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Projections onto the spaces of Toeplitz operators

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Abstract. Projections onto the spaces of all Toeplitz operators on the N-torus and the unit sphere are constructed. The constructions are also extended to generalized Toeplitz operators and applied to show hyperreflexivity results.

1. Introduction. Arveson [1] defined a projection from the algebra $B(H^2(\mathbb{T}))$ of all bounded linear operators on the Hardy space on the unit circle onto the space of all Toeplitz operators on $H^2(\mathbb{T})$. He used the classical Banach limit. We construct a projection from the algebra $B(H^2(\mathbb{T}^N))$ of all bounded linear operators on the Hardy space on the N-torus onto the space of all Toeplitz operators on $H^2(\mathbb{T}^N)$. We use the extension of the Banach limit to N-parameter sequences, given in Section 2.

In Section 4 we will use the above projection to show that the subspace of all Toeplitz operators on the N-torus is 2-hyperreflexive (for definition see Section 4). The single variable case was considered in [7].

A natural generalization of the unit circle is not only the N-torus but also the unit sphere $\partial\mathbb{B}_N$. In Section 5 we construct a projection from the algebra $B(H^2(\partial\mathbb{B}_N))$ of all bounded linear operators on the Hardy space on the unit sphere onto the space of all Toeplitz operators on $H^2(\partial\mathbb{B}_N)$.

In Section 6 we extend the idea of such a projection to generalized Toeplitz operators which were introduced in [10], [11]. We consider both one and multi-variable cases. In Section 7 we give a hyperreflexivity result for generalized Toeplitz operators. The one variable case was considered in [8].

2. Multi-variable Banach limit. There is a functional on all bounded sequences in ℓ^∞, which to any convergent sequence $\{x(n)\}_{n\in\mathbb{N}}$ assigns its

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limit (see e.g. [4]). It is called the *Banach limit*. We extend this idea to multi-variable bounded sequences in $\ell^\infty(\mathbb{N}^N)$.

THEOREM 2.1. *There is a linear functional* $\Lambda\colon \ell^\infty(\mathbb{N}^N,\mathbb{C}) \to \mathbb{C}$ (*resp.* $\Lambda\colon \ell^\infty(\mathbb{N}^N,\mathbb{R}) \to \mathbb{R}$) *such that*

(a) $\|\Lambda\| = 1$ *and* $\Lambda(\mathbf{1}) = 1$, *where* $\mathbf{1}$ *is the constantly* 1 *sequence*,
(b) *if* $x \in \ell^\infty(\mathbb{N}^N)$ *converges*, *then* $\Lambda(x) = \lim_{k\to\infty,\, k\in\mathbb{N}^N} x(k)$,
(c) *if* $x \in \ell^\infty(\mathbb{N}^N)$ *is nonnegative*, *i.e.* $x(k) \geq 0$ *for all* $k \in \mathbb{N}^N$, *then* $\Lambda(x) \geq 0$,
(d) *for any sequence* $x \in \ell^\infty(\mathbb{N}^N)$ *and any* $i = 1,\dots,N$, *let* $x^{(i)}$ *denote the sequence* $x^{(i)}(k) = x(k+e_i)$, *where* $e_i = (0,\dots,1,\dots,0)$; *then* $\Lambda(x) = \Lambda(x^{(i)})$.

Proof. First we deal with the case of a real-valued functional, i.e. $\Lambda\colon \ell^\infty(\mathbb{N}^N,\mathbb{R}) \to \mathbb{R}$. For each i, set $\mathcal{M}_i = \{x - x^{(i)} : x \in \ell^\infty(\mathbb{N}^N,\mathbb{R})\}$, where $x^{(i)}$ is defined as in (d). Note that $\mathcal{M}_i$ is a linear manifold. Let $\mathcal{M}$ be the subspace spanned by all $\mathcal{M}_i$, $i = 1,\dots,N$. We show first that

$$d(\mathbf{1},\mathcal{M}) = 1, \tag{1}$$

where d denotes the distance from the sequence $\mathbf{1}$ to the subspace $\mathcal{M}$. Since $0 \in \mathcal{M}$ we have $d(\mathbf{1},\mathcal{M}) \leq 1$. Assume that there are $\varepsilon > 0$ and $x_i \in \mathcal{M}_i$ and $\alpha_i \in \mathbb{R}$ with $\|\alpha_i\, x_i\|_\infty \leq M$, $i = 1,\dots,N$, such that

$$\Big\|\mathbf{1} - \sum_{i=1}^{N} \alpha_i (x_i - x_i^{(i)})\Big\|_\infty < 1 - \varepsilon.$$

In particular, for fixed $n \in \mathbb{N}$, for all $k = (k_1,\dots,k_N) \in \mathbb{N}^N$ such that $|k|_\infty = \max |k_i| \leq n$, we have

$$1 - \sum_{i=1}^{N} \alpha_i (x_i(k) - x_i^{(i)}(k)) < 1 - \varepsilon.$$

Summation over k gives

$$n^N - \sum_{i=1}^{N} \alpha_i \Big(\sum_{\substack{|k|_\infty \leq n \\ k_i = 1}} x_i(k) - \sum_{\substack{|k|_\infty \leq n \\ k_i = n}} x_i(k+e_i) \Big) < n^N - n^N \varepsilon.$$

Thus

$$n^N \varepsilon < \sum_{i=1}^{N} |\alpha_i| \Big(\sum_{\substack{|k|_\infty \leq n \\ k_i = 1}} |x_i(k)| + \sum_{\substack{|k|_\infty \leq n \\ k_i = n}} |x_i(k+e_i)| \Big) \leq 2n^{N-1} NM.$$

Hence $n\varepsilon < 2NM$ and we have a contradiction for n large enough, so (1) follows.

The Hahn–Banach theorem yields a linear functional Λ on $\ell^\infty(\mathbb{N}^N,\mathbb{R})$ such that $\Lambda(\mathbf{1}) = 1$, $\Lambda(\mathcal{M}) = 0$ and $\|\Lambda\| = 1$.

To see (b), for a given sequence $x \in \ell^\infty(\mathbb{N}^N, \mathbb{R})$ and for all multi-indices in $\mathbb{N}^N$, define by multi-induction the following sequences:

$$x_{(e_i)} = x^{(i)} \quad \text{and} \quad x_{(k+e_i)} = (x_{(k)})^{(i)}.$$

Note that the definition is correct since $(x_{(k-e_i)})^{(j)} = (x_{(k-e_j)})^{(i)}$. For fixed $k = (k_1, \dots, k_N) \in \mathbb{N}^N$ we have

$$\begin{aligned} x_{(k)} - x = (x_{(k)} - x_{(k-e_N)}) + \dots + (x_{(k_1,\dots,k_2,1)} - x_{(k_1,\dots,k_2,0)}) \\ + \dots + (x_{(k_1,0,\dots,0)} - x_{(k_1-1,0,\dots,0)}) + \dots + (x_{(1,0,\dots,0)} - x). \end{aligned}$$

Thus $\Lambda(x) = \Lambda(x_{(k)})$. If x is convergent and $\alpha = \lim_{k' \to \infty,\, k' \in \mathbb{N}^N} x(k')$, then

$$\begin{aligned} |\Lambda(x) - \alpha| &= |\Lambda(x_{(k)} - \alpha \cdot \mathbf{1})| \le \|x_{(k)} - \alpha \cdot \mathbf{1}\|_\infty \\ &\le \sup\{|x(k'_1, \dots, k'_N) - \alpha| : k'_i > k_i \text{ for all } i\}. \end{aligned}$$

Thus $|\Lambda(x) - \alpha|$ is arbitrarily small and we get (b).

Condition (c) and extension to the case of complex-valued sequences can be shown as in the single variable case (see for example [4]). ∎

3. Projection onto Toeplitz operators on the N-torus. Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Set $L^2(\mathbb{T}) = L^2(\mathbb{T}, m)$ and $L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Let $H^2(\mathbb{T})$ be the Hardy space corresponding to $L^2(\mathbb{T})$ and let $P_{H^2(\mathbb{T})}$ be the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. We denote by $H^\infty(\mathbb{T})$ the Hardy space corresponding to $L^\infty(\mathbb{T})$, i.e. the space of those functions from $L^\infty(\mathbb{T})$ which have an analytic extension to the whole unit disc $\mathbb{D}$.

For each $\varphi \in L^\infty(\mathbb{T})$ we define $T_\varphi \in B(H^2(\mathbb{T}))$ by $T_\varphi f = P_{H^2(\mathbb{T})}(\varphi f)$ for $f \in H^2(\mathbb{T})$. The operator T_φ is called a *Toeplitz operator* with symbol φ. Let $\mathcal{T}(\mathbb{T})$ denote the space of all Toeplitz operators, and $\mathcal{A}(\mathbb{T})$ the space of Toeplitz operators with symbols from $H^\infty(\mathbb{T})$. We have ([6, Corollary to Problem 194])

$$\mathcal{T}(\mathbb{T}) = \{A \in B(H^2(\mathbb{T})) : A = T_z^* A T_z\}, \tag{2}$$

and by [6, Problem 116],

$$\mathcal{A}(\mathbb{T}) = \{A \in B(H^2(\mathbb{T})) : A T_z = T_z A\}. \tag{3}$$

Similarly we denote the corresponding spaces on the N-torus, $L^2(\mathbb{T}^N)$, $L^\infty(\mathbb{T}^N)$, $H^2(\mathbb{T}^N)$, $H^\infty(\mathbb{T}^N)$ and the projection $P_{H^2(\mathbb{T}^N)} : L^2(\mathbb{T}^N) \to H^2(\mathbb{T}^N)$. For each $\varphi \in L^\infty(\mathbb{T}^N)$ we define the Toeplitz operator $T_\varphi \in B(H^2(\mathbb{T}^N))$ by $T_\varphi f = P_{H^2(\mathbb{T}^N)}(\varphi f)$. We denote by $\mathcal{T}(\mathbb{T}^N)$ the space of all Toeplitz operators and by $\mathcal{A}(\mathbb{T}^N)$ the space of all Toeplitz operators with symbols from $H^\infty(\mathbb{T}^N)$. By T_{z_i}, $i = 1, \dots, N$, we denote the multiplication operators by the independent variables. Since the operators T_{z_i} commute we can set $T_{z^k} = T_{z_1}^{k_1} \cdots T_{z^N}^{k_N}$ for $k = (k_1, \dots, k_n) \in \mathbb{N}^N$ $(z^k = z_1^{k_1} \cdots z_N^{k_N})$.

Similarly to the one variable case we have the following characterizations (see [9, Proposition 3.3]):

$$\mathcal{T}(\mathbb{T}^N) = \{A \in B(H^2(\mathbb{T}^N)) : A = T_{z_i}^* A T_{z_i},\ i = 1, \ldots, N\}, \tag{4}$$

$$\mathcal{A}(\mathbb{T}^N) = \{A \in B(H^2(\mathbb{T}^N)) : AT_{z_i} = T_{z_i} A,\ i = 1, \ldots, N\}. \tag{5}$$

We will construct a projection onto the space of all Toeplitz operators on the N-torus.

THEOREM 3.1. *There is a positive linear projection* $\pi\colon B(H^2(\mathbb{T}^N)) \to \mathcal{T}(\mathbb{T}^N)$ *such that*

(a) $\pi(I) = I$, $\|\pi\| = 1$,
(b) $\pi(T) = T$ *for* $T \in \mathcal{T}(\mathbb{T}^N)$,
(c) $\pi(AT_\varphi) = \pi(A)T_\varphi$ *for* $A \in B(H^2(\mathbb{T}^N))$ *and* $T_\varphi \in \mathcal{A}(\mathbb{T}^N)$,
(d) $\pi(A)$ *belongs to the weakly-closed convex hull of* $\{T_{z^k}^* A T_{z^k} : k \in \mathbb{N}^N\}$ *for* $A \in B(H^2(\mathbb{T}^N))$,
(e) $\pi(P_k) = 1$, *where* P_k *is the orthogonal projection on the range of* T_{z^k}.

Proof. For $A \in B(H^2(\mathbb{T}^N))$ and $x, y \in H^2(\mathbb{T}^N)$ we define

$$[x, y] = \Lambda(\{(T_{z^k}^*\, A\, T_{z^k}\, x, y)\}_{k \in \mathbb{N}^N}),$$

where Λ denotes the multi-variable Banach limit given in Theorem 2.1. Since $(x, y) \mapsto [x, y]$ is a bounded sesquilinear form, there is an operator $\pi(A) \in B(H^2(\mathbb{T}^N))$ such that

$$(\pi(A)x, y) = \Lambda(\{(T_{z^k}^*\, A\, T_{z^k}\, x, y)\}_{k \in \mathbb{N}^N}). \tag{6}$$

From the definition it is easy to see that $\pi(I) = I$. Note that for any i, by Theorem 2.1(d),

$$\begin{aligned}(T_{z_i}^* \pi(A) T_{z_i} x, y) &= (\pi(A) T_{z_i} x, T_{z_i} y) = \Lambda(\{(T_{z^k}^* A T_{z^k} T_{z_i} x, T_{z_i} y)\}_{k \in \mathbb{N}^N}) \\ &= \Lambda(\{(T_{z^{k+e_i}}^* A T_{z^{k+e_i}} x, y)\}_{k \in \mathbb{N}^N}) \\ &= \Lambda(\{(T_{z^k}^* A T_{z^k} x, y)\}_{k \in \mathbb{N}^N}) = (\pi(A)x, y).\end{aligned}$$

Thus $T_{z_1}^* \pi(A) T_{z_1} = \pi(A)$ and, by the characterization (4) of Toeplitz operators, we see that $\pi(A) \in \mathcal{T}(\mathbb{T}^N)$.

If $A \in \mathcal{T}(\mathbb{T}^N)$ then, by (4), $\{(T_{z^k}^*\, A\, T_{z^k} x, y)\}_{k \in \mathbb{N}^N} = \{(Ax, y)\}_{k \in \mathbb{N}^N}$ is a constant sequence for all x, y and $(\pi(A)x, y) = (Ax, y)$ by Theorem 2.1(b), thus $\pi(A) = A$.

Formula (6) also implies that π is positive. If (d) is not satisfied then for a given operator $A \in B(H^2(\mathbb{T}^N))$ there are $x, y \in H^2(\mathbb{T}^N)$ such that $(\pi(A)x, y) \neq 0$, but $(Bx, y) = 0$ for all B in the weakly-closed convex hull of $\{(T_{z^k}^* A T_{z^k}\, x, y) : k \in \mathbb{N}^N\}$. This contradicts Theorem 2.1(c). The remaining properties follow from formula (6). ∎

4. 2-hyperreflexivity of Toeplitz operators on the N-torus. As before, for a given complex separable Hilbert space $\mathcal{H}$ we denote by $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It is well known that the space of trace class operators τc is predual to $B(\mathcal{H})$ with the dual action $\langle A, f\rangle = \mathrm{tr}(Af)$ for $A \in B(\mathcal{H})$ and $f \in \tau c$. The trace norm in τc will be denoted by $\|\cdot\|_1$. Denote by F_k the set of operators of rank at most k. Every rank-one operator may be written as $x \otimes y$ for some $x, y \in \mathcal{H}$, where $(x \otimes y)z = (z, y)x$ for $z \in \mathcal{H}$. Moreover, $\mathrm{tr}(T(x \otimes y)) = (Tx, y)$.

Let $\mathcal{S} \subset B(\mathcal{H})$ be a norm-closed subspace. We denote by $d(T, \mathcal{S})$ the standard distance from an operator T to the subspace $\mathcal{S}$. It is known that when $\mathcal{S}$ is weak*-closed, $d(T, \mathcal{S}) = \sup\{|\mathrm{tr}(Tf)| : f \in \mathcal{S}_\perp,\ \|f\|_1 \le 1\}$, where $\mathcal{S}_\perp$ denotes the preannihilator of $\mathcal{S}$.

Recall that if $\mathcal{S}$ is a weak*-closed subspace of $B(\mathcal{H})$, then $\mathcal{S}$ is *reflexive* if and only if $\mathcal{S}_\perp$ is a closed linear span of rank-one operators contained in $\mathcal{S}_\perp$ (i.e., $\mathcal{S}_\perp = \mathrm{span}(\mathcal{S}_\perp \cap F_1)$). At the other extreme, if $\mathcal{S}_\perp \cap F_1 = \{0\}$ then we call $\mathcal{S}$ *transitive*. A weak*-closed subspace $\mathcal{S} \subset B(\mathcal{H})$ is called *k-reflexive* if $\mathcal{S}_\perp = \mathrm{span}(\mathcal{S}_\perp \cap F_k)$. We also have a stronger property: $\mathcal{S}$ is called *hyperreflexive* if there is a constant a such that

$$d(T, \mathcal{S}) \le a \sup\{|\langle T, x \otimes y\rangle| : x \otimes y \in \mathcal{S}_\perp,\ \|x \otimes y\|_1 \le 1\} \tag{7}$$

for all $T \in B(\mathcal{H})$, and *k-hyperreflexive* if there is a such that for any $T \in B(\mathcal{H})$,

$$d(T, \mathcal{S}) \le a\ \sup\{|\mathrm{tr}(Tf)| : f \in \mathcal{S}_\perp \cap F_k,\ \|f\|_1 \le 1\}. \tag{8}$$

The distance on the right hand side will be denoted by $\alpha_k(T, \mathcal{S})$. Let $\kappa_k(\mathcal{S})$ be the infimum of the constants a in (8); we call it the *k-hyperreflexivity constant*. For further properties of k-reflexivity and k-hyperreflexivity the reader is referred to [3] and [7].

Analyzing the spaces of all Toeplitz operators on the unit circle $\mathcal{T}(\mathbb{T})$ and on the N-torus $\mathcal{T}(\mathbb{T}^N)$ from the reflexivity point of view, note first that the characterizations (2) and (4) allow us to see that both spaces are weak*-closed. The space $\mathcal{T}(\mathbb{T})$ is transitive, but 2-reflexive (see [2]) and even 2-hyperreflexive (see [7]).

In [9] it was shown that $\mathcal{T}(\mathbb{T}^N)$ is transitive, thus not reflexive, but that it is 2-reflexive. Now we will show the stronger condition: 2-hyperreflexivity.

THEOREM 4.1. *The space of all Toeplitz operators on the torus $\mathcal{T}(\mathbb{T}^N)$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}(\mathbb{T}^N)) \le 2$.*

Proof. Let $A \in B(H^2(\mathbb{T}^N))$. Since $\pi(A)$ belongs to the weakly-closed convex hull of the set $\{T_{z^k}^* A T_{z^k} : k \in \mathbb{N}^N\}$, we have

$$\begin{aligned}d(A,\mathcal{T}(\mathbb{T}^N)) &\le \|A-\pi(A)\| \le \sup_{k\in\mathbb{N}^N}\|A-T_{z^k}^*AT_{z^k}\|\\ &\le \sup_{k\in\mathbb{N}^N}\sup\{|((A-T_{z^k}^*AT_{z^k})x,y)| : x,y\in H^2(\mathbb{T}^N),\ \|x\otimes y\|_1=1\}\\ &\le \sup_{k\in\mathbb{N}^N}\sup\{|(Ax,y)-(A\,z^kx,z^ky)| : x,y\in H^2(\mathbb{T}^N),\ \|x\otimes y\|_1=1\}\\ &\le \sup_{k\in\mathbb{N}^N}\sup\{|\mathrm{tr}(A(x\otimes y-z^kx\otimes z^ky))| : x,y\in H^2(\mathbb{T}^N),\ \|x\otimes y\|_1=1\}.\end{aligned}$$

Since $\mathrm{rank}(x\otimes y-z^kx\otimes z^ky)\le 2$ and $\|x\otimes y-z^kx\otimes z^ky\|_1\le 2$ if $\|x\otimes y\|_1=1$, it follows that $d(A,\mathcal{T}(\mathbb{T}^N))\le 2\,\alpha_2(A,\mathcal{T}(\mathbb{T}^N))$. ∎

5. Projection onto Toeplitz operators on the unit ball. Let $\mathbb{B}_N$ be the unit ball in $\mathbb{C}^N$ and denote by σ the normalized surface measure on the unit sphere $\partial\mathbb{B}_N$. We set $L^2(\partial\mathbb{B}_N)=L^2(\partial\mathbb{B}_N,\sigma)$ and $L^\infty(\partial\mathbb{B}_N)=L^\infty(\partial\mathbb{B}_N,\sigma)$ and denote by $H^2(\partial\mathbb{B}_N)$, $P_{H^2(\partial\mathbb{B}_N)}$ etc. the corresponding spaces and operators on $\partial\mathbb{B}_N$. Also the symbols T_{z_i} and T_{z^k} have the same meaning as before.

In [5] it was shown that

$$\mathcal{T}(\partial\mathbb{B}_N)=\Big\{A\in B(H^2(\partial\mathbb{B}_N)) : A=\sum_{i=1}^N T_{z_i}^*AT_{z_i}\Big\},\tag{9}$$

$$\mathcal{A}(\partial\mathbb{B}_N)=\{A\in B(H^2(\partial\mathbb{B}_N)) : AT_{z_i}=T_{z_i}A,\ i=1,\dots,N\}.\tag{10}$$

For a given operator $A\in B(H^2(\partial\mathbb{B}_N))$ we define by induction a sequence $\{A^{(n)}\}_{n\in\mathbb{N}}$ in $B(H^2(\partial\mathbb{B}_N))$:

$$A^{(0)}=A,\quad A^{(n+1)}=\sum_{i=1}^N T_{z_i}^*A^{(n)}T_{z_i}.\tag{11}$$

Note that $I^{(n)}=I$ and if $T\in\mathcal{T}(\partial\mathbb{B}_N)$, then $T^{(n)}=T$ by (9). Moreover, by (10),

$$(A\,T_\varphi)^{(n)}=A^{(n)}\,T_\varphi\quad\text{for } A\in B(H^2(\partial\mathbb{B}_N)) \text{ and } T_\varphi\in\mathcal{A}(\partial\mathbb{B}_N).\tag{12}$$

LEMMA 5.1. *If* $A\in B(H^2(\partial\mathbb{B}_N))$, *then* $\|A^{(n)}\|\le 2\|A\|$.

Proof. For $k=(k_1,\dots,k_N)\in\mathbb{N}^N$ we write $k!=k_1!\cdots k_N!$ and $|k|=k_1+\dots+k_N$. One can easily note that

$$A^{(n)}=\sum_{|k|=n}\frac{n!}{k!}\,T_{z^k}^*AT_{z^k}.\tag{13}$$

If $x\in H^2(\partial\mathbb{B}_N)$, then

$$|(A^{(n)}x,x)|\le\sum_{|k|=n}\frac{n!}{k!}\,|(Az^kx,z^kx)|\le\|A\|\sum_{|k|=n}\frac{n!}{k!}\,\|z^kx\|^2$$

$$= \|A\| \sum_{|k|=n} \frac{n!}{k!} \int_{\partial\mathbb{B}_N} |z^k x(z)|^2 \, d\sigma(z) = \|A\| \int_{\partial\mathbb{B}_N} |x(z)|^2 \sum_{|k|=n} \frac{n!}{k!} |z^k|^2 \, d\sigma(z)$$

$$= \|A\| \int_{\partial\mathbb{B}_N} |x(z)|^2 (|z_1|^2 + \cdots + |z_n|^2)^n \, d\sigma(z) = \|A\| \, \|x\|^2.$$

Thus the numerical range satisfies $w(A^{(n)}) \le \|A\|$ and $\|A^{(n)}\| \le 2\|A\|$ by [6]. ∎

THEOREM 5.2. *There is a positive linear projection $\pi\colon B(H^2(\partial\mathbb{B}_N)) \to \mathcal{T}(\partial\mathbb{B}_N)$ such that*

(a) $\pi(I) = I$, $\|\pi\| \le 2$,
(b) $\pi(T) = T$ *for* $T \in \mathcal{T}(\partial\mathbb{B}_N)$,
(c) $\pi(AT_\varphi) = \pi(A)T_\varphi$ *for* $A \in B(H^2(\partial\mathbb{B}_N))$ *and* $T_\varphi \in \mathcal{A}(\partial\mathbb{B}_N)$,
(d) $\pi(A)$ *belongs to the weakly-closed convex hull of* $\{A^{(n)} : n \in \mathbb{N}\}$ *for* $A \in B(H^2(\partial\mathbb{B}_N))$.

Proof. For $A \in B(H^2(\partial\mathbb{B}_N))$ and $x, y \in H^2(\partial\mathbb{B}_N)$ we define

$$[x, y] = \Lambda(\{(A^{(n)}x, y)\}_{n\in\mathbb{N}}),$$

where Λ denotes the one-dimensional Banach limit (see Theorem 2.1). Note that $\{(A^{(n)}x, y)\}_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathbb{C})$ by Lemma 5.1. Since $(x, y) \mapsto [x, y]$ is a bounded sesquilinear form, there is an operator $\pi(A) \in B(H^2(\partial\mathbb{B}_N))$ such that

$$(\pi(A)x, y) = \Lambda(\{(A^{(n)}x, y)\}_{n\in\mathbb{N}}). \tag{14}$$

Since $I^{(n)} = I$, we have $\pi(I) = I$.

Now, by Theorem 2.1(d), we get

$$\Big(\sum_{i=1}^N T_{z_i}^* \pi(A) T_{z_i} x, y\Big) = \sum_{i=1}^N (\pi(A) z_i x, z_i y)$$
$$= \sum_{i=1}^N \Lambda(\{(A^{(n)} z_i x, z_i y)\}_{n\in\mathbb{N}}) = \Lambda\Big(\Big\{\Big(\sum_{i=1}^N T_{z_i}^* A^{(n)} T_{z_i} x, y\Big)\Big\}_{n\in\mathbb{N}}\Big)$$
$$= \Lambda\big(\{(A^{(n+1)}x, y)\}_{n\in\mathbb{N}}\big) = (\pi(A)x, y).$$

Thus $\pi(A) \in \mathcal{T}(\partial\mathbb{B}_N)$ by (9).

If $A \in \mathcal{T}(\partial\mathbb{B}_N)$ then $A^{(n)} = A$ for all n, and thus $\pi(A) = A$. Property (c) is a consequence of (12), and the proof of (d) is similar to that of Theorem 3.1(d). ∎

6. Projection onto generalized Toeplitz operators. The idea of generalized Toeplitz operators is to replace in the characterization (2) the backward shift T_z^* by any contraction. Precisely, for given contractions S, T

in $B(\mathcal{H})$, an operator $X \in B(\mathcal{H})$ is called a *generalized Toeplitz operator with respect to* S *and* T if $X = SXT^*$. These operators were investigated in [11]. The space of all such operators is denoted by $\mathcal{T}(S,T)$. Note that this definition implies weak*-closedness of $\mathcal{T}(S,T)$.

In [10] this idea was extended to two variables. It is easy to extend it to the multi-variable case. Having in mind the characterization (4) of Toeplitz operators on the N-torus we make the following definition. For given N-tuples $\mathbf{S} = (S_1, \ldots, S_N)$ and $\mathbf{T} = (T_1, \ldots, T_N)$ of commuting contractions on $\mathcal{H}$, an operator $X \in B(\mathcal{H})$ is called a *generalized Toeplitz operator with respect to* $\mathbf{S}$ *and* $\mathbf{T}$ if $X = S_i X T_i^*$ for $i = 1, \ldots, N$. The space of all such operators is denoted by $\mathcal{T}(\mathbf{S}, \mathbf{T})$. It is also weak*-closed. For a given commuting N-tuple $\mathbf{S} = (S_1, \ldots, S_N)$ we set $\mathbf{S}^k = S_1^{k_1} \cdots S_N^{k_N}$ for $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$.

Now we extend the definition of the projection considered in Section 3 to generalized Toeplitz operators. We formulate the theorem for arbitrary N, but even the case $N = 1$ is worth noting.

Theorem 6.1. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be two* N*-tuples of commuting contractions on* $\mathcal{H}$. *There is a linear projection* $\pi\colon B(\mathcal{H}) \to \mathcal{T}(\mathbf{S}, \mathbf{T})$ *such that*

(a) $\|\pi\| \le 1$,
(b) $\pi(X) = X$ *for* $X \in \mathcal{T}(\mathbf{S}, \mathbf{T})$,
(c) *if* $A \in B(\mathcal{H})$ *then* $\pi(A)$ *belongs to the weakly-closed convex hull of* $\{\mathbf{S}^k A \mathbf{T}^{*k} : k \in \mathbb{N}^N\}$.

Proof. Let Λ be the functional from Theorem 2.1. For $A \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$ we define

$$(\pi(A)x, y) = \Lambda(\{(\mathbf{S}^k A \mathbf{T}^{*k} x, y)\}_{k \in \mathbb{N}^N}).$$

To check the details, one can follow the proof of Theorem 3.1. ∎

7. 2-hyperreflexivity of generalized Toeplitz operators. The reflexive behavior of the space $\mathcal{T}(S,T)$ of generalized Toeplitz operators depends on the contractions S, T. For example if the underlying Hilbert space is the Hardy space on the unit circle and $S = T = T_z^*$ then $\mathcal{T}(T_z^*, T_z^*) = \mathcal{T}(\mathbb{T})$ is transitive. On the other hand, the space $\mathcal{T}(S,T)$ might be even (hyper)reflexive. For example, if $S = T = I_H$ then $\mathcal{T}(I_H, I_H) = B(\mathcal{H})$, which is (hyper)reflexive. However, we can estimate the reflexive behavior even for arbitrary N by

Theorem 7.1. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be two* N*-tuples of commuting contractions on* $\mathcal{H}$. *Then* $\mathcal{T}(\mathbf{S}, \mathbf{T})$ *is* 2*-hyperreflexive.*

Proof. By Theorem 6.1(c), for any $A \in B(\mathcal{H})$, $\pi(A)$ belongs to the weakly-closed convex hull of $\{\mathbf{S}^k A \mathbf{T}^{*k} : k \in \mathbb{N}^N\}$. As in the proof of Theorem

4.1 we can show that

$$d(A,\mathcal{T}(\mathbf{S},\mathbf{T})) \le \|A-\pi(A)\| \le \sup_{k\in\mathbb{N}^N} \|A-\mathbf{S}^k A\mathbf{T}^{*k}\|$$
$$\le \sup_{k\in\mathbb{N}^k} \sup\{|\mathrm{tr}(A(x\otimes y-\mathbf{T}^{*k}x\otimes \mathbf{S}^{*k}y))| : \|x\otimes y\|_1 = 1\}.$$

Since $\mathrm{rank}(x\otimes y-\mathbf{T}^{*k}x\otimes \mathbf{S}^{*k}y) \le 2$ and $\|x\otimes y-\mathbf{T}^{*k}x\otimes \mathbf{S}^{*k}y\|_1 \le 2$ for $\|x\otimes y\|_1 = 1$, we have

$$d(A,\mathcal{T}(\mathbf{S},\mathbf{T})) \le 2\alpha_2(A,\mathcal{T}(\mathbf{S},\mathbf{T})). \ \blacksquare$$

Theorem 7.1 for $N = 1$ is also a consequence of [8].

Added in proof. D. Timotin (private communication) has shown that the norm of the projection in Theorem 5.2 is equal to 1, $\|\pi\| = 1$.

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