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An intermediate value theorem in ordered Banach spaces

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Abstract. We prove an intermediate value theorem for certain quasimonotone increasing functions in ordered Banach spaces, under the assumption that each nonempty order bounded chain has a supremum.

1. Introduction. Let E be a real Banach space ordered by a cone K. A *cone* K is a nonempty closed convex subset of E such that $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y :\Leftrightarrow y - x \in K$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. Let K^* denote the dual wedge of K, that is, the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ $(x \geq 0)$.

For $D \subseteq E$ a function $f : D \to E$ is called *quasimonotone increasing* (in the sense of Volkmann [19]) if

$$x, y \in D,\ x \leq y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \ \Rightarrow\ \varphi(f(x)) \leq \varphi(f(y)).$$

For quasimonotone increasing functions several intermediate value (or equivalently fixed point) theorems are known, for special spaces [4], [8], [14], [15], under order conditions [6], [18], and under compactness conditions [6], [9], [10], [13, VIII.6], [18]. For an application of such intermediate value theorems to boundary value problems see [7].

In this paper we will prove the following version under the assumption that the order defined by K (or K for short) has the following property:

(C) *Each chain $C \subseteq E$, $C \neq \emptyset$, which is order bounded above has a supremum.*

Theorem 1. *Let E be ordered by a cone K with property* (C), *let $D \subseteq E$ be open, and let $f : D \to E$ be locally Lipschitz continuous and quasimonotone increasing. Moreover let $a, b \in D$ satisfy*

$$a \leq b, \quad [a, b] \subseteq D, \quad \textit{and} \quad f(b) \leq 0 \leq f(a).$$

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Then

$$\min f^{-1}(0) \cap [a,b] \quad \textit{and} \quad \max f^{-1}(0) \cap [a,b]$$

exist.

Remarks. 1. Condition (C) is valid in particular if K is regular (that is, each increasing and order bounded sequence in E is convergent; see [2, Lemma 2] or [11, Lemma 1]). For regular cones a related intermediate value theorem is valid (see [6]). On the other hand, condition (C) implies that K is normal (that is, $0 \le x \le y$ implies $\|x\| \le \gamma\|y\|$ for some constant $\gamma \ge 1$; see [1, Lemma 2]), but normality for itself is not sufficient to guarantee the intermediate value property. We repeat the following example from [6] for the convenience of the reader:

Let $E = c(\mathbb{N},\mathbb{R})$ be the Banach space of all convergent real sequences $x = (x_k)_{k\in\mathbb{N}}$, endowed with the supremum norm and ordered by the cone K of all nonnegative sequences, which is normal. Let $f : E \to E$ be defined by

$$f(x) = (0, 1, x_1, x_2, x_3, \dots) - x.$$

Then f is Lipschitz continuous and quasimonotone increasing, and

$$f((1)_{k\in\mathbb{N}}) = (-1, 0, 0, 0, \dots) \le 0 \le (1, 2, 0, 0, 0, \dots) = f((-1)_{k\in\mathbb{N}}),$$

but $f(z) = 0$ is unsolvable in $c(\mathbb{N},\mathbb{R})$, since the only coordinatewise solution is

$$z = (0, 1, 0, 1, 0, 1, \dots).$$

2. An example of a nonregular cone with property (C) is the cone of all nonnegative sequences in $l^\infty(\mathbb{N},\mathbb{R})$. More generally, let J be a nonempty set and let $(F_j)_{j\in J}$ be a family of Banach spaces, each ordered by a regular cone K_j. Consider

$$E = \{x = (x_j)_{j\in J} : x_j \in F_j\ (j \in J),\ \|x\| = \sup_{j\in J} \|x_j\| < \infty\}$$

ordered by the cone

$$K = \{x \in E : x_j \in K_j\ \ (j \in J)\}.$$

Then K has property (C) (see [11, Lemma 2]).

2. Preliminaries. To prove Theorem 1 we will make use of the following theorems. The first is a result on differential inequalities due to Volkmann [20, Satz 2], and two of its immediate consequences on dynamical systems:

Theorem 2. *Let E be ordered by a cone K, let $D \subseteq E$ be open, let $f : D \to E$ be locally Lipschitz continuous and quasimonotone increasing, and let $u(\cdot, x) : [0, T_x) \to D$ denote the solution of $u'(t) = f(u(t))$, $u(0) = x$ (nonextendable to the right). Then:*

1. *If* $v, w : [0, T) \to D$ *satisfy* $v'(t) - f(v(t)) \le w'(t) - f(w(t))$ $(t \in [0, T))$ *and* $v(0) \le w(0)$*, then* $v(t) \le w(t)$ $(t \in [0, T))$*.*
2. $x, y \in D$, $x \le y \Rightarrow u(t, x) \le u(t, y)$ $(t \in [0, \min\{T_x, T_y\}))$*.*
3. $x \in D$, $f(x) \ge 0$ $[\le 0] \Rightarrow t \mapsto u(t, x)$ *is increasing* [*decreasing*] *on* $[0, T_x)$*.*

Second, we will use the following versions of Bourbaki's and Tarski's fixed point theorems (see [3], [5, Proposition 1], [12]). For a function $g : \Omega \to \Omega$ we set

$$\mathrm{Fix}(g) := \{x \in \Omega : g(x) = x\}.$$

THEOREM 3. *Let* $\Omega \neq \emptyset$ *be an ordered set such that each chain* $\emptyset \neq C \subseteq \Omega$ *has a supremum. Let* $g : \Omega \to \Omega$ *satisfy* $x \le g(x)$ $(x \in \Omega)$*. Then* $\mathrm{Fix}(g) \neq \emptyset$*.*

THEOREM 4. *Let* $\Omega \neq \emptyset$ *be an ordered set such that* $\min \Omega$ *exists, and such that each chain* $\emptyset \neq C \subseteq \Omega$ *has a supremum. Let* $g : \Omega \to \Omega$ *be increasing. Then* $\min \mathrm{Fix}(g)$ *exists.*

3. Proof of Theorem 1. We consider the set

$$\Omega := \{x \in [a, b] : f(x) \ge 0,\ x \le z\, (z \in f^{-1}(0) \cap [a, b])\}.$$

First, observe that $a \in \Omega$, so $\Omega \neq \emptyset$. Next, let $x \in [a, b]$. According to Theorem 2 we have

$$u(t, x) \in [a, b] \quad (t \in [0, T_x)).$$

If in addition $f(x) \ge 0$ then $t \mapsto u(t, x)$ is increasing on $[0, T_x)$, so

$$f(u(t, x)) \ge 0 \quad (t \in [0, T_x)),$$

and if in addition

$$x \le z\ (z \in f^{-1}(0) \cap [a, b])$$

then

$$u(t, x) \le u(t, z) = z \quad (t \in [0, T_x),\ z \in f^{-1}(0) \cap [a, b]).$$

Thus $x \in \Omega$ implies $u([0, T_x), x) \subseteq \Omega$. Note that $u([0, T_x), x)$ is a chain in Ω for each $x \in \Omega$.

Let $\emptyset \neq C \subseteq \Omega$ be a chain with $c := \sup C$. We prove $c \in \Omega$. Clearly $c \in [a, b]$. According to Theorem 2 we have

$$x \le u(t, c) \quad (t \in [0, T_c),\ x \in C),$$

and therefore

$$c \le u(t, c) \quad (t \in [0, T_c)).$$

Hence $u'(0, c) = f(c) \ge 0$. Moreover

$$x \le z \quad (x \in C,\ z \in f^{-1}(0) \cap [a, b]),$$

thus

$$c \le z \quad (z \in f^{-1}(0) \cap [a,b]),$$

and summing up we have $c \in \Omega$.

We define

$$g : \Omega \to \Omega, \quad g(x) = \sup u([0,T_x),x).$$

Now, $x \le g(x)$ $(x \in \Omega)$, and according to Theorem 3, g has a fixed point $\underline{z} \in \Omega$. Since $t \mapsto u(t,\underline{z})$ is increasing on $[0,T_{\underline{z}})$ we conclude $T_{\underline{z}} = \infty$ and

$$u(t,\underline{z}) = \underline{z} \quad (t \in [0,\infty)),$$

hence $f(\underline{z}) = 0$. To prove the minimality of $\underline{z}$ observe that $z \in [a,b]$, $f(z) = 0$ implies $\underline{z} \le z$ by the definition of Ω. Thus $\underline{z} = \min f^{-1}(0) \cap [a,b]$.

To prove the existence of a greatest solution in $[a,b]$ of $f(z) = 0$ we consider

$$h : -D \to E, \quad h(x) = -f(-x).$$

Now, h is locally Lipschitz continuous, quasimonotone increasing, and

$$h(-a) \le 0 \le h(-b).$$

Thus, in $[-b,-a]$ the equation $h(z) = 0$ has a smallest solution w, and $\overline{z} := -w = \max f^{-1}(0) \cap [a,b]$. ∎

Remark. If it is assumed in addition that $f(B)$ is bounded for each bounded subset $B \subseteq D$ then $T_x = \infty$ for each $x \in [a,b]$, and the proof above can be changed by applying Theorem 4 to $\Omega = \{x \in [a,b] : f(x) \ge 0\}$ and $g : \Omega \to \Omega$ defined by $g(x) = u(T,x)$ for any fixed $T > 0$.

4. Discontinuous functions. Following the idea in [18] we can extend Theorem 1 the following way.

Let $D \subseteq E$ be open, let $a,b \in D$ satisfy $a \le b$, $[a,b] \subseteq D$, and let

$$F : D \times [a,b] \to E, \quad f : [a,b] \to E$$

satisfy

(a) $x \mapsto F(x,y)$ is locally Lipschitz continuous and quasimonotone increasing for each $y \in [a,b]$,
(b) $y \mapsto F(x,y)$ is monotone increasing for each $x \in D$,
(c) $f(x) = F(x,x)$ $(x \in [a,b])$, and $f(b) \le 0 \le f(a)$.

Under these assumptions f is quasimonotone increasing, and allows upward jumps (see [18]). We have

Theorem 5. *Let E be ordered by a cone K with property* (C), *let $D \subseteq E$ be open, let $a \le b$ with $[a,b] \subseteq D$, and let $F : D \times [a,b] \to E$ and $f : [a,b] \to E$ satisfy* (a)–(c) *above. Then*

$$\min f^{-1}(0) \cap [a,b] \quad \textit{and} \quad \max f^{-1}(0) \cap [a,b]$$

exist.

5. Proof of Theorem 5. Let $y \in [a,b]$. Then

$$F(b,y) \leq f(b) \leq 0 \leq f(a) \leq F(a,y).$$

According to Theorem 1 the mapping $x \mapsto F(x,y)$ has in $[a,b]$ a smallest zero $g(y)$. We obtain a function $g : [a,b] \to [a,b]$ and we prove that g is increasing. Indeed, let $y,z \in [a,b]$ with $y \leq z$. Now

$$F(g(z),y) \leq F(g(z),z) = 0 \leq F(a,y).$$

Thus $x \mapsto F(x,y)$ has in $[a,g(z)]$ a zero v, which is a zero in $[a,b]$. Therefore

$$g(y) \leq v \leq g(z).$$

According to Theorem 4 (applied to $\Omega = [a,b]$) $\underline{z} := \min \operatorname{Fix}(g)$ exists, and clearly $f(\underline{z}) = 0$. Now, let $z \in [a,b]$ satisfy $f(z) = 0$. Then z is a zero of $x \mapsto F(x,z)$ in $[a,b]$, hence $g(z) \leq z$. Thus $g([a,z]) \subseteq [a,z]$, and so g has a fixed point w in $[a,z]$ which is a fixed point in $[a,b]$. Thus

$$\underline{z} = \min \operatorname{Fix}(g) \leq w \leq z.$$

Therefore $\underline{z} = \min f^{-1}(0) \cap [a,b]$.

Application of this state of knowledge to $H : (-D) \times [-b,-a] \to E$ and $h : [-b,-a] \to E$ defined by

$$H(x,y) = -F(-x,-y), \quad h(x) = H(x,x)$$

proves the existence of $\overline{z} = \max f^{-1}(0) \cap [a,b]$. ∎

6. Example. Let F be a Banach space ordered by a regular cone K_F with nonempty interior, let $E = l^\infty(\mathbb{Z}, F)$ be ordered by the cone

$$K = \{(x_n)_{n\in\mathbb{Z}} : x_n \in K_F \ (n \in \mathbb{Z})\},$$

and let $q : F \to F$ be locally Lipschitz continuous and quasimonotone increasing. We can apply Theorem 1 to prove

THEOREM 6. *Let $(w_n)_{n\in\mathbb{Z}} \in E$, and let $a,b \in F$ be such that*

$$a \leq b, \qquad q(b) \leq w_n \leq q(a) \qquad (n \in \mathbb{Z}).$$

Then the second order difference equation

$$z_{n+1} - 2z_n + z_{n-1} + q(z_n) = w_n \quad (n \in \mathbb{Z})$$

has in $[(a)_{n\in\mathbb{Z}}, (b)_{n\in\mathbb{Z}}]$ a smallest and a greatest solution.

Proof. According to Remark 2. the order on E defined by K has property (C). We consider $f : E \to E$ defined by

$$f((x_n)_{n\in\mathbb{Z}}) = (x_{n+1} - 2x_n + x_{n-1} + q(x_n) - w_n)_{n\in\mathbb{Z}}.$$

It is clear that f is locally Lipschitz continuous and, using Uhl's criterion for quasimonotonicity [17, Theorem 2], it is not hard to see that f is quasimonotone increasing. We have

$$f((b)_{n\in\mathbb{Z}}) = (q(b) - w_n)_{n\in\mathbb{Z}} \le (0)_{n\in\mathbb{Z}} \le (q(a) - w_n)_{n\in\mathbb{Z}} = f((a)_{n\in\mathbb{Z}}).$$

Thus, according to Theorem 1, the maximum and the minimum of

$$f^{-1}((0)_{n\in\mathbb{Z}}) \cap [(a)_{n\in\mathbb{Z}}, (b)_{n\in\mathbb{Z}}]$$

exist. ∎

Consider for example $F = \mathbb{R}^3$ ordered by the ice-cream cone

$$K_F = \{x = (\xi, \eta, \zeta) : \zeta \ge \sqrt{\xi^2 + \eta^2}\},$$

and $q : F \to F$ defined by

$$q(\xi, \eta, \zeta) = \begin{pmatrix} -\eta - 2\xi\zeta \\ \xi - 2\eta\zeta \\ -\xi^2 - \eta^2 - \zeta^2 \end{pmatrix}.$$

Obviously q is locally Lipschitz continuous, and q is quasimonotone increasing since $q'(\xi, \eta, \zeta) : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear quasimonotone increasing mapping for each $(\xi, \eta, \zeta) \in \mathbb{R}^3$ (see [16, Theorem 3.31]). Since $p = (0, 0, \lambda) \in \operatorname{Int} K$ for each $\lambda > 0$, and since $q(0, 0, \lambda) = -(0, 0, \lambda^2)$ we can apply Theorem 6 if $(w_n)_{n\in\mathbb{Z}}$ is a bounded sequence in $-K_F$, by setting

$$a = (0, 0, 0), \qquad b = (0, 0, \lambda),$$

with $\lambda > 0$ sufficiently large.

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