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On sectional curvature of a Riemannian manifold with semi-symmetric metric connection

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Abstract. We prove that if the sectional curvature of an n-dimensional pseudo-symmetric manifold with semi-symmetric metric connection is independent of the orientation chosen then the generator of such a manifold is gradient and also such a manifold is subprojective in the sense of Kagan.

1. Introduction. Let (M_n, g) be an n-dimensional differentiable manifold of class C^∞ with the metric tensor g, the Riemannian connection ∇ and a smooth linear connection ∇^* on M_n. A smooth linear connection ∇^* on M_n is said to be *semi-symmetric* if its torsion tensor T satisfies the relation

$$T(X,Y) = w(Y)X - w(X)Y \tag{1}$$

where w is a smooth linear differential form and X and Y are any smooth vector fields on M_n, [Y1]. The concept of a semi-symmetric connection has been studied on Kenmotsu manifolds [PD1], almost contact manifolds [DS], Sasakian manifolds [PD2] and Riemannian manifolds [D]. It is known [Y1] that if ∇^* is a semi-symmetric metric connection then

$$\nabla^*_X Y = \nabla_X Y + w(Y)X - g(X,Y)\rho, \tag{2}$$

$$g(X,\rho) = w(X), \tag{3}$$

for any vector fields X and Y. Further, it is also known [Y1] that if R^* and R denote of the curvature tensors of the smooth linear connection ∇^* and the Levi-Civita connection ∇, respectively, then

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z &- \alpha(Y,Z)X + \alpha(X,Z)Y \\ &- g(Y,Z)AX + g(X,Z)AY \end{aligned} \tag{4}$$

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where α is a tensor field of type $(0,2)$ defined by

$$\alpha(X,Y)=(\nabla_X w)(Y)-w(X)w(Y)+\tfrac{1}{2}w(\rho)g(X,Y) \tag{5}$$

and A is a tensor field of type $(1,1)$ defined by

$$g(AX,Y)=\alpha(X,Y) \tag{6}$$

for any vector fields X and Y.

We shall use the following results in the next section:

In a local coordinate system, equations (4), (5) and (6) can be written as follows:

$$R^*_{ijkh}=R_{ijkh}-P_{jk}g_{ih}+P_{ik}g_{jh}-P_{ih}g_{jk}+P_{jh}g_{ik} \tag{7}$$

where

$$P_{jk}=\nabla_j w_k-w_jw_k+\tfrac{1}{2}g_{jk}w_hw^h,\qquad P^h_k=P_{km}g^{mh}. \tag{8}$$

From (7), we have (see [Y1])

$$R^*_{ih}=R_{ih}-(n-2)P_{ih}-\alpha g_{ih}, \tag{9}$$

$$R^*=R-2(n-1)\alpha, \tag{10}$$

where

$$\alpha=g^{ih}P_{ih}. \tag{11}$$

M. C. Chaki [CH] introduced a type of non-flat Riemannian manifold (M_n,g) $(n\geq 2)$ whose curvature tensor R_{hijk} satisfies the condition

$$\nabla_l R_{hijk}=2\lambda_l R_{hijk}+\lambda_h R_{lijk}+\lambda_i R_{hljk}+\lambda_j R_{hilk}+\lambda_k R_{hijl} \tag{12}$$

where λ_l is a non-zero vector which is called the *generator* of the manifold. Such a manifold is called *pseudo-symmetric* and is denoted by $(PS)_n$.

A Riemannian manifold is called an *Einstein manifold* if its Ricci tensor is proportional to its metric.

Moreover, an n-dimensional manifold with a semi-symmetric metric connection is called an *Einstein manifold with a semi-symmetric metric connection* if the symmetric part of the Ricci tensor is proportional to the metric, i.e.,

$$R^*_{(ij)}=\lambda g_{ij} \tag{13}$$

where λ is a scalar function.

Now, we can state the following lemma which will be used in our subsequent work:

Lemma. *Suppose that S is a $(0,2)$ covariant tensor. If for all linearly independent vectors X and Y,*

$$S_{\alpha\beta\lambda\mu}X^\alpha Y^\beta X^\lambda Y^\mu=0, \tag{14}$$

then

$$S_{\alpha\beta\lambda\mu} + S_{\lambda\mu\alpha\beta} + S_{\alpha\mu\lambda\beta} + S_{\lambda\beta\alpha\mu} = 0. \tag{15}$$

Here X^α and Y^β are the contravariant components of X and Y, respectively, [LR].

2. Sectional curvatures of a Riemannian manifold having a semi-symmetric metric connection. Let $P(x^k)$ be any point of $M_n(\nabla^*, g)$ and denote by X^α, Y^α the components of two linearly independent vectors $X, Y \in T_P(M_n)$. These vectors determine a two-dimensional subspace (plane) π in $T_P(M_n)$.

The scalar

$$K^*(\pi) = \frac{R^*_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu}{(g_{\beta\lambda} g_{\alpha\mu} - g_{\alpha\lambda} g_{\beta\mu}) X^\alpha Y^\beta X^\lambda Y^\mu} \tag{16}$$

is called the *sectional curvature* of $M_n(\nabla^*, g)$ at P with respect to the plane π.

From (16), it follows that

$$S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu = 0 \tag{17}$$

where we have put

$$S_{\alpha\beta\lambda\mu} = R^*_{\alpha\beta\lambda\mu} - K^*(\pi)(g_{\beta\lambda} g_{\alpha\mu} - g_{\alpha\lambda} g_{\beta\mu}). \tag{18}$$

Assume that at any point $P \in M_n(\nabla^*, g)$, the sectional curvatures for all planes in $T_P(M_n)$ are the same. A two-dimensional Riemannian manifold having semi-symmetric metric connection need not be considered, since it has only one plane at each point. Then, according to the Lemma, the condition (15) gives

$$\begin{aligned} R^*_{\alpha\beta\lambda\mu} + R^*_{\lambda\mu\alpha\beta} + R^*_{\alpha\mu\lambda\beta} + R^*_{\lambda\beta\alpha\mu} = 2K^*(\pi)(g_{\mu\alpha} g_{\lambda\beta} + g_{\alpha\beta} g_{\mu\lambda}) \\ - 4K^*(\pi) g_{\alpha\lambda} g_{\beta\mu}. \end{aligned} \tag{19}$$

Multiply the equation (19) by $g^{\alpha\mu}$ to find

$$\frac{R^*_{\lambda\beta} + R^*_{\beta\lambda}}{2} = (n-1)K^*(\pi) g_{\lambda\beta}. \tag{20}$$

This can be rewritten in the form

$$R^*_{(\lambda\beta)} = (n-1)K^*(\pi) g_{\lambda\beta} \tag{21}$$

where

$$R^*_{(\lambda\beta)} = \frac{R^*_{\lambda\beta} + R^*_{\beta\lambda}}{2}. \tag{22}$$

Transvecting (21) by $g^{\lambda\beta}$, we get

$$R^* = n(n-1)K^*(\pi). \tag{23}$$

From (9), we have

$$R^*_{[\lambda\beta]} = (2-n)P_{[\lambda\beta]}. \tag{24}$$

Since the sectional curvatures at $P \in M_n(\nabla^*, g)$ are the same for all planes in $T_P(M_n)$, by using (16), we have

$$R^*_{\alpha\beta\lambda\mu} = K^*(\pi)(g_{\beta\lambda}g_{\alpha\mu} - g_{\alpha\lambda}g_{\beta\mu}). \tag{25}$$

Multiplying (25) by $g^{\alpha\mu}$ and summing over α and μ, we get

$$R^*_{\lambda\beta} = K^*(\pi)(n-1)g_{\lambda\beta}. \tag{26}$$

From (8), (21), (25) and (26), it follows that

$$R^*_{[\lambda\beta]} = 0, \tag{27}$$

$$\nabla_{[\lambda}w_{\beta]} = 0. \tag{28}$$

(21) means that $M_n(\nabla^*, g)$ is an Einstein manifold with a semi-symmetric metric connection. (28) implies that the 1-form w is closed.

With the help of (7), (8) and (28), we find that

$$R^*_{\alpha\beta\lambda\mu} + R^*_{\beta\lambda\alpha\mu} + R^*_{\lambda\alpha\beta\mu} = 0, \tag{29}$$

i.e., the first Bianchi identity holds for the linear connection.

From (9) and (10) we have

$$P_{ij} = -\lambda_{ij} - \frac{R^*_{ih}}{n-2} - \frac{R^*g_{ih}}{2(n-1)(n-2)} \tag{30}$$

where

$$\lambda_{ij} = -\frac{1}{n-2}R_{ij} + \frac{1}{2(n-1)(n-2)}Rg_{ij}. \tag{31}$$

From (21), (23) and (27), we have $R^*_{ih} = R^*g_{ih}/n$. Then, by using (30), we find

$$P_{ij} = -\lambda_{ij} - \frac{R^*g_{ij}}{2n(n-1)}. \tag{32}$$

By the aid of the equations (7), (23) and (32), we get

$$R^*_{ijkh} = C_{ijkh} + K^*(\pi)(g_{ih}g_{jk} - g_{ik}g_{jh}). \tag{33}$$

By using (25) and (33), we can easily see that this space is conformally flat.

In [I], by using a different method, it has been shown that if a Riemannian manifold admits a semi-symmetric metric connection with closed π constant curvature, then the manifold is conformally flat.

Since this manifold is conformally flat, we have

$$R_{ijkh} = \frac{1}{(n-2)}(g_{jk}R_{ih} - g_{ik}R_{jh} + g_{ih}R_{jk} - g_{jh}R_{ik}) - \frac{1}{(n-1)(n-2)}R(g_{jk}g_{ih} - g_{jh}g_{ik}). \tag{34}$$

By using (31), the equation (34) can be rewritten as

$$R_{ijkh} = -g_{jk}\lambda_{ih} - g_{ih}\lambda_{jk} + g_{ik}\lambda_{jh} + g_{jh}\lambda_{ik}. \tag{35}$$

If we multiply the equation (12) by g^{hk}, we obtain

$$2\lambda_l R_{jk} + \lambda_j R_{lk} + \lambda_k R_{jl} + \lambda_h g^{ih}(R_{ljki} + R_{ijkl}) = \nabla_l R_{jk}. \tag{36}$$

Multiplying (36) by g^{jk}, we find

$$2\lambda_l R + 4\lambda_i g^{ih} R_{lh} = \nabla_l R. \tag{37}$$

By cyclic permutation of the indices l, j and k and by using the last two equations and (36), we have the relation

$$\lambda_l R_{jk} + \lambda_j R_{kl} + \lambda_k R_{lj} = \tfrac{1}{4}(\nabla_l R_{jk} + \nabla_j R_{kl} + \nabla_k R_{lj}). \tag{38}$$

It is known [CH] that a conformally flat $(PS)_n$ $(n \geq 3)$ cannot be of zero scalar curvature and in a conformally flat $(PS)_n$, it is also known [T] that

$$R_{ij} = \frac{R-t}{n-1}g_{ij} + \frac{nt-R}{(n-1)\lambda_p\lambda^p}\lambda_i\lambda_j \tag{39}$$

where R denotes the scalar curvature and t is a scalar.

The expression (39) can be written as

$$R_{ij} = \theta g_{ij} + \beta v_i v_j \tag{40}$$

where

$$\theta = \frac{R-t}{n-1}, \quad \beta = \frac{nt-R}{n-1}, \quad \lambda^h R_{hk} = t\lambda_k, \quad v_i = \frac{\lambda_i}{\sqrt{\lambda_m\lambda^m}} \tag{41}$$

and v_i is a unit vector.

Thus, from (34) and (40), we have

$$R_{ijkl} = b(-g_{jl}v_iv_k + g_{jk}v_iv_l - g_{ik}v_jv_l + g_{il}v_jv_h) + a(g_{il}g_{jk} - g_{jl}g_{ik}) \tag{42}$$

where $a = \frac{R-2t}{(n-1)(n-2)}$ and $b = \frac{nt-R}{(n-1)(n-2)}$.

D. Smaranda [S] calls a Riemannian manifold whose curvature tensor satisfies (42) a *manifold of almost constant curvature.* Hence, we have the following theorem:

THEOREM 2.1. *If a $(PS)_n$ admits a semi-symmetric metric connection with constant sectional curvature then this manifold is of almost constant curvature.*

For a conformally flat $(PS)_n$, the following condition holds [T]:

$$\lambda^j \nabla_l R_{jk} = \lambda^j \lambda_j R_{lk} + \frac{3n-2}{n-1} t \lambda_l \lambda_k - \frac{t}{n-1} g_{lk} \lambda^j \lambda_j. \tag{43}$$

Taking the covariant derivative of $(41)_3$ with respect to x^m and using equation (43), we find

$$\begin{aligned}\lambda^h \lambda_h R_{km} + \frac{3n-2}{n-1} t \lambda_m \lambda_k - \frac{t}{n-1} g_{km} \lambda^h \lambda_h \\ = \lambda_k \nabla_m t + t \nabla_m \lambda_k - R_{hk} \nabla_m \lambda^h.\end{aligned} \tag{44}$$

From (40), (41) and (44), we get

$$\begin{aligned}\frac{R-t}{n-1} g_{km} \lambda^h \lambda_h + \frac{nt-R}{n-1} \lambda_k \lambda_m + \frac{(3n-2)t}{n-1} \lambda_k \lambda_m - \frac{t}{n-1} g_{km} \lambda^h \lambda_h \\ = \lambda_k \nabla_m t + t \nabla_m \lambda_k - \frac{R-t}{n-1} g_{kh} \nabla_m \lambda^h - \frac{nt-R}{(n-1)\lambda_i \lambda^i} \lambda_h \lambda_k \nabla_m \lambda^h.\end{aligned} \tag{45}$$

If we multiply (45) by λ^k then we find

$$\nabla_m t = 4t\lambda_m. \tag{46}$$

With the help of (37) and (40), we get

$$\nabla_l R = 2((n+2)\theta + 3\beta)\lambda_l. \tag{47}$$

From equation (47), it is clear that the covariant vector λ_l is a gradient. Thus, we have the following theorem:

THEOREM 2.2. *If a $(PS)_n$ admits a semi-symmetric metric connection with constant sectional curvature then the covariant vector λ_l of this manifold is a gradient.*

Now, for a conformally flat manifold $(PS)_n$, we have (see [DG])

$$v_l \nabla_k \beta - v_k \nabla_l \beta + \beta(\nabla_k v_l - \nabla_l v_k) = 0. \tag{48}$$

By using $(41)_2$ and (46), we obtain

$$v_l \nabla_k \beta - v_k \nabla_l \beta = 0. \tag{49}$$

By using (48) and (49), we get

$$\beta = 0 \quad \text{or} \quad \nabla_k v_l - \nabla_l v_k = 0. \tag{50}$$

If $\beta = 0$ then the manifold is flat. This contradicts the hypotheses. Thus, from (50),

$$\nabla_k v_l - \nabla_l v_k = 0. \tag{51}$$

It is known [DG] that the covariant vector v_i of a conformally flat $(PS)_n$ is a proper concircular vector field. Hence, we have the following theorem:

THEOREM 2.3. *A $(PS)_n$ admitting a semi-symmetric metric connection with a constant sectional curvature has a proper concircular vector field.*

It is known [A] that if a conformally flat manifold admits a proper concircular vector field then the manifold is a subprojective manifold in the sense of Kagan. Thus, we can state the following theorem:

THEOREM 2.4. *If a $(PS)_n$ admits a semi-symmetric metric connection with a constant sectional curvature then this manifold is subprojective.*

In [Y3], K. Yano proved that for a Riemannian manifold to admit a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form

$$ds^2 = (dx^1)^2 + c^q g^*_{\alpha\beta} dx^\alpha dx^\beta \tag{52}$$

where

$$g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\nu) \tag{53}$$

are functions of x^ν $(\alpha, \beta, \nu = 2, 3, \dots, n)$ and $q = q(x^1) \neq \text{const}$ is a function of x^1 only. Since a conformally flat $(PS)_n$ admits a proper concircular vector field v_i, the manifold under consideration is the warped product $1 \times_{e^q} M^*$ where (M^*, g^*) is an $(n-1)$-dimensional Riemannian manifold.

Since this manifold is conformally flat, from (34), the following equation is satisfied:

$$\nabla_k R_{jl} - \nabla_l R_{jk} = \frac{1}{2(n-1)}(g_{jl}\nabla_k R - g_{jk}\nabla_l R). \tag{54}$$

Gębarowski [G] proved that the warped product $1 \times_{e^q} M^*$ satisfies (52) if and only if M^* is an Einstein manifold.

Thus, we can state the following theorem:

THEOREM 2.5. *If a $(PS)_n$ admits a semi-symmetric metric connection with a constant sectional curvature then this manifold is the warped product $1 \times_{e^q} M^*$ where M^* is an Einstein manifold.*

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