)} \ge 1, \qquad \forall \vartheta \in (\beta, \vartheta_0),\\ &\liminf_{t\to\infty} \left(\frac{\vartheta}{\alpha}\right)^{p-1} \frac{g((1-\alpha\vartheta)t)}{f((\vartheta-\beta)t)} \le 1, \qquad \forall \vartheta \in (\vartheta_0, 1), \end{aligned}$$

then for any nonnegative solution (u, v) *of* (1.1),

$$\lim_{x\to\partial\Omega} \frac{u}{w_{\vartheta_0}} = \frac{\alpha}{\vartheta_0}, \qquad \lim_{x\to\partial\Omega} \frac{v}{w_{\vartheta_0}} = 1,$$

where w_{ϑ_0} is the minimal positive solution to (1.8) *with $\vartheta = \vartheta_0$.*

(c) *If g satisfies* (1.7) *and f is larger than g at infinity in the sense that*

$$\lim_{t\to\infty} \frac{f(\varepsilon t)}{g(t)} = \infty \quad \textit{for any given } \varepsilon > 0,$$

then any nonnegative solution (u, v) *of* (1.1) *satisfies*

$$\lim_{x\to\partial\Omega} \frac{u}{\varpi} = 1, \qquad \lim_{x\to\partial\Omega} \frac{v}{\varpi} = \beta,$$

where ϖ is the minimal positive solution to the problem

$$\begin{cases} \Delta_p \varpi = g((1-\alpha\beta)\varpi), & x \in \Omega,\\ \varpi = \infty, & x \in \partial\Omega. \end{cases}$$

Finally, we mention that it is difficult to get uniqueness results for (1.1) when $p \neq 2$ because of the nonlinearity of the operator. But when $p = 2$, we use another method different from that of [DD, Collary 1.5] to prove the uniqueness of solution without the condition of Ω being a ball.

THEOREM 1.3. *Make the same assumptions as in Theorem* 1.2 *and suppose the problem* (1.1) *has positive solutions. Assume in addition that $p = 2$ and*

$$\tag{1.12} f(t)/t,\ g(t)/t \textit{ are strictly increasing on } (0, \infty).$$

Then the positive solution is unique.

The paper is organized as follows. In Section 2 we give a comparison principle and prove Theorems 1.1. In Section 3 the asymptotic behavior for the solutions to (1.1) is proved. Uniqueness results for $p = 2$ are given in Section 4.

2. Necessary and sufficient conditions for existence. The following comparison lemma ([M09, Lemma 2.1]), proved in [DL, OT], will be useful.

LEMMA 2.1. *Let $h = h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in x and nondecreasing in t. Let $u, v \in W^{1,p}(\Omega)$ satisfy*

$$-\Delta_p u + h(x,u) \le -\Delta_p v + h(x,v), \qquad x \in \Omega.$$

If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

The proof of the existence of solutions in Theorem 1.1 follows a standard scheme where one first solves the system with a finite boundary condition m and then lets $m \to \infty$. The former step can be carried out for more general f and g (see Lemma 2.2 below). The solution (u,v) obtained in this way is called the *minimal solution*. Here minimality refers to the following property: take any open set $D \subseteq \Omega$ and $(\bar{u}, \bar{v}) \in W^{1,p}(D)$ satisfying

$$\begin{cases} \Delta_p \bar{u} \le g(\bar{u}, \bar{v}), & x \in D, \\ \Delta_p \bar{v} \le f(\bar{u}, \bar{v}), & x \in D, \\ \bar{u} \ge 0, \quad \bar{v} \ge 0, & x \in D, \\ \bar{u} \ge u, \quad \bar{v} \ge v, & x \in \partial D. \end{cases} \tag{2.1}$$

Then

$$u \le \bar{u}, \qquad v \le \bar{v} \quad \text{in } D.$$

LEMMA 2.2. *Suppose that f, g are two nonnegative C^1 functions such that $f(0,0) = g(0,0) = 0$ and $\partial g/\partial v, \partial f/\partial u \le 0$. Given $m > 0$, the system*

$$\begin{cases} \Delta_p u = g(u,v), & x \in \Omega, \\ \Delta_p v = f(u,v), & x \in \Omega, \\ u = v = m, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

admits a unique minimal nonnegative solution (u,v).

Proof. Choose $a, b > 0$ sufficiently large such that the functions

$$u \mapsto g(u,v) - au, v \mapsto f(u,v) - bv \text{ are decreasing for } 0 \le u, v \le m. \tag{2.3}$$

Define $u_0 = 0$, $v_0 = 0$ and for $k \ge 1$,

$$\begin{cases} \Delta_p u_k - a u_k = g(u_{k-1}, v_{k-1}) - a u_{k-1}, & x \in \Omega, \\ \Delta_p v_k - b v_k = f(u_{k-1}, v_{k-1}) - b v_{k-1}, & x \in \Omega, \\ u_k = v_k = m, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

We claim that

$$0 \le u_{k-1} \le u_k \le m \quad \text{and} \quad 0 \le v_{k-1} \le v_k \le m \quad \text{in } \Omega.$$

Indeed, this is straightforward if $k = 1$. Take $k \ge 2$ and assume by induction that $u_{k-2} \le u_{k-1}, v_{k-2} \le v_{k-1}$ in Ω. Then, from $\partial g/\partial v \le 0$ and (2.3), we have

$$\begin{aligned}\Delta_p u_k - au_k - (\Delta_p u_{k-1} - au_{k-1})& \\ &= g(u_{k-1}, v_{k-1}) - g(u_{k-2}, v_{k-2}) - a(u_{k-1} - u_{k-2}) \\ &\le g(u_{k-1}, v_{k-2}) - g(u_{k-2}, v_{k-2}) - a(u_{k-1} - u_{k-2}) \\ &\le 0 \quad \text{ in } \Omega.\end{aligned}$$

By Lemma 2.1 it follows that $u_{k-1} \le u_k$ in Ω. The remaining inequalities are obtained similarly. Thus we obtain uniform local bounds for u_k, v_k, and hence $u_k, v_k \in C^{1,\eta}(\bar{\Omega})$ (see [DB, G09, L, T]). So it is standard to deduce that the limits

$$u = \lim_{k\to\infty} u_k, \qquad v = \lim_{k\to\infty} v_k$$

in $C^1_{\mathrm{loc}}(\Omega)$ give a weak solution to (2.2).

Minimality. Let $D \subset \Omega$ be open and suppose $(\bar{u}, \bar{v}) \in (C(\bar{D}))^2$ satisfies (2.1). Choose a, b large enough so that $g(u, v) - au$ is decreasing in u and $f(u, v) - bv$ is decreasing in v for all u, v in the range $0 \le u, v \le M$ with $M \ge \max\{m, \max_{\bar{D}} \bar{u}, \max_{\bar{D}} \bar{v}\}$.

Consider u_k, v_k defined by (2.4). Now we show that $\bar{u} \ge u_k, \bar{v} \ge v_k$ in D for all k. By induction, if $\bar{u} \ge u_{k-1}, \bar{v} \ge v_{k-1}$ in D then it is easy to get

$$\Delta_p \bar{u} - a\bar{u} - (\Delta_p u_k - au_k) \le g(\bar{u}, \bar{v}) - a\bar{u} - g(u_{k-1}, v_{k-1}) + au_{k-1} \le 0 \text{ in } D$$

and hence $\bar{u} \ge u_k$ in D by Lemma 2.1. ■

By [D06, Remark 6.7], we can get the following two propositions which will be used in the proof of Theorem 1.1.

Proposition 2.3. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, positive and nondecreasing function and* (1.2) *has a solution in some bounded domain $\Omega \subset \mathbb{R}^N$, then* (1.4) *holds.*

Proposition 2.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a constant $s_0 > 0$ and a continuous, nondecreasing function $h : [s_0, \infty) \to \mathbb{R}$ such that*

$$f(u) \ge h(u) > 0 \quad \text{for } u \ge s_0, \qquad \int_{s_0}^{\infty} \Big[\int_{s_0}^{t} h(s)\, ds\Big]^{-1/p} dt < \infty.$$

If there exists some $v_* \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\Delta_p v_* \ge f(v_*), \qquad x \in \Omega,$$

then the boundary blow-up problem (1.2) *has at least one solution $u \in C^1(\Omega)$ satisfying $u \ge v_*$ in Ω. Moreover, it has a minimal solution u_* and a maximal solution u^* among all such solutions.*

Proof of Theorem 1.1. Necessity. Suppose that (u, v) is a nonnegative solution to (1.1) and for given $\gamma > 0$, set $w = \min\{\gamma u, v\}$. Let χ_A denote the characteristic function of a set A. By Kato's inequality ([H, Theorem 3.1]),

$$\begin{aligned}\Delta_p w &\le \gamma^{p-1}\Delta_p u\chi_{[\gamma u<v]} + \Delta_p v\chi_{[\gamma u>v]}\\ &= \gamma^{p-1}g(u-\alpha v)\chi_{[\gamma u<v]} + f(v-\beta u)\chi_{[\gamma u>v]}\\ &\le \gamma^{p-1}g((1-\alpha\gamma)u)\chi_{[\gamma u<v]} + f((1-\beta/\gamma)v)\chi_{[\gamma u>v]}\\ &= \gamma^{p-1}g\biggl(\frac{1-\alpha\gamma}{\gamma}w\biggr)\chi_{[\gamma u<v]} + f((1-\beta/\gamma)w)\chi_{[\gamma u>v]}\\ &\le \max\biggl\{\gamma g\biggl(\frac{1-\alpha\gamma}{\gamma}w\biggr), f((1-\beta/\gamma)w)\biggr\} =: h_1(w).\end{aligned}$$

Hence w is a supersolution to the problem

$$\text{(2.5)}\qquad \begin{cases} \Delta_p u = h_1(u), & x\in\Omega,\\ u=\infty, & x\in\partial\Omega.\end{cases}$$

Now we investigate the Dirichlet boundary problem

$$\text{(2.6)}\qquad \begin{cases} \Delta_p u = h_1(u), & x\in\Omega,\\ u=m, & x\in\partial\Omega.\end{cases}$$

Since 0 and w are sub- and supersolutions to (2.6) and $h_1(u)$ is nondecreasing in u, the problem (2.6) has a nonnegative solution by [W, Theorem 9.5.2]. Then let $m\to\infty$. It is standard to deduce that the problem (2.5) admits a solution and hence h_1 must satisfy the Keller–Osserman condition (1.4) by Proposition 2.3. Choosing $\gamma = 1/\alpha$ implies that f satisfies (1.4) and $\alpha\beta<1$. Then, choosing $\gamma=\beta$ implies that g satisfies (1.4) and $\alpha\beta<1$ too.

Sufficiency. Consider the minimal solution (u_m, v_m) to the truncated problem

$$\text{(2.7)}\qquad \begin{cases} \Delta_p u = g(u-\alpha v), & x\in\Omega,\\ \Delta_p v = f(v-\beta u), & x\in\Omega,\\ u=v=m, & x\in\partial\Omega,\end{cases}$$

where $m>0$. Such a solution can be easily constructed by the method of upper and lower solutions (see Lemma 2.2). Let $\gamma\in(\beta,1/\alpha)$ and set

$$w_m = \max\{\gamma u_m, v_m\}.$$

Then

$$\begin{aligned}\Delta_p w_m &\ge \gamma^{p-1}\Delta_p u_m\chi_{[\gamma u_m>v_m]} + \Delta_p v_m\chi_{[\gamma u_m<v_m]}\\ &= \gamma^{p-1}g(u_m-\alpha v_m)\chi_{[\gamma u_m>v_m]} + f((1-\beta/\gamma)v_m)\chi_{[\gamma u_m<v_m]}\\ &\ge \gamma^{p-1}g\biggl(\frac{1-\alpha\gamma}{\gamma}w_m\biggr)\chi_{[\gamma u_m>v_m]} + f((1-\beta/\gamma)w_m)\chi_{[\gamma u_m<v_m]}\\ &\ge h_2(w_m),\end{aligned}$$

where $h_2(w) = \min\{\gamma g(\frac{1-\alpha\gamma}{\gamma}w), f((1-\beta/\gamma)w)\}$. Since h_2 satisfies (1.3) and (1.4), by Proposition 2.4, the boundary blow-up problem $\Delta_p w = h_2(w)$ in Ω, $w = \infty$ on $\partial\Omega$ has a maximal solution w. By comparison, $w_m \le w$ in Ω for all $m > 0$. Hence $\{u_m\}, \{v_m\}$ remain bounded on compact sets of Ω as $m \to \infty$, and by standard elliptic estimates they converge—up to a subsequence—in $C^1_{\mathrm{loc}}(\Omega)$ to a solution of (1.1). ∎

REMARK 2.5. The proof of Theorem 1.1 implies that whenever solutions exist, one of them is minimal in the class of nonnegative solutions. Moreover this solution (u, v) satisfies

$$u \ge \alpha v \quad \text{and} \quad v \ge \beta u \quad \text{in } \Omega. \tag{2.8}$$

Indeed let us show that the minimal nonnegative solution (u_m, v_m) to (2.7) satisfies $u_m \ge \alpha v_m$ in Ω. To this end, let us recall that $u_m = \lim_{k\to\infty} u_{m,k}$, $v_m = \lim_{k\to\infty} v_{m,k}$, where $u_{m,k}, v_{m,k}$ are defined recursively by (2.4) starting with the trivial solutions, with $g(u, v) = g(u - \alpha v)$ and $f(u, v) = f(v - \beta u)$. We choose $a = b$ large so that (2.3) is satisfied. We claim that $u_{m,k} \ge \alpha v_{m,k}$. Proceeding inductively, assume $u_{m,k-1} \ge \alpha v_{m,k-1}$. Then

$$\begin{aligned}\Delta_p u_{m,k} - a u_{m,k} &= g(u_{m,k-1} - \alpha v_{m,k-1}) - a(u_{m,k-1} - \alpha v_{m,k-1}) - a\alpha v_{m,k-1}\\ &\le -a\alpha v_{m,k-1},\end{aligned}$$

while

$$\Delta_p \alpha v_{m,k} - a\alpha v_{m,k} = \alpha^{p-1} f(v_{m,k-1} - \beta u_{m,k-1}) - a\alpha v_{m,k-1} \ge -a\alpha v_{m,k-1}.$$

By Lemma 2.1, $u_{m,k} \ge \alpha v_{m,k}$ in Ω. For the other inequality in (2.8) we may proceed similarly, but this time it is convenient to work with $\widetilde{u}_{m,k}, \widetilde{v}_{m,k}$ defined by (2.4) with the boundary conditions $\widetilde{u}_{m,k} = m$ and $\widetilde{v}_{m,k} = \beta m$ on $\partial\Omega$. The limit of $\widetilde{u}_{m,k}, \widetilde{v}_{m,k}$ as $k \to \infty$ and then as $m \to \infty$ is the minimal nonnegative solution to the system, as can be seen by comparison.

3. Asymptotic behavior. In order to get the asymptotic behavior near the boundary of the solution to the system (1.1), we need some results in the scalar case. We know that when f satisfies (1.3) and the Keller–Osserman condition (1.4), the problem (1.2) admits a minimal solution u_* and a maximal solution u^*.

LEMMA 3.1 ([DG02, Proposition 3.3] and [M, Corollary 4.5]). *Assume f satisfies* (1.3) *and* (1.7). *Then for any solution u of* (1.2) *we have*

$$\lim_{x\to\partial\Omega} \frac{u(x)}{\psi(d(x))} = 1,$$

where $\psi = \phi^{-1}$ and ϕ is the function appearing in (1.7).

The following two lemmas are very similar to [DD, Lemmas 3.2 and 3.3], and the methods of proof are the same, so we omit the details. In the following, the notation $m \sim n$ means $\lim_{t\to t_0} m(t)/n(t) = 1$, and t_0 may be infinity.

Lemma 3.2. *Suppose $f_1 \sim f_2$ at infinity and that f_1 satisfies* (1.3) *and* (1.7). *Let u_1 and u_2 be any two solutions to* (1.2) *with nonlinearity being f_1 and f_2 respectively. Then*

$$\lim_{x\to\partial\Omega} \frac{u_1(x)}{u_2(x)} = 1.$$

Lemma 3.3. *Assume f satisfies* (1.3) *and* (1.7). *Given $\gamma > 0$, let u_γ denote any solution of*

$$\Delta_p u_\gamma = f(\gamma u_\gamma) \quad \text{in } \Omega, \qquad u_\gamma = \infty \qquad \text{on } \partial\Omega.$$

Then

$$\limsup_{\gamma\to 1} \limsup_{x\to\partial\Omega} \frac{u_\gamma}{u_1} \le 1 \le \liminf_{\gamma\to 1} \liminf_{x\to\partial\Omega} \frac{u_\gamma}{u_1}.$$

Proof of Theorem 1.2(a). By Lemma 3.1, it is enough to prove (1.10) for the minimal nonnegative solution (u, v) to (1.1). Now let w_1 be the minimal nonnegative solution to (1.8) with $\vartheta = 1$. For simplicity we write $w = w_1$. First we note that

$$w \le v \le u/\alpha. \tag{3.1}$$

Indeed, for the minimal solution (u, v), we always have $u \ge \alpha v$ by (2.8). Consequently,

$$\Delta_p v = f(v - \beta u) \le f((1 - \alpha\beta)v).$$

On the other hand, the fact that

$$\Delta_p w = f((1 - \alpha\beta)w)$$

yields $w \le v$ since w is the minimal nonnegative solution. Let

$$z_\vartheta = \max\{\vartheta u/\alpha, v\},$$

where $\alpha\beta < \vartheta < 1$. By Kato's inequality we have

$$\Delta_p z_\vartheta \ge h_\vartheta(z_\vartheta)$$

with

$$h_\vartheta(w) = \min\left\{ \left(\frac{\vartheta}{\alpha}\right)^{p-1} g\left(\frac{\alpha - \alpha\vartheta}{\vartheta} w\right), f\left(\frac{\vartheta - \alpha\beta}{\vartheta} w\right) \right\}. \tag{3.2}$$

Let w_ϑ be the minimal solution to (1.8) and $\widetilde{w}_\vartheta$ be the maximal solution to

$$\Delta_p \widetilde{w}_\vartheta = h_\vartheta(\widetilde{w}_\vartheta) \quad \text{in } \Omega, \qquad \widetilde{w}_\vartheta = \infty \quad \text{on } \partial\Omega.$$

Then $z_\vartheta \le \widetilde{w}_\vartheta$ in Ω. Note that under condition (1.9), we have $h_\vartheta(w) = f\big(\frac{\vartheta - \alpha\beta}{\vartheta} w\big)$ for large w. It follows from Lemma 3.2 that

$$\lim_{x\to\partial\Omega} \frac{w_\vartheta}{\widetilde{w}_\vartheta} = 1 \quad \text{and} \quad \limsup_{x\to\partial\Omega} \frac{z_\vartheta}{w_\vartheta} \le 1$$

for any $\vartheta \in (\beta, 1)$. It follows that

$$\limsup_{x\to\partial\Omega} \frac{z_\vartheta}{w} \le \limsup_{x\to\partial\Omega} \frac{z_\vartheta}{w_\vartheta} \limsup_{x\to\partial\Omega} \frac{w_\vartheta}{w} \le \limsup_{x\to\partial\Omega} \frac{w_\vartheta}{w}.$$

Letting now $\vartheta \to 1$ and using Lemma 3.3 we deduce that

$$\limsup_{\vartheta\to 1} \limsup_{x\to\partial\Omega} \frac{z_\vartheta}{w} \le 1$$

This together with (3.1) yields the conclusion. The method of proof of (c) is the same; we omit the details.

Proof of Theorem 1.2(b). First, it is easy to see that (1.1) has a non-negative boundary blow-up solution (u, v). Now we use Kato's inequality with

$$z_\vartheta = \max\{\vartheta u/\alpha, v\},$$

where $\alpha\beta < \vartheta < \vartheta_0$, to get

$$\Delta_p z_\vartheta \ge h_\vartheta(z_\vartheta)$$

with h_ϑ given by (3.2). By assumption (1.11), given $\varepsilon > 0$, we have $h_\vartheta(t) \ge (1-\varepsilon) f\big(\frac{\vartheta - \alpha\beta}{\vartheta} t\big)$ for t large. In particular, there exists $\delta > 0$ such that in $V := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \delta\}$,

$$\Delta_p z_\vartheta \ge (1-\varepsilon) f\left(\frac{\vartheta - \alpha\beta}{\vartheta} z_\vartheta\right).$$

Let $w_{\varepsilon,\vartheta}$ denote the maximal solution of

$$\begin{cases} \Delta_p w_{\varepsilon,\vartheta} = (1-\varepsilon) f\big(\frac{\vartheta-\alpha\beta}{\vartheta} w_{\varepsilon,\vartheta}\big), & x \in V, \\ w_{\varepsilon,\vartheta} = \infty, & x \in \partial V. \end{cases}$$

Then $z_\vartheta \le w_{\varepsilon,\vartheta}$ in V. By Lemma 3.3,

$$\limsup_{\varepsilon\to 0,\, \vartheta\to\vartheta_0} \limsup_{x\to\partial\Omega} \frac{w_{\varepsilon,\vartheta}}{w} \le 1,$$

where w is the minimal solution of (1.8) with $\vartheta = \vartheta_0$. Thus,

$$\limsup_{\vartheta\to\vartheta_0} \limsup_{x\to\partial\Omega} \frac{z_\vartheta}{w} \le 1. \tag{3.3}$$

Let $\vartheta \in (\vartheta_0, 1)$ and $\widetilde{z}_\vartheta = \min\{\vartheta u/\alpha, v\}$. Then, as before, $\widetilde{z}_\vartheta \ge \widetilde{w}_{\varepsilon,\vartheta}$, where now $\widetilde{w}_{\varepsilon,\vartheta}$ is the minimal solution of

$$\begin{cases} \Delta_p \widetilde{w}_{\varepsilon,\vartheta} = (1-\varepsilon) f\big(\frac{\vartheta-\alpha\beta}{\vartheta} \widetilde{w}_{\varepsilon,\vartheta}\big), & x \in V, \\ \widetilde{w}_{\varepsilon,\vartheta} = \infty, & x \in \partial\Omega, \\ \widetilde{w}_{\varepsilon,\vartheta} = \tau, & x \in \partial V \setminus \partial\Omega, \end{cases}$$

and $\tau > 0$ is a fixed small constant. Using Lemma 3.3 one proves that

$$\liminf_{\varepsilon\to 0,\, \vartheta\to\vartheta_0} \liminf_{x\to\partial\Omega} \frac{\widetilde{w}_{\varepsilon,\vartheta}}{w} \ge 1,$$

hence

$$\liminf_{\vartheta\to\vartheta_0}\liminf_{x\to\partial\Omega}\frac{\widetilde{z}_\vartheta}{w}\ge 1. \tag{3.4}$$

Collecting (3.3) and (3.4) shows that the theorem is proved. ∎

4. Uniqueness. Now we use a different method from [DD, Corollary 1.5] to prove the uniqueness of solution to (1.1) provided that $p=2$ and f,g satisfy the conditions in Theorem 1.2, and (1.12). Our first uniqueness result is concerned with the problem (1.1) with finite boundary conditions. To this end, Lemma 8 in [G08] will be used. For the readers' convenience, we state it below.

LEMMA 4.1. *Let $f,g\in C(\overline{\Omega})$ and $u,v\in C^{1,\eta}(\overline{\Omega})$ be weak solutions to $\Delta_p u=f$, $\Delta_p v=g$ in Ω with $u\le v$ and $u=v$ at some point of Ω. Assume moreover that $u<v$ on $\partial\Omega$. Then there exists $x_0\in\Omega$ such that $u(x_0)=v(x_0)$ and $f(x_0)\le g(x_0)$.*

With the same method as in the proof of [DD, Lemma 4.1], we have

LEMMA 4.2. *Let (u,v) be the minimal boundary blow-up solution of (1.1) with $p=2$. Then*

$$u>\alpha v,\qquad v>\beta u\qquad \text{in } \Omega$$

and

$$\lim_{x\to\partial\Omega}(u(x)-\alpha v(x))=+\infty,\qquad \lim_{x\to\partial\Omega}(v(x)-\beta u(x))=+\infty.$$

LEMMA 4.3. *Assume that the condition (1.12) holds. Let (u_1,v_1) and (u_2,v_2) be positive solutions to the problem*

$$\begin{cases} \Delta u=g(u-\alpha v), & x\in\Omega,\\ \Delta v=f(v-\beta u), & x\in\Omega,\\ u=\varphi(x),\quad v=\psi(x), & x\in\partial\Omega,\end{cases} \tag{4.1}$$

and (u_2,v_2) be the minimal solution. Then $u_1=u_2$, $v_1=v_2$ in $\bar{\Omega}$.

Proof. Since (u_2,v_2) is the minimal solution to (4.1) and is positive, we can select a large k so that

$$u_1\le ku_2,\qquad v_1\le kv_2,\qquad x\in\Omega. \tag{4.2}$$

Choose the least k with this property, and assume $k>1$. Then one of the two inequalities in (4.2) is not strict. Assume it is the second one. We can apply Lemma 4.1 to obtain a point $x_0\in\Omega$ with $v_1(x_0)=kv_2(x_0)$ and

$$f(v_1(x_0)-\beta u_1(x_0))\le kf(v_2(x_0)-\beta u_2(x_0)).$$

On the other hand, by Lemma 4.2, we have $u_2(x_0)>\alpha v_2(x_0)$, $v_2(x_0)>\beta u_2(x_0)$. As $f(t)/t$ is increasing in $t>0$, and $u_1(x_0)\le ku_2(x_0)$, $v_1(x_0)=$

$kv_2(x_0)$, it follows that

$$f(v_1(x_0) - \beta u_1(x_0)) \geq f(kv_2(x_0) - \beta k u_2(x_0)) > kf(v_2(x_0) - \beta u_2(x_0)).$$

This contradiction shows $k \leq 1$, that is, $u_1 \leq u_2, v_1 \leq v_2$. This concludes the proof. ■

Proof of Theorem 1.3. Let (u_1, v_1) and (u_2, v_2) be two positive solutions of the problem (1.1) and (u_2, v_2) be the minimal solution. By Lemma 4.2, we have $u_2 > \alpha v_2, v_2 > \beta u_2$ in Ω. Moreover, by Theorem 1.2,

$$\lim_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} = \lim_{d(x)\to 0} \frac{v_1(x)}{v_2(x)} = 1.$$

Thus, for every $0 < \varepsilon < 1$, there exists $\delta > 0$ such that when $d(x) \leq \delta$,

$$(1-\varepsilon)u_2 \leq u_1 \leq (1+\varepsilon)u_2, \quad (1-\varepsilon)v_2 \leq v_1 \leq (1+\varepsilon)v_2. \tag{4.3}$$

Now we set $\Omega_\delta = \{x \in \Omega : d(x) > \delta\}$, and consider the problem

$$\begin{cases} \Delta w = g(w - \alpha z), & x \in \Omega_\delta, \\ \Delta z = f(z - \beta w), & x \in \Omega_\delta, \\ w = u_1, \quad z = v_1, & x \in \partial\Omega_\delta. \end{cases} \tag{4.4}$$

Since $f(t)/t, g(t)/t$ are increasing functions, it is not difficult to see that the pairs $((1+\varepsilon)u_2, (1+\varepsilon)v_2)$, $((1-\varepsilon)u_2, (1-\varepsilon)v_2)$ are upper and lower solutions to (4.4). So, the problem (4.4) has at least one solution (u, v) and satisfies

$$(1-\varepsilon)u_2 \leq u \leq (1+\varepsilon)u_2, \quad (1-\varepsilon)v_2 \leq v \leq (1+\varepsilon)v_2 \quad \text{in } \Omega_\delta.$$

On the other hand, thanks to Lemma 4.3, the problem (4.4) has a unique positive solution, which is precisely (u_1, v_1). Thus, (4.3) is valid in Ω_δ. Therefore, (4.3) holds for all $x \in \Omega$. Letting ε go to zero in (4.3) we arrive at $u_1 = u_2, v_1 = v_2$, which proves uniqueness. ■

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