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The Siciak–Zahariuta extremal function as the envelope of disc functionals

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Abstract. We establish disc formulas for the Siciak–Zahariuta extremal function of an arbitrary open subset of complex affine space. This function is also known as the pluricomplex Green function with logarithmic growth or a logarithmic pole at infinity. We extend Lempert's formula for this function from the convex case to the connected case.

Introduction. The *Siciak–Zahariuta extremal function* V_X of a subset X of complex affine space $\mathbb{C}^n$ is defined as the supremum of all entire plurisubharmonic functions u of minimal growth with $u|X \leq 0$. It is also called the *pluricomplex Green function* of X *with logarithmic growth* or *with a logarithmic pole at infinity* (although this is a bit of a misnomer if X is not bounded). A plurisubharmonic function u on $\mathbb{C}^n$ is said to have *minimal growth* (and belong to the class $\mathcal{L}$) if $u - \log^+ \|\cdot\|$ is bounded above on $\mathbb{C}^n$. If X is open and nonempty, then $V_X \in \mathcal{L}$. More generally, if X is not pluripolar, then the upper semicontinuous regularization V_X^* of V_X is in $\mathcal{L}$, and if X is pluripolar, then $V_X^* = \infty$. Siciak–Zahariuta extremal functions play a fundamental role in pluripotential theory and have found important applications in approximation theory, complex dynamics, and elsewhere. For a detailed account of the basic theory, see [K, Chapter 5]. For an overview of some recent developments, see [Pl].

The extremal functions of pluripotential theory are usually defined as suprema of classes of plurisubharmonic functions with appropriate properties. The theory of disc functionals, initiated by Poletsky in the late 1980s [P1, PS], offers a different approach to extremal functions, realizing them as envelopes of disc functionals. A *disc functional* on a complex manifold Y

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is a map H into $\overline{\mathbb{R}} = [-\infty, \infty]$ from the set of analytic discs in Y, that is, holomorphic maps from the open unit disc $\mathbb{D}$ into Y. We usually restrict ourselves to analytic discs that extend holomorphically to a neighbourhood of the closed unit disc. The *envelope* EH of H is the map $Y \to \overline{\mathbb{R}}$ that takes a point $x \in Y$ to the infimum of the values $H(f)$ for all analytic discs f in Y with $f(0) = x$. Disc formulas have been proved for such extremal functions as largest plurisubharmonic minorants, including relative extremal functions, and pluricomplex Green functions of various sorts, and used to establish properties of these functions that had proved difficult to handle via the supremum definition. Some of this work has been devoted to extending to arbitrary complex manifolds results that were first proved for domains in $\mathbb{C}^n$. See for instance [BS, E, EP, LLS, LS1, LS2, P2, P3, R, RS].

In the convex case, there is a disc formula for the Siciak–Zahariuta extremal function due to Lempert [M, Appendix]. The main motivation for the present work was to generalize Lempert's formula. Because of the growth condition in the definition of the Siciak–Zahariuta extremal function, we did not see how to fit it into the theory of disc functionals until we realized, from a remark of Guedj and Zeriahi [GZ], that minimal growth is nothing other than quasi-plurisubharmonicity with respect to the current of integration along the hyperplane at infinity. This observation is implicit in the proof of Theorem 1, which presents a family of new disc formulas for the Siciak–Zahariuta extremal function of an arbitrary open subset of affine space. Theorem 2 contains more such formulas. Our main result, Theorem 3, establishes Lempert's formula, in the following slightly modified form, for every connected open subset of affine space. The formula is easily seen to fail for disconnected sets in general.

Theorem. *The Siciak–Zahariuta extremal function V_X of a connected open subset X of $\mathbb{C}^n$ is given by the disc formula*

$$V_X(z) = -\sup_f \sum_{f(\zeta)\in H_\infty} \log|\zeta|, \qquad z \in \mathbb{C}^n,$$

where f runs through all analytic discs in $\mathbb{P}^n$ with $f(\mathbb{T}) \subset X$ and $f(0) = z$.

Here, $\mathbb{T}$ denotes the unit circle and H_∞ denotes the hyperplane at infinity in complex projective space $\mathbb{P}^n$.

Let us summarize our approach. We let X be an open subset of $\mathbb{C}^n$ and seek a disc formula for V_X. If we have a good upper semicontinuous majorant for V_X on $\mathbb{C}^n$, so good that V_X is its largest plurisubharmonic minorant, then we have a disc formula for V_X as the so-called Poisson envelope of the majorant. If B is a ball in X, say the unit ball, then such a majorant is easily seen to be given as zero on X and $V_B = \log\|\cdot\|$ outside X. The first main idea is to introduce certain good sets of analytic discs in $\mathbb{P}^n$, adapted to X, and get many more good majorants for V_X as the envelopes of a new disc functional (called J below) over such sets. The second main idea is that the disc formulas for V_X thus obtained are in fact closely related to Lempert's formula in the convex case, even though they look quite different at first sight. The relationship appears when we modify the Poisson functional by adding to it the nonnegative functional J, balancing this by taking the envelope over the larger class of all analytic discs in $\mathbb{P}^n$. We show that the envelope is still V_X. If we restrict to analytic discs in $\mathbb{P}^n$ with boundary in X, the Poisson term disappears and J alone remains. This is essentially Lempert's formula, so the envelope is still V_X if X is convex. A proof of Lempert's formula as stated above, assuming only that X is connected, concludes the paper. The proof relies on a judicious choice of a good set of analytic discs, as well as a fundamental argument in the theory of disc functionals, Poletsky's proof of the plurisubharmonicity of the Poisson envelope, adapted here to a somewhat different purpose.

Good sets of analytic discs and the first disc formula. If Y is a complex manifold, then we denote by $\mathcal{A}_Y$ the set of *analytic discs* in Y, that is, the set of maps $\overline{\mathbb{D}} \to Y$ that extend holomorphically to a neighbourhood of $\overline{\mathbb{D}}$. If $H : \mathcal{A}_Y \to \overline{\mathbb{R}}$ is a disc functional on Y and $\mathcal{B} \subset \mathcal{A}_Y$, then the *envelope of H with respect to $\mathcal{B}$* is the function $E_{\mathcal{B}}H : Y \to \overline{\mathbb{R}}$ with

$$E_{\mathcal{B}}H(y) = \inf\{H(f) : f \in \mathcal{B},\ f(0) = y\}, \qquad y \in Y.$$

We usually write EH for $E_{\mathcal{A}_Y}H$ and simply call it the *envelope of H*.

Perhaps the most important example of a disc functional is the Poisson functional $f \mapsto \int_{\mathbb{T}} \varphi \circ f \, d\sigma$ associated to an upper semicontinuous function $\varphi : Y \to [-\infty, \infty)$ (here, σ is the normalized arc length measure on the unit circle $\mathbb{T}$). Its envelope is the largest plurisubharmonic minorant of φ on Y. This was first proved for domains in affine space by Poletsky [P1], and later, with a different proof, by Bu and Schachermayer [BS], and finally generalized to all complex manifolds by Rosay [R].

We view $\mathbb{C}^n$ as the subset of $\mathbb{P}^n$ with projective coordinates $[z_0 : \cdots : z_n]$ where $z_0 \neq 0$ and write H_∞ for the hyperplane at infinity where $z_0 = 0$. We define a disc functional J on $\mathbb{P}^n$ by the formula

$$J(f) = -\sum_{\zeta \in f^{-1}(H_\infty)} m_{f_0}(\zeta) \log|\zeta| \geq 0, \qquad f \in \mathcal{A}_{\mathbb{P}^n}.$$

Here, $m_{f_0}(\zeta)$ denotes the multiplicity of the intersection of f with H_∞, that is, the order of the zero of the component f_0 at ζ when f is expressed as $[f_0 : \cdots : f_n]$ in projective coordinates. When the zeros of f_0 are not isolated, that is, $f(\mathbb{D}) \subset H_\infty$, we set $J(f) = \infty$, and when $f(\mathbb{D}) \cap H_\infty = \emptyset$, we set $J(f) = 0$.

To indicate the relevance of J to the Siciak–Zahariuta extremal function, let $X \subset \mathbb{C}^n$ be open and $f \in \mathcal{A}_{\mathbb{P}^n}$ have $f(\mathbb{T}) \subset X$. For simplicity, we assume that f sends only one point $\zeta \in \mathbb{D}$ to H_∞ and $m_{f_0}(\zeta) = 1$. Let ρ be the reciprocal and τ be an automorphism of $\mathbb{D}$ interchanging 0 and ζ. Then $g = f \circ \tau \circ \rho : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C}^n$ is holomorphic with a simple pole at infinity and $g(\mathbb{T}) \subset X$. Hence, $V_X \circ g$, extended as zero across $\mathbb{D}$, is a subharmonic function on $\mathbb{C}$ of minimal growth, so $V_X \circ g \leq V_{\mathbb{D}} = \log|\cdot|$ and

$$V_X(f(0)) = V_X(g(1/\zeta)) \leq -\log|\zeta| = J(f).$$

A subset $\mathcal{B}$ of $\mathcal{A}_{\mathbb{P}^n}$ is called *good* with respect to an open subset X of $\mathbb{C}^n$ if:

(1) $f(\mathbb{T}) \subset X$ for every $f \in \mathcal{B}$,
(2) for every $z \in \mathbb{C}^n$, there is a disc in $\mathcal{B}$ with centre z,
(3) for every $x \in X$, the constant disc at x is in $\mathcal{B}$, and
(4) the envelope $E_{\mathcal{B}}J$ is upper semicontinuous on $\mathbb{C}^n$ and has minimal growth, that is, $E_{\mathcal{B}}J - \log^+\|\cdot\|$ is bounded above on $\mathbb{C}^n$.

Note that by (2), $0 \leq E_{\mathcal{B}}J < \infty$ on $\mathbb{C}^n$, by (3), $E_{\mathcal{B}}J = 0$ on X, and clearly $E_{\mathcal{B}}J = \infty$ on H_∞. Property (4) may be hard to verify directly, but Proposition 2 gives a useful sufficient condition for it to hold. Roughly speaking, if $\mathcal{B}$ contains a disc centred at each point of $\mathbb{P}^n$, then (4) holds if (but not only if) discs in $\mathcal{B}$ can be varied continuously.

THEOREM 1. *Let X be an open subset of $\mathbb{C}^n$ and $\mathcal{B}$ be a good set of analytic discs in $\mathbb{P}^n$ with respect to X. Then the Siciak–Zahariuta extremal function V_X of X is the envelope of the disc functional $H_{\mathcal{B}}$ on $\mathbb{P}^n$ defined by the formula*

$$H_{\mathcal{B}}(f) = J(f) + \int_{\mathbb{T}\setminus f^{-1}(X)} E_{\mathcal{B}}J \circ f \, d\sigma, \qquad f \in \mathcal{A}_{\mathbb{P}^n}.$$

REMARKS. 1. We define $V_X = \infty$ on H_∞ and it is clear that $EH_{\mathcal{B}} = \infty$ on H_∞. Since $E_{\mathcal{B}}J = 0$ on X, the integral above might as well be taken over all of $\mathbb{T}$. The disc functional $H_{\mathcal{B}}$ is thus given as the Poisson functional of $E_{\mathcal{B}}J$ minus the Lelong-like functional $-J$ (see [LS2] for the definition of the Lelong functional). Envelopes of disc functionals associated to complex subspaces in the way that $-J$ is associated to H_∞ are Green functions of a type studied in [RS].

2. Using Proposition 2, it is easy to see that the largest subset $\mathcal{B} \subset \mathcal{A}_{\mathbb{P}^n}$ that is good with respect to X is the set $\mathcal{A}^X_{\mathbb{P}^n}$ of all $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset X$. This yields the smallest possible $E_{\mathcal{B}}J$ in the second term of the formula for $H_{\mathcal{B}}$. We write H_X for $H_{\mathcal{A}^X_{\mathbb{P}^n}}$, so H_X is the smallest disc functional $H_{\mathcal{B}}$ where $\mathcal{B} \subset \mathcal{A}_{\mathbb{P}^n}$ is good with respect to X. Other choices of $\mathcal{B}$ make the second term explicitly computable and yield information about almost extremal

discs (see Propositions 4 and 5). Theorem 2 shows that V_X is in fact the envelope of $H_{\mathcal{B}}$ over analytic discs in $\mathbb{C}^n$ only; for such discs, J vanishes.

3. By a result of Lempert [M, Appendix], if X is convex, then V_X is the envelope of $H_{\mathcal{B}}$ over analytic discs in $\mathbb{P}^n$ with boundary in X and at most one simple pole; for such discs, the second term vanishes, leaving only J. We discuss this in detail later in the paper. For disconnected X, it is generally not true that $V_X = E_{\mathcal{A}^X_{\mathbb{P}^n}} J$. For example, suppose X is the disjoint union of two nonempty convex open sets Y and Z. Then

$$E_{\mathcal{A}^X_{\mathbb{P}^n}} J = \min\{E_{\mathcal{A}^Y_{\mathbb{P}^n}} J, E_{\mathcal{A}^Z_{\mathbb{P}^n}} J\} = \min\{V_Y, V_Z\}$$

is not even plurisubharmonic in general (but it does provide an upper bound for V_X).

Proof of Theorem 1. Let $\pi : Z = \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection. Write $Z_0 = \pi^{-1}(H_\infty) = \{z \in Z : z_0 = 0\}$. The advantage of working on Z rather than on $\mathbb{P}^n$ is that the pullback of the current of integration along H_∞ has a global plurisubharmonic potential $\varphi(z) = \log|z_0|$ on Z. Note that if $x \in \mathbb{P}^n$ and $z \in Z$ with $\pi(z) = x$, then every analytic disc in $\mathbb{P}^n$ centred at x lifts to an analytic disc in Z centred at z. Hence, as f runs through all analytic discs in Z with $f(0) = z$, $\pi \circ f$ runs through all analytic discs g in $\mathbb{P}^n$ with $g(0) = x$.

Let $\pi^*\mathcal{B}$ be the set of analytic discs f in Z with $\pi \circ f \in \mathcal{B}$. Define a function ψ on $Z \setminus Z_0$ by the formula

$$\psi(z) = \inf\Big\{\int_{\mathbb{T}} \varphi \circ f\, d\sigma : f \in \pi^*\mathcal{B},\ f(0) = z\Big\}, \qquad z \in Z \setminus Z_0.$$

By the defining property (2) of a good set of analytic discs and plurisubharmonicity of φ, we have $\varphi \le \psi < \infty$ on $Z \setminus Z_0$, and by property (3), $\psi = \varphi$ on $\pi^{-1}(X)$.

If $f \in \pi^*\mathcal{B}$ and $f(0) \notin Z_0$, then the Riesz Representation Theorem applied to the subharmonic function $\varphi \circ f = \log|f_0|$ gives

$$\varphi(f(0)) = \int_{\mathbb{T}} \varphi \circ f\, d\sigma + \frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\, \Delta(\varphi \circ f).$$

Also,

$$\frac{1}{2\pi}\int_{\mathbb{D}} \log|\cdot|\, \Delta(\varphi \circ f) = \sum_{\zeta \in f_0^{-1}(0)} m_{f_0}(\zeta) \log|\zeta| = -J(\pi \circ f),$$

so

$$\int_{\mathbb{T}} \varphi \circ f\, d\sigma = \varphi(f(0)) + J(\pi \circ f),$$

and

$$\psi = \varphi + E_{\mathcal{B}} J \circ \pi \qquad \text{on } Z \setminus Z_0.$$

By property (4), $E_{\mathcal{B}}J$ is upper semicontinuous on $\mathbb{C}^n$, so $\psi : Z \setminus Z_0 \to \mathbb{R}$ is upper semicontinuous. The minimal growth condition on $E_{\mathcal{B}}J$ means that $E_{\mathcal{B}}J \circ \pi + \varphi = \psi$ is locally bounded above at Z_0. Hence, the upper semicontinuous extension $\psi^* : Z \to [-\infty, \infty)$, which we shall simply call ψ, is well defined, and we have $\varphi \le \psi$ on Z and $\psi = \varphi$ on $\pi^{-1}(X)$. The key property of ψ is that if u is a plurisubharmonic function on Z and $u \le \varphi$ on $\pi^{-1}(X)$, then $u \le \psi$ on $Z \setminus Z_0$ by property (1), so $u = (u|Z \setminus Z_0)^* \le \psi$ on all of Z. The converse is clear since $\psi = \varphi$ on $\pi^{-1}(X)$.

Now $u \in \mathcal{L}$ if and only if $u \circ \pi + \varphi$ is plurisubharmonic on Z. Namely, the minimal growth condition that defines $\mathcal{L}$ means that $u \circ \pi + \varphi$, which is plurisubharmonic on $Z \setminus Z_0$, is locally bounded above at Z_0, which in turn means that $u \circ \pi + \varphi$ extends uniquely to a plurisubharmonic function on all of Z.

Hence, $u \in \mathcal{L}$ and $u \le 0$ on X if and only if $u \circ \pi + \varphi$ is plurisubharmonic on Z and $u \circ \pi + \varphi \le \varphi$ on $\pi^{-1}(X)$, that is, $u \circ \pi + \varphi \le \psi$ on Z. Thus, clearly, $V_X \circ \pi + \varphi \le \psi$. Also, since $\psi - \varphi = E_{\mathcal{B}}J \circ \pi$ on $Z \setminus Z_0$ is invariant under homotheties, so is its largest plurisubharmonic minorant $P_{Z \setminus Z_0}(\psi - \varphi)$ on $Z \setminus Z_0$. Since φ is pluriharmonic on $Z \setminus Z_0$,

$$P_{Z \setminus Z_0}(\psi - \varphi) = P_{Z \setminus Z_0}(\psi) - \varphi,$$

so $P_{Z \setminus Z_0}(\psi) = u \circ \pi + \varphi$, where $u \in \mathcal{L}$ and $u \le 0$ on X. Therefore, $u \le V_X$ and $P_Z(\psi) \le P_{Z \setminus Z_0}(\psi) \le V_X \circ \pi + \varphi$ on $Z \setminus Z_0$, so $P_Z(\psi) \le V_X \circ \pi + \varphi$ on Z.

This shows that $V_X \circ \pi + \varphi$ is the largest plurisubharmonic minorant of ψ on Z, so Poletsky's theorem yields a disc formula for $V_X \circ \pi + \varphi$ as the Poisson envelope of ψ on Z. For $z \in Z \setminus Z_0$, it follows that $V_X(\pi(z))$ is the infimum over all analytic discs f in Z with $f(0) = z$ of the numbers

$$\int_{\mathbb{T}} \psi \circ f \, d\sigma - \varphi(z).$$

By the Riesz Representation Theorem,

$$\begin{aligned} \int_{\mathbb{T}} \psi \circ f \, d\sigma - \varphi(z) &= \int_{\mathbb{T}} (\psi - \varphi) \circ f \, d\sigma + J(\pi \circ f) \\ &= J(\pi \circ f) + \int_{\mathbb{T} \setminus f^{-1}(\pi^{-1}(X))} E_{\mathcal{B}}J \circ \pi \circ f \, d\sigma. \end{aligned}$$

Note that $f^{-1}(Z_0) \cap \mathbb{T}$ is finite, so the second and third integrals are equal. This shows that $V_X(x) = EH_{\mathcal{B}}(x)$ for all $x \in \mathbb{C}^n$. For $x \in H_\infty$, this is obvious, as mentioned in Remark 1 above. ∎

The multiplicity factor in the definition of J may be omitted without affecting Theorem 1 with $\mathcal{B} = \mathcal{A}^X_{\mathbb{P}^n}$, that is, without changing $E_{\mathcal{A}^X_{\mathbb{P}^n}}J$ or EH_X.

PROPOSITION 1. *Let $X \subset \mathbb{C}^n$ be open and $f \in \mathcal{A}_{\mathbb{P}^n}$ have $f(0) \notin H_\infty$. Then there is $g \in \mathcal{A}_{\mathbb{P}^n}$ with $g(0) = f(0)$, $m_{g_0} = 1$ on $g^{-1}(H_\infty)$, and $J(f) = J(g)$, such that g is uniformly as close to f on $\overline{\mathbb{D}}$ as we wish, so in particular, if $f(\mathbb{T}) \subset X$, then $g(\mathbb{T}) \subset X$.*

Proof. Now f intersects H_∞ in finitely many points $a_1, \ldots, a_k \in \mathbb{D} \setminus \{0\}$ with multiplicities $m_j = m_{f_0}(a_j)$. Let $\widetilde{f} \in \mathcal{A}_Z$ be a lifting of f. By exactly the same argument as in the proof of Lemma 3.1 in [LS2], taking the function α there to be the characteristic function of Z_0 in Z, we obtain $\widetilde{g} \in \mathcal{A}_Z$ arbitrarily uniformly close to $\widetilde{f}$ on $\overline{\mathbb{D}}$ such that $\widetilde{g}(0) = \widetilde{f}(0)$, the zeros $c_1, \ldots, c_m$ of $\widetilde{g}_0$ in $\mathbb{D}$ all have multiplicity 1, their number m equals $m_1 + \cdots + m_k$, and

$$\sum_{j=1}^{m} \log |c_j| = \sum_{j=1}^{k} m_j \log |a_j|.$$

Finally, take $g = \pi \circ \widetilde{g}$. ∎

Further results on good sets of analytic discs. Using the proof of Theorem 1, we now present a sufficient condition for a set of analytic discs to be good.

PROPOSITION 2. *Let X be an open subset of $\mathbb{C}^n$ and $\mathcal{B}$ be a subset of $\mathcal{A}_{\mathbb{P}^n}$ with the following two properties:*

- (2′) *For every $z \in \mathbb{P}^n$, there is a disc in $\mathcal{B}$ with centre z.*
- (4′) *Discs in $\mathcal{B}$ can be varied continuously, that is, for every $f \in \mathcal{B}$ there is a map from a neighbourhood U of $f(0)$ into $\mathcal{B}$, continuous as a map $\overline{\mathbb{D}} \times U \to \mathbb{P}^n$, taking each $x \in U$ to a disc centred at x and taking $f(0)$ to f.*

Then $\mathcal{B}$ has property (4) *in the definition of a good set of analytic discs.*

Hence, if $\mathcal{B}$ satisfies (1), (2′), (3), *and* (4′), *then $\mathcal{B}$ is good with respect to X.*

Proof. Define

$$\psi(z) = \inf\Big\{ \int_{\mathbb{T}} \varphi \circ f \, d\sigma : f \in \pi^*\mathcal{B},\ f(0) = z \Big\}, \qquad z \in Z \setminus Z_0,$$

as in the proof of Theorem 1. Properties (2) and (4′) imply $\psi : Z \setminus Z_0 \to \mathbb{R}$ is upper semicontinuous. Now $E_{\mathcal{B}} J \circ \pi = \psi - \varphi$ on $Z \setminus Z_0$, so $E_{\mathcal{B}} J$ is upper semicontinuous on $\mathbb{C}^n$. Moreover, $E_{\mathcal{B}} J$ has minimal growth since ψ is locally bounded above at Z_0 by (2′) and (4′). ∎

The next result provides an interesting class of examples of good sets of analytic discs.

PROPOSITION 3. *Let X be a connected open subset of $\mathbb{C}^n$ and let β be a free homotopy class of loops in X, that is, of continuous maps $\mathbb{T} \to X$.*

Let $\mathcal{B}$ be the set of analytic discs f in $\mathbb{P}^n$ such that $f(\mathbb{T}) \subset X$ and $f|\mathbb{T} \in \beta$. Then $\mathcal{B}$ has properties (1), (2′), *and* (4′). *If β is the trivial class*, *then $\mathcal{B}$ also satisfies* (3), *so $\mathcal{B}$ is good with respect to X.*

Proof. Only (2′) is not obvious. Let $z \in \mathbb{C}^n$ and a continuous map $\alpha : \mathbb{T} \to X$ be a representative for β. Rational functions on $\mathbb{C}$ whose poles lie outside $\mathbb{T} \cup \{0\}$ are uniformly dense among continuous functions $\mathbb{T} \cup \{0\} \to \mathbb{C}$ (see e.g. [AW, Theorem 2.8]). Therefore, for each $\varepsilon > 0$, we obtain rational functions $f_1, \ldots, f_n$ without poles on $\mathbb{T} \cup \{0\}$, defining an analytic disc $f = (f_1, \ldots, f_n)$ in $\mathbb{P}^n$, such that $f(0) = z$ and $f|\mathbb{T}$ is within ε of α, so $f|\mathbb{T}$ is freely homotopic to α in X if ε is small enough. If $z \in H_\infty$, we reduce to the previous case by moving z into $\mathbb{C}^n$ by an automorphism of $\mathbb{P}^n$ close to the identity. ∎

Majorants for the Siciak–Zahariuta function and the second disc formula. Let X be an open subset of $\mathbb{C}^n$ and $\mathcal{B}$ be a good set of analytic discs in $\mathbb{P}^n$ with respect to X. By Theorem 1, $V_X = EH_{\mathcal{B}}$. Clearly, $EH_{\mathcal{B}} \le E_{\mathcal{B}} H_{\mathcal{B}} = E_{\mathcal{B}} J$, so $V_X \le E_{\mathcal{B}} J$. Moreover, if u is a plurisubharmonic function on $\mathbb{C}^n$ with $u \le E_{\mathcal{B}} J$, then $u \le 0$ on X by property (3) in the definition of a good set of analytic discs, and u has minimal growth by property (4), so $u \le V_X$. This shows that V_X is the largest plurisubharmonic minorant, and hence the Poisson envelope, of $E_{\mathcal{B}} J$ on $\mathbb{C}^n$. It follows that $E_{\mathcal{B}} J$ is plurisubharmonic if and only if $E_{\mathcal{B}} J = V_X$. We have proved the following result.

Theorem 2. *Let X be an open subset of $\mathbb{C}^n$ and $\mathcal{B}$ be a good set of analytic discs in $\mathbb{P}^n$ with respect to X. Then V_X is the largest plurisubharmonic minorant of $E_{\mathcal{B}} J$ on $\mathbb{C}^n$. Consequently*, *for every $z \in \mathbb{C}^n$*,

$$V_X(z) = \inf \int_{\mathbb{T} \setminus f^{-1}(X)} E_{\mathcal{B}} J \circ f \, d\sigma,$$

where the infimum is taken over all analytic discs f in $\mathbb{C}^n$ with $f(0) = z$.

The third disc formula and almost extremal discs. Let X be an open subset of $\mathbb{C}^n$. The simple disc formula for V_X mentioned in the introduction is in fact a special case of Theorem 2. Namely, suppose B is a closed ball with centre a and radius $r > 0$ contained in X. As is well known, $V_B = \log \| \cdot - a \| - \log r$ outside B. Setting $w = V_B$ outside X and $w = 0$ on X, we obtain an upper semicontinuous majorant $w : \mathbb{C}^n \to [0, \infty)$ for V_X. It is easily seen that V_X is the largest plurisubharmonic minorant and hence the Poisson envelope of w. Let us record this fact.

Proposition 4. *Let X be an open subset of $\mathbb{C}^n$ containing the closed ball with centre a and radius $r > 0$. Then V_X is the envelope of the disc* *functional H_r on $\mathbb{C}^n$ defined by the formula*

$$H_r(f) = \int\limits_{\mathbb{T}\setminus f^{-1}(X)} \log\|f-a\|\,d\sigma - \sigma(\mathbb{T}\setminus f^{-1}(X))\log r, \qquad f\in\mathcal{A}_{\mathbb{C}^n}.$$

For simplicity, let us assume that a is the origin. Let $\mathcal{B}$ contain the constant analytic discs in X as well as the analytic discs g_z in $\mathbb{P}^n$ with

$$g_z(\zeta) = \frac{\|z\|+r\zeta}{r+\|z\|\zeta}\,\frac{r}{\|z\|}\,z$$

for each $z\in\mathbb{C}^n\setminus X$. Note that g_z is centred at z, lies in the projective line through z and the origin, and has its boundary on the sphere of radius r centred at the origin. Also, g_z sends one point in $\mathbb{D}$ to H_∞, namely $-r/\|z\|$. Hence, $E_{\mathcal{B}}J(z) = J(g_z) = \log\|z\| - \log r$ if $z\in\mathbb{C}^n\setminus X$, and $E_{\mathcal{B}}J = 0$ on X, so $E_{\mathcal{B}}J = w$. The defining conditions for $\mathcal{B}$ to be a good set of analytic discs are easily verified. This shows that Proposition 4 is a special case of Theorem 2. Note that the good set $\mathcal{B}$ has neither property (2′) nor (4′) in Proposition 2.

By the disconnected example in Remark 3 above, the following description of almost extremal discs may be said to be optimal. Namely, we cannot always obtain V_X as the envelope of H_X or, equivalently, of J over analytic discs in $\mathbb{P}^n$, let alone $\mathbb{C}^n$, with boundaries in X. (Recall that H_X was introduced as shorthand for $H_{\mathcal{A}^X_{\mathbb{P}^n}}$ in Remark 2.)

PROPOSITION 5. *Let X be a nonempty open subset of $\mathbb{C}^n$. Let K be a compact subset of X and $z\in\mathbb{C}^n$. For each $\varepsilon>0$, there is an analytic disc f in $\mathbb{C}^n$ centred at z such that*

$$V_X(z) \le H_X(f) = \int\limits_{\mathbb{T}\setminus f^{-1}(X)} E_{\mathcal{A}^X_{\mathbb{P}^n}} J\circ f\,d\sigma < V_X(z)+\varepsilon$$

and

$$\sigma(\mathbb{T}\setminus f^{-1}(X\setminus K)) < \varepsilon.$$

Proof. Say $X\setminus K$ contains a closed ball of radius $R>0$. By Proposition 4 applied to $X\setminus K$, for each $0<r\le R$, there is $f_r\in\mathcal{A}_{\mathbb{C}^n}$ with $f_r(0)=z$ and

$$V_X(z)\le H_X(f_r)\le H_{X\setminus K}(f_r)\le H_r(f_r) < V_{X\setminus K}(z)+\varepsilon = V_X(z)+\varepsilon.$$

Thus, as $r\to 0$, we must have $\sigma(\mathbb{T}\setminus f_r^{-1}(X\setminus K))\to 0$, so we take $f=f_r$ with r small enough. ∎

Since $V_X\le E_{\mathcal{A}^X_{\mathbb{P}^n}}J$, Proposition 5 has the curious consequence, for every open subset X of $\mathbb{C}^n$, that V_X is its own Poisson envelope with respect to analytic discs in $\mathbb{C}^n$ that take all but an arbitrarily small piece of the circle $\mathbb{T}$ into X.

Relationship to the work of Lempert in the convex case. We will now describe the relationship between Lempert's disc formula for the Siciak–Zahariuta extremal function in the convex case, an account of which was provided by Momm in [M, Appendix], and our first disc formula.

Let K be a strictly convex compact subset of $\mathbb{C}^n$ with real-analytic boundary and let $z \in \mathbb{C}^n \setminus K$. Lempert's formula states that $V_K(z)$ is the infimum of the numbers $\log r$ over all holomorphic maps $f : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}^n$ with a continuous extension to $\mathbb{T}$ such that $f(\mathbb{T}) \subset K$, $f(r) = z$ with $r > 1$, and $\|f\|/|\cdot|$ is bounded, meaning that f has at most a simple pole at ∞. (Furthermore, extremal maps exist and can be described explicitly.) Precomposing f with the reciprocal, we see that $V_K(z)$ is the infimum of the numbers $-\log|\zeta|$ over all $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset K$ and $f(\zeta) = z$ such that f maps into $\mathbb{C}^n$ except for at most a simple pole at 0. Precomposing f with an automorphism of $\mathbb{D}$ that interchanges 0 and ζ, we see that $V_K(z)$ is the infimum of the numbers $-\log|\zeta|$ over all $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset K$ and $f(0) = z$ such that f maps into $\mathbb{C}^n$ except for at most a simple pole at ζ. For such a map f, we have $-\log|\zeta| = J(f)$.

Let X be a convex open subset of $\mathbb{C}^n$. Then X can be written as the increasing union of relatively compact open subsets X_n, $n \geq 1$, such that the closure $\overline{X}_n$ is strictly convex with real-analytic boundary. Namely, take a strictly convex exhaustion function of X, such as the sum of $\|\cdot\|^2$ and the reciprocal of the Euclidean distance to the boundary, and Weierstrass-approximate it by a polynomial; the generic sublevel sets of the polynomial will be smooth.

Now V_X is the decreasing limit of V_{X_n} and hence also the decreasing limit of $V_{\overline{X}_n}$ as $n \to \infty$. Therefore, by Lempert's formula, $V_X(z)$ for $z \in \mathbb{C}^n \setminus X$, and thus obviously for all $z \in \mathbb{P}^n$, is the infimum of the numbers $J(f)$ over all $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset X$ and $f(0) = z$ such that f maps into $\mathbb{C}^n$ except for at most one simple pole. By Theorem 1, in between this infimum and $V_X(z)$ is the infimum of $J(f)$ over the larger class of $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(\mathbb{T}) \subset X$ and $f(0) = z$.

Thus, Lempert's formula can be stated as the following strengthening of Theorem 1 for the convex case.

Lempert's formula. *Let X be a convex open subset of $\mathbb{C}^n$. Then the Siciak–Zahariuta extremal function of X is the envelope of J with respect to the set of analytic discs in $\mathbb{P}^n$ with boundary in X and at most one simple pole. It follows that*

$$V_X = E_{\mathcal{A}^X_{\mathbb{P}^n}} J.$$

We conclude the paper by proving Lempert's formula for any connected open subset X of $\mathbb{C}^n$ in the slightly weakened form $V_X = E_{\mathcal{A}^X_{\mathbb{P}^n}} J$, which we will henceforth refer to as Lempert's formula. As remarked earlier, this formula is easily seen to fail for disconnected sets in general. We do not know whether the stronger, original form of Lempert's formula, using only analytic discs with at most one simple pole, extends to all connected sets.

Lempert's formula for arbitrary domains. Let X be a connected open subset of $\mathbb{C}^n$. We may assume that X is neither empty nor all of $\mathbb{C}^n$ (otherwise, Lempert's formula is obvious). From now on, $\mathcal{B}$ will denote the set of analytic discs in $\mathbb{P}^n$ containing all the constant discs in X and every disc

$$f_{z,w,r} : \zeta \mapsto w + \frac{\|z-w\| + r\zeta}{r + \|z-w\|\zeta}\, \frac{r}{\|z-w\|}\,(z-w)$$

in $\mathbb{P}^n$, where $z \in \mathbb{C}^n \setminus X$, $w \in X$, and r is less than the Euclidean distance $d(w, \partial X)$ from w to the boundary ∂X of X. Observe that $f_{z,w,r}$ is injective, centred at z, takes one point to H_∞, namely $-r/\|z-w\|$, lies on the projective line through z and w, and maps $\mathbb{T}$ onto the circle with centre w and radius r on this line. It is easy to verify that

$$E_{\mathcal{B}}J = \inf_{w\in X} \log^+ \frac{\|\cdot - w\|}{d(w,\partial X)} = \inf\{V_B : B \text{ is a ball in } X\}.$$

It follows that $\mathcal{B}$ is a good set of analytic discs in $\mathbb{P}^n$ with respect to X. Note that $E_{\mathcal{B}}J$ is not plurisubharmonic in general: just consider an annulus.

If X is smoothly bounded, then using balls touching the boundary from the inside, we see that $E_{\mathcal{B}}J = 0$ on ∂X. Now $V_X \le E_{\mathcal{A}^X_{\mathbb{P}^n}} J \le E_{\mathcal{B}}J$ on $\mathbb{C}^n$, so $V_X = E_{\mathcal{A}^X_{\mathbb{P}^n}} J$ on $\overline{X}$. Since every domain can be exhausted by smoothly bounded domains and Lempert's formula is preserved by increasing unions, it suffices to prove the formula on $\mathbb{C}^n \setminus \overline{X}$ assuming $E_{\mathcal{B}}J = 0$ on ∂X. The argument is based on the following result.

Lemma. *Let X be a connected open subset of $\mathbb{C}^n$ and $\mathcal{B}$ be as above. For every analytic disc h in $\mathbb{C}^n \setminus \overline{X}$, continuous function $v \ge E_{\mathcal{B}}J$ on $\mathbb{C}^n \setminus \overline{X}$, and $\varepsilon > 0$, there is $g \in \mathcal{A}^X_{\mathbb{P}^n}$ with $g(0) = h(0)$ and*

$$J(g) \le \int_{\mathbb{T}} v \circ h \, d\sigma + \varepsilon.$$

Fixing $z \in \mathbb{C}^n \setminus \overline{X}$ and taking the infimum over all v, ε, and h with $h(0) = z$ as in the Lemma, we see that $E_{\mathcal{A}^X_{\mathbb{P}^n}} J$ is no larger than the Poisson envelope, that is, the largest plurisubharmonic minorant $P_{\mathbb{C}^n\setminus\overline{X}} E_{\mathcal{B}}J$, of $E_{\mathcal{B}}J$ on $\mathbb{C}^n \setminus \overline{X}$. Now

$$P_{\mathbb{C}^n\setminus\overline{X}} E_{\mathcal{B}}J = P_{\mathbb{C}^n} E_{\mathcal{B}}J | \mathbb{C}^n \setminus \overline{X}.$$

Namely, if u is plurisubharmonic on $\mathbb{C}^n \setminus \overline{X}$ and $u \le E_{\mathcal{B}}J$, then, after replacing u by $\max\{u, 0\}$ and using the assumption that $E_{\mathcal{B}}J = 0$ on ∂X, we can extend u to a plurisubharmonic function on all of $\mathbb{C}^n$ by setting $u = 0$

on $\overline{X}$. Then $u \le E_{\mathcal{B}}J$ on $\mathbb{C}^n$, so $u \le P_{\mathbb{C}^n} E_{\mathcal{B}}J$. This proves one inequality; the other is obvious. Finally, by the remarks preceding Theorem 2, $P_{\mathbb{C}^n} E_{\mathcal{B}}J = V_X$ since $\mathcal{B}$ is good. Thus, given the Lemma, we have established Lempert's formula:

THEOREM 3. *The Siciak–Zahariuta extremal function of a connected open subset X of $\mathbb{C}^n$ is the envelope of J with respect to the set of analytic discs in $\mathbb{P}^n$ with boundary in X, that is,*

$$V_X = E_{\mathcal{A}^X_{\mathbb{P}^n}} J.$$

It remains to prove the Lemma. Our argument is an adaptation of Poletsky's orginal proof of the plurisubharmonicity of the Poisson envelope. See [P1] or [LS1, Section 2]. We proceed as if we were trying to show that $E_{\mathcal{B}}J$ was plurisubharmonic.

Proof of the Lemma. Take $\zeta_0 \in \mathbb{T}$ and set $z_0 = h(\zeta_0)$. By the definition of $\mathcal{B}$, there exist $w_0 \in X$ and $r_0 < d(w_0, \partial X)$ with

$$J(f_{z_0,w_0,r_0}) = \log(\|z_0 - w_0\|/r_0) < E_{\mathcal{B}}J(z_0) + \varepsilon.$$

By continuity, there exists an open arc I_0 containing ζ_0 such that

$$J(f_{h(\zeta),w_0,r_0}) = \log(\|h(\zeta) - w_0\|/r_0) < v(h(\zeta)) + \varepsilon/2, \qquad \zeta \in I_0.$$

By compactness, there exist a cover of $\mathbb{T}$ by open arcs $I_1, \ldots, I_m$, points $w_1, \ldots, w_m$ in X, and $r_1, \ldots, r_m > 0$ such that $r_j < d(w_j, \partial X)$ and

$$J(f_{h(\zeta),w_j,r_j}) = \log(\|h(\zeta) - w_j\|/r_j) < v(h(\zeta)) + \varepsilon/2$$

for $\zeta \in I_j$, $j = 1, \ldots, m$. There exist $A \subset \{1, \ldots, m\}$ and closed arcs $J_j \subset I_j$, $j \in A$, which cover $\mathbb{T}$ and have disjoint interiors. By possibly renumbering the arcs and splitting the interval I_j containing 1, we may assume that $A = \{1, \ldots, m\}$ and

$$J_j = \{e^{i\theta} : \theta \in [a_j, a_{j+1}]\}, \qquad \text{where } 0 = a_1 < a_2 < \cdots < a_{m+1} = 2\pi.$$

Then

$$\sum_{j=1}^{m} \int_{J_j} J(f_{h(\zeta),w_j,r_j})\, d\sigma(\zeta) < \int_{\mathbb{T}} v \circ h \, d\sigma + \varepsilon/2. \tag{1}$$

Since X is connected, we can join w_j and w_{j+1} by a C^∞ path $\alpha_j : [0,1] \to X$ with $\alpha_j(0) = w_j$, $\alpha_j(1) = w_{j+1}$, and choose a C^∞ function $\beta_j : [0,1] \to (0,\infty)$ with $\beta_j(0) = r_j$, $\beta_j(1) = r_{j+1}$, and $\beta_j < d(\alpha_j, \partial X)$. Here we take $w_{m+1} = w_1$ and $r_{m+1} = r_1$. We may assume that the derivatives of all orders of α_j and β_j vanish at 0 and 1. We choose

$$C > \sum_{j=1}^{m} \sup_{\zeta \in J_j,\, t \in [0,1]} |J(f_{h(\zeta),w_j,r_j}) - J(f_{h(\zeta),\alpha_j(t),\beta_j(t)})| \tag{2}$$

and $\delta > 0$ such that $C\delta < \varepsilon/2$ and $\delta < \min_j(a_{j+1} - a_j)$. We split each arc J_j into the subarcs $K_j = \{e^{i\theta} : \theta \in [a_j, a_{j+1} - \delta]\}$ and $L_j = \{e^{i\theta} : \theta \in [a_{j+1} - \delta, a_{j+1}]\}$, and define the C^∞ loop $\gamma : \mathbb{T} \to X$ by

$$\gamma(\zeta) = \begin{cases} w_j, & \zeta \in K_j,\ j = 1, \dots, m, \\ \alpha_j((\theta - a_{j+1} + \delta)/\delta), & \zeta = e^{i\theta} \in L_j,\ j = 1, \dots, m, \end{cases}$$

the C^∞ function $\varrho : \mathbb{T} \to (0, \infty)$ by

$$\varrho(\zeta) = \begin{cases} r_j, & \zeta \in K_j,\ j = 1, \dots, m, \\ \beta_j((\theta - a_{j+1} + \delta)/\delta), & \zeta = e^{i\theta} \in L_j,\ j = 1, \dots, m, \end{cases}$$

and, finally, the C^∞ family

$$F(\cdot, \zeta) = f_{h(\zeta), \gamma(\zeta), \varrho(\zeta)}, \qquad \zeta \in \mathbb{T},$$

of analytic discs in $\mathcal{A}^X_{\mathbb{P}^n}$. By (1) and (2),

$$\int_{\mathbb{T}} J(F(\cdot, \zeta))\, d\sigma(\zeta) < \sum_{j=1}^{m} \int_{J_j} J(f_{h(\zeta), w_j, r_j})\, d\sigma(\zeta) + C\delta < \int_{\mathbb{T}} v \circ h\, d\sigma + \varepsilon. \tag{3}$$

We take the lifting $\widetilde{h} = (1, h) \in \mathcal{A}_Z$ of h to $Z = \mathbb{C}^{n+1} \setminus \{0\}$ by the projection $\pi : Z \to \mathbb{P}^n$, and the lifting $\widetilde{f}_{z,w,r}$ of $f_{z,w,r}$ given by

$$\widetilde{f}_{z,w,r}(\xi) = (\|z - w\|\xi/r + 1, (\|z - w\|\xi/r + 1)w + (r\xi/\|z - w\| + 1)(z - w)).$$

Then the lifting $\widetilde{F}(\cdot, \zeta) = \widetilde{f}_{h(\zeta), \gamma(\zeta), \varrho(\zeta)}$ of F satisfies $\widetilde{F}(0, \cdot) = \widetilde{h}$ on $\mathbb{T}$.

Take $r > 1$ such that $h \in \mathcal{O}(D_r, \mathbb{C}^n)$ and $F(\cdot, \zeta) \in \mathcal{O}(D_r, \mathbb{P}^n)$ for all $\zeta \in \mathbb{T}$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$, and define $\widetilde{F}_j \in \mathcal{O}(D_r \times (D_r \setminus \{0\}), \mathbb{C}^{n+1})$, $j \geq 1$, by

$$\widetilde{F}_j(\xi, \zeta) = \widetilde{h}(\zeta) + \sum_{k=-j}^{j} \left(\frac{1}{2\pi} \int_0^{2\pi} (\widetilde{F}(\xi, e^{i\theta}) - \widetilde{h}(e^{i\theta})) e^{-ik\theta}\, d\theta \right) \zeta^k.$$

Since the function $\theta \mapsto \widetilde{F}(\xi, e^{i\theta}) - \widetilde{h}(e^{i\theta})$ is C^∞ with period 2π, its Fourier series converges uniformly on $\mathbb{R}$ to the function itself. Hence, the sequence $(\widetilde{F}_j)$ converges uniformly on $\{\xi\} \times \mathbb{T}$ for each $\xi \in D_r$. The convergence is uniform on $D_t \times \mathbb{T}$ for each $t \in (1, r)$. In fact, an integration by parts of the integral above shows that it can be estimated by

$$\frac{1}{k^2} \max_{\xi \in D_t, \theta \in \mathbb{R}} |\partial^2 (\widetilde{F}(\xi, e^{i\theta}) - \widetilde{h}(e^{i\theta}))/\partial\theta^2|, \qquad k \neq 0.$$

Fixing $t \in (1, r)$, since $\widetilde{F}(D_r \times \mathbb{T}) \subset Z$, $F(\mathbb{T} \times \mathbb{T}) \subset X$, and $\widetilde{F}_j \to \widetilde{F}$ uniformly on $D_t \times \mathbb{T}$, we have $\widetilde{F}_j(D_t \times \mathbb{T}) \subset Z$ and $\widetilde{F}_j(\mathbb{T} \times \mathbb{T}) \subset \pi^{-1}(X)$ if j is large enough. For such j, define $F_j = \pi \circ \widetilde{F}_j : D_t \times \mathbb{T} \to \mathbb{P}^n$. The 0th coordinate of $\widetilde{F}$ is $\widetilde{F}_0(\xi, \cdot) = \|h - \gamma\|\xi/\varrho + 1$, so the 0th coordinate of $\widetilde{F}_j$ is $\widetilde{F}_{j0}(\xi, \cdot) = \chi_j \xi + 1$, where $\theta \mapsto \chi_j(e^{i\theta})$ is the jth partial sum of the Fourier series of $\theta \mapsto \|h(e^{i\theta}) - \gamma(e^{i\theta})\|/\varrho(e^{i\theta})$. Hence,

$$J(F_j(\cdot,\zeta)) = \log|\chi_j(\zeta)| \to \log(\|h(\zeta) - \gamma(\zeta)\|/\varrho(\zeta)) = J(F(\cdot,\zeta))$$

uniformly for $\zeta \in \mathbb{T}$. Thus, by (3),

$$\int_{\mathbb{T}} J(F_j(\cdot,\zeta))\,d\sigma(\zeta) < \int_{\mathbb{T}} v \circ h\,d\sigma + \varepsilon$$

for j large enough. We now fix j so large that these properties hold.

For every $\xi \in D_r$, the map $\zeta \mapsto \widetilde{F}_j(\xi,\zeta) - \widetilde{h}(\zeta)$ has a pole of order at most j at the origin, and for every $\zeta \in D_r$, $\zeta \neq 0$, the map $\xi \mapsto \widetilde{F}_j(\xi,\zeta) - \widetilde{h}(\zeta)$ has a zero at the origin. Hence, $(\xi,\zeta) \mapsto \widetilde{F}_j(\xi\zeta^k,\zeta)$ extends to a holomorphic map $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \to \mathbb{C}^{n+1}$ for every $k \geq j$.

Since $\widetilde{F}_j(0,\zeta) = \widetilde{h}(\zeta) \in Z$ for all $\zeta \in D_r$, $\zeta \neq 0$, there is $\delta > 0$ such that $\widetilde{F}_j(\xi\zeta^k,\zeta) \in Z$ for all $k \geq j$ and $(\xi,\zeta) \in D_\delta \times \overline{\mathbb{D}}$. Since $\widetilde{F}_j(\xi,\zeta) \in Z$ for all $(\xi,\zeta) \in \overline{\mathbb{D}} \times \mathbb{T}$, there is $\tau < 1$ such that $\widetilde{F}_j(\xi,\zeta) \in Z$ for all $(\xi,\zeta) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_\tau)$, so $\widetilde{F}_j(\xi\zeta^k,\zeta) \in Z$ for all $(\xi,\zeta) \in \overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus D_\tau)$ and all $k \geq j$. Choose $k \geq j$ large enough that $|\xi\zeta^k| < \delta$ for all $(\xi,\zeta) \in \overline{\mathbb{D}} \times D_\tau$. Then there is $s \in (1,t)$ such that $\widetilde{F}_j(\xi\zeta^k,\zeta) \in Z$ for all $(\xi,\zeta) \in D_s \times D_s$.

Now define $\widetilde{G} \in \mathcal{O}(D_s \times D_s, Z)$ by $\widetilde{G}(\xi,\zeta) = \widetilde{F}_j(\xi\zeta^k,\zeta)$ and let $G = \pi \circ \widetilde{G}$. In the proof of Theorem 1, we observed that if $\widetilde{f} = (f_0,\dots,f_n) \in \mathcal{A}_Z$ is a lifting of $f \in \mathcal{A}_{\mathbb{P}^n}$ and $f_0(0) \neq 0$, then

$$J(f) = \int_{\mathbb{T}} \varphi \circ \widetilde{f}\,d\sigma - \varphi(\widetilde{f}(0)),$$

where, as before, $\varphi(z) = \log|z_0|$ for $z \in \mathbb{C}^{n+1}$. Now $\widetilde{G}(0,\cdot) = \widetilde{h} = (1,h)$, so $\varphi(\widetilde{G}(0,\cdot)) = 0$. Therefore,

$$\begin{aligned}\int_{\mathbb{T}} J(G(\cdot,\zeta))\,d\sigma(\zeta) &= \int_{\mathbb{T}^2} \varphi \circ \widetilde{G}\,d(\sigma\times\sigma) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(\widetilde{F}_j(e^{i(t+k\theta)}, e^{i\theta}))\,dt\,d\theta \\ &= \int_{\mathbb{T}^2} \varphi \circ \widetilde{F}_j\,d(\sigma\times\sigma) = \int_{\mathbb{T}} J(F_j(\cdot,\zeta))\,d\sigma(\zeta) \\ &< \int_{\mathbb{T}} v \circ h\,d\sigma + \varepsilon.\end{aligned}$$

By the Mean Value Theorem, there is $\theta_0 \in [0,2\pi]$ such that

$$\begin{aligned}\int_{\mathbb{T}} J(G(\cdot,\zeta))\,d\sigma(\zeta) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(\widetilde{G}(e^{i(\theta+t)}, e^{it}))\,dt\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\widetilde{G}(e^{i(\theta_0+t)}, e^{it}))\,dt.\end{aligned}$$

Now define $\widetilde{g}(\zeta) = \widetilde{G}(e^{i\theta_0}\zeta, \zeta)$ for $\zeta \in D_s$, and $g = \pi \circ \widetilde{g}$. Then

$$\widetilde{g}(0) = \widetilde{G}(0,0) = (1, h(0)),$$

so $g(0) = h(0)$, and

$$g(\mathbb{T}) \subset \pi(\widetilde{G}(\mathbb{T} \times \mathbb{T})) \subset X,$$

so $g \in \mathcal{A}^X_{\mathbb{P}^n}$. Also,

$$\begin{aligned} J(g) &= \int_{\mathbb{T}} \varphi \circ \widetilde{g}\, d\sigma = \frac{1}{2\pi}\int_0^{2\pi} \varphi(\widetilde{G}(e^{i\theta_0}e^{it}, e^{it}))\, dt = \int_{\mathbb{T}} J(G(\cdot,\zeta))\, d\sigma(\zeta) \\ &< \int_{\mathbb{T}} v \circ h\, d\sigma + \varepsilon, \end{aligned}$$

and the proof is complete. ■

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