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Weighted Bernstein–Markov property in $\mathbb{C}^n$

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Abstract. We study the weighted Bernstein–Markov property for subsets in $\mathbb{C}^n$ which might not be bounded. An application concerning approximation of the weighted Green function using Bergman kernels is also given.

1. Introduction. Let E be a Borel (not necessarily bounded) non-pluripolar subset of $\mathbb{C}^n$, $\omega \geq 0$ be an upper semicontinuous (usc for short) function defined on E and μ be a positive Borel measure on E. We say that ω is an *admissible weight* if:

(i) $\{\omega > 0\}$ is non-pluripolar.
(ii) $\sup_{z\in E} |z|\omega(z) < \infty$.

The aim of this article is to study conditions guaranteeing that the triple (E, μ, ω) has the Bernstein–Markov property. Recall that (E, μ, ω) is said to have the *Bernstein–Markov property* if there is strong comparability between L^2 and L^∞ norms of *weighted* polynomials on E. More precisely, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for every $P \in \mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials of n complex variables,

$$\|\omega^{\deg P} P\|_E \leq C_\varepsilon (1+\varepsilon)^d \|\omega^{\deg P} P\|_{L^2(E,\mu)}.$$

Here $\|\omega^d P\|_E$ and $\|\omega^d P\|_{L^2(E,\mu)}$ denote the sup norm and the L^2 norm with respect to $d\mu$ of the weighted polynomial $\omega^d P$.

In the previous papers of Bloom and Levenberg ([BL2], [Le]), the Bernstein–Markov property has already been considered for unbounded sets. However, in addition to (i), the following conditions are assumed in their work:

(ii′) $\lim_{|z|\to\infty} z\omega(z) = 0$.
(iii) There exists $d_0 \geq 1$ such that $\int_E |p_d|^2 \omega^{2d}\, d\mu < \infty$ for every $d > d_0$ and all $p_d \in \mathcal{P}_d$, the set of polynomials of degree d in $\mathbb{C}^n$.

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Note that (ii′) is much stronger than (ii). Thus the main point in our work is that we only require ω to satisfy the mild estimate (ii) in the case E is unbounded.

Now we describe in more detail the content of the paper. In Section 3, we apply Bloom's results [Bl3] to connect the Bernstein–Markov property for (E, μ, ω) to that for (Z, ν) where Z is a bounded circular set in $\mathbb{C}^{n+1}$ and ν is a positive Borel measure on Z. In the main theorem of that section, Theorem 3.3, we translate the mass density condition in Bloom–Levenberg's theorem (e.g. Theorem 2.4) from the pair (Z, ν) to the original triple (E, μ, ω). We also indicate in Proposition 3.5 a simpler situation when the Bernstein–Markov property for (E, μ, ω) with unbounded E might be reduced, after taking inversion maps, to the case when E is bounded. Next, we start Section 4 by considering the situation when ω is supposed to satisfy (ii)′. A sufficient condition for the Bernstein–Markov property in this special case is given in Corollary 4.2, which is based on Proposition 4.1. We also apply this result to provide a Bernstein–Markov type estimate in the case where $E = \mathbb{C}^n, \mu = \lambda_{2n}$ (the Lebesgue measure) and ω is just an admissible weight in our sense. The main result of that section is Theorem 4.5. More explicitly, by embedding E into the projective space $\mathbb{CP}^n$, we deal with the Bernstein–Markov property of (E, μ, ω) when E is assumed to be locally regular in $\mathbb{CP}^n$. Theorem 4.5 implies, in particular, that (E, μ, ω) has the Bernstein–Markov property if E is locally regular in $\mathbb{CP}^n$ and μ is, roughly speaking, determining on a subset of E. For the notion of determining measures, see the next section. We also give a version of the Bernstein–Markov property in the case where μ is determining but E is not assumed to be regular (see Proposition 4.8). In the final result of the section, following the work of Bloom and Shiffman [BSh], we give an application of the Bernstein–Markov property to the convergence of certain Bergman kernels to weighted Green functions. This problem was studied earlier in [BSh] when E is *bounded.*

2. Preliminaries. We first recall the definition of Lelong classes in $\mathbb{C}^n$. The set of plurisubharmonic functions with logarithmic growth is given by

$$\mathcal{L}(\mathbb{C}^n) = \{u \text{ plurisubharmonic on } \mathbb{C}^n : u(z) \le \log^+ |z| + C\},$$

where C is a constant depending on u, but not on z. Similarly,

$$\mathcal{L}^+(\mathbb{C}^n) = \{u \in \mathcal{L}(\mathbb{C}^n) : u(z) \ge \log^+ |z| - C\},$$

where again C may depend on u. For brevity, we write $\mathcal{L}$ (resp. $\mathcal{L}^+$) for $\mathcal{L}(\mathbb{C}^n)$ (resp. $\mathcal{L}^+(\mathbb{C}^n)$). Lelong classes are used to define global extremal functions associated to E, Q. More precisely, the *weighted Green function* of E with weight ω is defined as

$$V_{E,Q}(z) := \sup\{u(z) : u \in \mathcal{L}, u|_E \le Q\},$$

where $Q := -\log\omega$. We will use Q and ω alternatively. In the unweighted case (i.e., $Q = 0$), $V_{E,Q}$ becomes Siciak's extremal function V_E.

For a subset G of $\mathbb{C}^n$ and a function $f : G \to [-\infty, \infty)$, we let f^* denote its usc regularization, defined by

$$f^*(\xi) = \overline{\lim_{z\to\xi,\, z\in G}} f(z), \qquad \forall \xi \in \bar{G}.$$

It is easy to see that if E is non-pluripolar and ω is an admissible weight then $V_{E,Q}^* \in \mathcal{L}^+$. We will sometimes use the following weighted version of the classical Bernstein–Walsh inequality: For every Borel set $F \subset E$,

$$\|\omega^d P\|_E \le \|\omega^d P\|_F e^{\sup_E V_{F,Q}}.$$

We say that a Borel set E is *regular* if V_E is continuous on $\mathbb{C}^n$, and E is *locally regular* at the point $a \in \bar{E}$ if for every $r > 0$ small enough, the function $V_{E\cap B(a,r)}$ is continuous at a, where $B(a,r)$ is the ball centered at a and of radius r. The set E is locally regular if it is locally regular at every point $a \in \bar{E}$. It is known that E is regular if and only if $V_E^* = 0$ on E. Moreover, if E is locally regular and compact then, by a result of Siciak [Sic1], $V_{E,Q}$ is continuous for every continuous function Q.

Now, we recall some facts about the connection between quasi-plurisubharmonic functions on the projective space $\mathbb{CP}^n$ and functions in the Lelong class $\mathcal{L}$. More explicitly, let $\theta := \frac{1}{2} dd^c \log(1 + |z|^2)$ be the Fubini–Study Kähler form on $\mathbb{CP}^n$. Denote by $\mathrm{PSH}(\mathbb{CP}^n, \theta)$ the set of upper semicontinuous functions $\varphi : X \to [-\infty, +\infty)$ such that $\theta + dd^c\varphi \ge 0$ on $\mathbb{CP}^n$. Then there exists a one-to-one correspondence between $\mathcal{L}$ and $\mathrm{PSH}(\mathbb{CP}^n, \theta)$ induced by the natural mapping

$$\varphi \in \mathcal{L} \mapsto \overline{\varphi}(z) = \begin{cases} \varphi(z) - \frac{1}{2}\log(1 + |z|^2) & \text{if } z \in \mathbb{C}^n, \\ \overline{\lim_{\mathbb{C}^n \ni w \to z}} [\varphi(w) - \frac{1}{2}\log(1 + |w|^2))] & \text{if } z \in H_\infty, \end{cases}$$

where $H_\infty = \mathbb{CP}^n \backslash \mathbb{C}^n$ denotes the hyperplane at infinity. One can easily see that

$$\varphi \in \mathcal{L}^+ \quad \text{if only if} \quad \overline{\varphi} \in \mathrm{PSH} \cap L^\infty(\mathbb{CP}^n, \theta).$$

It is also not hard to check that

$$\overline{V_{E,Q}^*} = (\sup\{\psi(z) : \psi \in \mathrm{PSH}(\mathbb{CP}^n, \theta),\, \psi \le \overline{Q} \text{ on } E\})^*,$$

where $\overline{Q}(z) = Q(z) - \frac{1}{2}\log(1 + |z|^2)$.

Next, given $E \subset \mathbb{C}^n$ and a real valued function Q defined on E, we define the closure of E in $\mathbb{CP}^n$ as follows:

$$\overline{E}_{\mathbb{CP}^n} := \{z \in \mathbb{CP}^n : \exists \{w_j\} \subset E \text{ such that } w_j \to z \text{ in } \mathbb{CP}^n\},$$

and we set

$$\overline{Q}_*(z) = \underline{\lim}_{w\to z} \overline{Q}(w), \qquad \overline{Q}^*(z) = \overline{\lim_{w\to z}} \overline{Q}(w).$$

For more background on the above material, we refer the reader to [GZ], [BSt] and [BT].

Next, we recall some facts about pluripolar sets. A subset X in $\mathbb{C}^n$ is said to be *pluripolar* if for every $a \in X$, there exist a neighbourhood U of a and a plurisubharmonic function u with $u \not\equiv -\infty$ on U such that $u \equiv -\infty$ on $X \cap U$. An important theorem of Josefson asserts that it is possible to choose $U = \mathbb{C}^n$. Later on, Siciak [Sic1] improved this result by proving that u can be taken in the class $\mathcal{L}$. The usage of pluripolar sets arises naturally from Bedford–Taylor's theorem on negligible sets which says that for every family $\{u_\alpha\}_{\alpha \in I}$ of plurisubharmonic functions locally bounded from above on an open set Ω, the set $\{z \in \Omega : u(z) < u^*(z)\}$ is pluripolar, where $u(z) := \sup_{\alpha \in I} u_\alpha(z)$. Using this result, it is proved in [CKL] that for every sequence E_j of Borel sets that increase to a *bounded* set E, we have $V^*_{E_j} \downarrow V^*_E$. A generalization of this fact to weighted Green functions is given in Lemma 4.6. See also Proposition 3 in [BSt] for an analogous result when $\mathbb{C}^n$ is replaced by a compact Kähler manifold.

A property $\mathcal{P}$ is said to hold quasi everywhere (q.e. for short) on a set E if $\mathcal{P}$ is true outside a pluripolar subset of E. For a comprehensive discussion of pluripolar sets, we refer the reader to the monograph [Kl].

Now, we deal with notions which are relevant to the Bernstein–Markov property. The following kind of measures was introduced by Siciak [Sic2].

Definition 2.1. Let E be Borel, non-pluripolar subset of $\mathbb{C}^n$ and ω be an admissible weight on E. We say that a positive Borel measure μ on E is (E,ω)-*determining* if for every Borel subset F of E such that $\mu(E \setminus F) = 0$ we have $V^*_{F,Q} = V^*_{E,Q}$.

The role of determining measures is highlighted in the following important result.

Theorem 2.2. *Let E be a compact non-pluripolar subset of $\mathbb{C}^n$ and ω be an admissible weight on E. Assume that $\omega > 0$ on E and $V^*_{E,Q} \le Q$ on E. Then for every (E,ω)-determining measure μ, the triple (E,μ,ω) has the Bernstein–Markov property.*

The above theorem was proved by Siciak [Sic2] (see also [Le, Proposition 2.5]) under the stronger assumption that Q is *continuous* on E. However, the proof given there works equally well in our case. Later on, we will provide a generalization of Theorem 2.2 in the case where E is unbounded and μ is determining on a subset of E.

It is easy to see that if μ is a determining measure for (E,ω) then so is $f\mu$ for every positive continuous function f. Let $E = \bar{D}$, where D is an open set (possibly unbounded) in $\mathbb{C}^n$ such that ∂D is $\mathcal{C}^1$ smooth. Then for every admissible weight function ω, the Lebesgue measure $d\lambda_{2n}$ and the surface measure $dV_{\partial D}$ are determining for (E,ω). More subtle examples of determining measures are provided by the following well known fact.

PROPOSITION 2.3. *Let E be a closed, non-pluripolar subset of $\mathbb{C}^n$ and ω be an admissible weight on E. Then the measure $\mu := (dd^c V^*_{E,Q})^n$ is (E,ω)-determining.*

Proof. Since $\mu(E\setminus F)=0$ and E is non-pluripolar, we infer that F is also non-pluripolar. On the other hand, since $V^*_{E,Q}\le V^*_{F,Q}$ on $\mathbb{C}^n$, we see that $V^*_{F,Q}\in\mathcal{L}^+$. Note that $V^*_{F,Q}=V^*_{E,Q}$ a.e. on $E=\operatorname{supp}\mu$ with respect to the measure μ. Thus by Lemma 6.5 in [BT] we get $V^*_{F,Q}\le V^*_{E,Q}$ on $\mathbb{C}^n$. Hence equality holds everywhere on $\mathbb{C}^n$.

It is of interest to find a mass density condition on μ implying that (E,μ,ω) has the Bernstein–Markov property. In the unweighted case $(\omega=1)$, an efficient condition is provided by the following result due to Bloom and Levenberg [BL1].

THEOREM 2.4 (Bloom–Levenberg's theorem). *Let E be a regular compact set in $\mathbb{C}^n$ and μ be a positive Borel measure on E. Assume that there exists a constant $T>0$ such that $V^*_{E_r}$ goes to 0 q.e. on E when $r\to 0+$, where*

$$E_r := \{z\in E : \mu(E\cap\mathbb{B}(z,r))\ge r^T\}.$$

Then (E,μ) has the Bernstein–Markov property.

Proof. For the reader's convenience, we indicate briefly how Theorem 2.4 follows from Theorems 1.1 and 2.1 of [BL1]. Assume that $\bar E\subset\mathbb{B}$, the unit ball in $\mathbb{C}^n$. By step VI in the proof of Theorem 1.1 of [BL1], $u_r\to u$ pointwise on $\mathbb{B}$, where u_r (resp. u) is the relative extremal function of E_r (resp. E). Using again Theorem 1.1 of [BL1] we obtain $\lim_{j\to\infty} C(E_r)=C(E)$, where $C(F)$ is the relative capacity of the Borel set $F\subset\mathbb{B}$. It now follows from the *proof* of Theorem 2.1 of [BL1] that (E,μ) has the Bernstein–Markov property.

REMARKS. (a) Theorem 2.4 generalizes Theorem 4.1 of [Bl1]. In the latter theorem, the measure μ is assumed to be sufficiently dense on a set of full relative capacity. The essence of Theorem 2.4 is that, in fact, one only needs a type of denseness in the mean.

(b) By making the constant T larger, we can see that Theorem 2.4 still holds if we replace the balls $\mathbb{B}(z,r)$ by the polydisks $\Delta(z,r)$.

(c) According to Theorem 4.2 in [Bl1] every determining measure μ (in the sense of Definition 2.1) on a regular compact set E satisfies the mass density condition of [Bl1, Theorem 4.1]. Thus (E,μ) has the Bernstein–Markov property. We recover the unweighted case of Theorem 2.2.

(d) Bloom constructed a discrete measure (a countable linear combination of Dirac measures) μ such that (E,μ) has the Bernstein–Markov property ([Bl1, Example 4.1]). Note that, being a measure carried by a pluripolar set, μ *cannot* be determining for E.

Before closing this section, we remark that in previous work of Bloom and Levenberg (e.g., [BL2] and [Le]), when E is unbounded, the weight ω is supposed to decrease fast enough, i.e., $\lim_{|z|\to\infty} z\omega(z) = 0$. Under this strong growth condition, $V_{E,Q} = V_{E\cap\mathbb{B}(0,r),Q}$ for r large enough. Thus we return to the bounded case.

3. Reductions to the bounded case. We start with the following basic result, due in essence to Bloom, relating the Bernstein–Markov property of unbounded sets to bounded ones.

THEOREM 3.1. *Let ω be an admissible usc weight on E. Define*

$$Z := \{(tz_1,\dots,tz_n,t) \in \mathbb{C}^{n+1} : (z_1,\dots,z_n) \in E,\, t \in \mathbb{C},\, |t| = \omega(z)\}.$$

Let ν be a measure on Z defined by $d\nu := dm_t \otimes d\mu$, where dm_t is the normalized Lebesgue measure on the circle $\{|t| = \omega(z)\}$. Then the following assertions are equivalent:

(a) (E,μ,ω) *has the Bernstein–Markov property* (*in* $\mathbb{C}^n$).
(b) (Z,ν) *has the Bernstein–Markov property* (*in* $\mathbb{C}^{n+1}$).

The definition of ν means that for every continuous function φ with compact support in $\mathbb{C}^{n+1}\setminus\{t=0\}$,

$$\int_{\mathbb{C}^{n+1}} \varphi\, d\nu = \int_E \Big(\int_{|t|=\omega(z)} \varphi\, dm_t\Big)\, d\mu(z).$$

Notice that Z is always bounded since $\sup_{z\in E}|z|\omega(z) < \infty$. Thus it makes sense to talk about the Bernstein–Markov property of (Z,ν).

Proof. The above theorem is essentially Theorem 3.1 in [Bl3]. For the reader's convenience, we sketch some details. For a polynomial P of degree d in $\mathbb{C}[z_1,\cdots,z_n]$, we define the following *homogeneous* polynomial of degree d in $\mathbb{C}[z_1,\cdots,z_n,t]$:

$$\tilde{P}(z_1,\cdots,z_n,t) := t^d P(z/t).$$

It is not hard to show the following facts ([Bl3, Lemmas 3.1 and 2.1]):

(i) $\|\tilde{P}\|_{L^2(Z,\nu)} = \|\omega^d P\|_{L^2(E,\mu)}$.
(ii) $\|\tilde{P}(z,t)\|_Z = \|\omega^d P\|_E$.

From the above relations, we can prove the equivalence of (a) and (b).

To utilize the above result, the first problem is to decide when $\bar{Z}$ is regular. We will give a characterization of regularity of $\bar{Z}$ in Proposition 4.4. For the moment, we are content with the following simple observations:

PROPOSITION 3.2. *Let E, ω, μ and Z be as in Theorem* 3.1. *Suppose that E is closed and $\bar{Z}$ is regular in $\mathbb{C}^{n+1}$. Then:*

(a) *$V_Z = V_{\bar{Z}}$ on $\mathbb{C}^{n+1}$.*
(b) *$V_{E,Q}$ is continuous on $\mathbb{C}^n$.*

Proof. (a) For *E bounded*, the formula below follows from Theorem 2.1 and Proposition 2.2 of [Bl3]:

$$V_Z(tz,t) = \max\{V_{E,Q}(z) + \log|t|, 0\}, \quad \forall z \in \mathbb{C}^n,\ \forall t \in \mathbb{C}. \tag{1}$$

However, (1) also holds in our context. This is a consequence of a more general result due to Branker and Stawiska (see [BSt, Theorem 2]). For the reader's convenience, we give a direct proof of this crucial formula. Since Z is circular and bounded, we need only consider homogeneous polynomials in the definition of V_Z: if we write a polynomial p of degree d as $p = \sum_{0\le j\le d} p_j$ where p_j is a homogeneous polynomial of degree j then by the Cauchy inequalities we obtain

$$\|p_j\|_Z \le \|p\|_Z, \quad \forall 0 \le j \le d.$$

Next it suffices to apply Siciak's H-principle ([BL2, p. 61]) to obtain (1). Returning to our setup, since E is closed, $\bar{Z} \setminus Z \subset \mathbb{C}^n \times \{0\}$ is pluripolar in $\mathbb{C}^{n+1}$. Therefore $V_{\bar{Z}} = V^*_{\bar{Z}} = V^*_Z$ on $\mathbb{C}^{n+1}$. In particular $V^*_Z = 0$ on Z and $V_{\bar{Z}} = V^*_Z = V_Z$ is continuous on $\mathbb{C}^{n+1}$.

(b) We just apply Bloom's formula (1) and (a) to get continuity of $V_{E,Q}$.

REMARKS. (i) Let E be a closed, locally regular subset of $\mathbb{C}^n$ and $\omega > 0$ a continuous admissible weight on E. Assume that for every $\xi \in \bar{Z}\setminus(Z\cup\{0\})$ there exists a complex line l passing through ξ such that $l \cap \bar{Z}$ contains a continuous curve $\gamma([0,1])$ with $\gamma(0) = \xi$ and $\gamma((0,1]) \subset l \cap Z$. Then $\bar{Z}$ is regular. Indeed, since E is closed and locally regular, by a result of Siciak we have $V^*_{E,Q} \le Q$ on E. Combining this with Bloom's formula (1), we obtain $V^*_Z = 0$ on Z. By Proposition 3.2(a) we have $V^*_{\bar{Z}} = 0$ on Z. Next, we fix $\xi \in \bar{Z} \setminus (Z \cup \{0\})$. Choose a complex line l as in the assumption. Denote by u the restriction of $V^*_{\bar{Z}}$ to l. We may regard u as a subharmonic function on $\mathbb{C}$. By the above reasoning $u(z) = 0$ for every $z \in l \cap Z$. According to a classical result of potential theory saying that a continuous curve is non-thin at every point (see [Ra, Theorem 3.8.3]) we have

$$V^*_{\bar{Z}}(\xi) = u(\xi) = 0.$$

Finally, we consider the case $\xi = 0$. Fix $z^* \in E$ and define $v(t) = V^*_{\bar{Z}}(tz^*, t)$. Then $v \ge 0$ is subharmonic on $\mathbb{C}$, and $v = 0$ on the circle $|t| = \omega(z^*)$. So by the maximum principle,

$$V^*_{\bar{Z}}(0) = v(0) = 0.$$

Thus $V^*_{\bar{Z}} = 0$ on $\bar{Z}$ and hence $\bar{Z}$ is regular as claimed.

(ii) We construct below an explicit example of a pair (E, ω) satisfying the assumptions given in (a). Let $E = \mathbb{C}$. We choose a continuous function $\omega > 0$ on E satisfying $\omega(z) = |\mathrm{Re}(1/z)|$ for $|z| \geq 1$ and $0 \leq \omega(z) \leq 1$ for $|z| \leq 1$. This is possible, because by Tietze's extension theorem, we may extend continuously, preserving the norm, the function $|\mathrm{Re}(1/z)|$ from the unit circle $|z| = 1$ to the closed unit disk $|z| \leq 1$. Note that $|z|\omega(z) \leq 1$ for all $z \in \mathbb{C}$. Thus ω is a continuous admissible weight on E. We claim that the pair (E, ω) satisfies the assumptions in (a). Fix $\xi := (\alpha, 0) \in \bar{Z} \setminus (Z \cup \{0\})$. Then $0 < |\alpha| \leq 1$. Let l be the complex line $\{(\alpha, t) : t \in \mathbb{C}\}$. It is easy to see that

$$\{(\alpha, t) : |t| = \omega(\alpha/t) = |\mathrm{Re}(t/\alpha)|, \, 0 < |t| < |\alpha|\} \subset l \cap Z.$$

It follows easily from the above relation and from the properties of α that $l \cap \bar{Z}$ contains a segment $\gamma([0, 1])$ with $\gamma((0, 1]) \subset l \cap Z$ and $\gamma(0) = \xi$. The claim now follows.

We now apply Theorem 3.1 in conjunction with Theorem 2.4 to obtain a certain sufficient condition for the Bernstein–Markov property of (E, μ, ω) in terms of convergence of sequences of weighted Green functions.

THEOREM 3.3. *Under the notation of Theorem* 3.1, *assume that $\bar{Z}$ is regular* (*in $\mathbb{C}^{n+1}$*). *Denote by α the function*

$$\alpha(z) := 1 + |z| + \|\omega\|_E, \qquad z \in \mathbb{C}^n.$$

Assume that $V^*_{E_r,Q} \to V^*_{E,Q}$ pointwise on E as $r \to 0+$, where

$$E_r := \left\{ z^0 \in E : \frac{1}{\alpha(z^0)\sqrt{\omega(z^0)}} \int_{E \cap \Delta_\omega(z^0, r/\alpha(z^0))} \frac{d\mu(z)}{\sqrt{\omega(z)}} \geq r^T \right\},$$

and $\Delta_\omega(z^0, r) := \{z \in \Delta(z^0, r) : |\omega(z) - \omega(z_0)| < r/2\}$. Then (E, μ, ω) has the Bernstein–Markov property.

Proof. For $r > 0$ we set

$$Z_r := \{\tilde{z} \in Z : \nu(\bar{Z} \cap \tilde{\Delta}(\tilde{z}, r)) \geq r^{T+2}\},$$

where $\tilde{\Delta}(\tilde{z}, r)$ is the polydisks in $\mathbb{C}^{n+1}$, centred at $\tilde{z}$ and radius r. We will show that $V^*_{Z_r} \to 0$ pointwise on Z as $r \to 0+$. Granted this, we conclude the proof as follows. Since $\bar{Z} \setminus Z \subset \mathbb{C}^n \times \{0\}$ is pluripolar we have $V^*_{Z_r} \to 0$ q.e. on $\bar{Z}$. Thus $V^*_{Z'_r} \to 0$ q.e. on $\bar{Z}$ where

$$Z'_r := \{\tilde{z} \in \bar{Z} : \nu(\bar{Z} \cap \tilde{\Delta}(\tilde{z}, r)) \geq r^{T+2}\}.$$

By Bloom–Levenberg's theorem (see the remark following Theorem 2.4), $(\bar{Z},\nu)$ has the Bernstein–Markov property. Using Theorem 3.1 we find that (E,μ,ω) also has the Bernstein–Markov property.

Thus, the key point is to check that $V^*_{Z_r} \to 0$ on Z as $r \to 0+$. Let $\pi : \mathbb{C}^{n+1} \setminus \{z_{n+1} = 0\} \to \mathbb{C}^n$ be defined by

$$\pi(z_1,\dots,z_{n+1}) := (z_1/z_{n+1},\dots,z_n/z_{n+1}).$$

Since Z is circular, we can easily check by the definition of ν that Z_r is also circular. Then it follows from Bloom's formula (1) that

$$V^*_{Z_r}(tz,t) = \max\{V^*_{\pi(Z_r),Q}(z) + \log|t|, 0\}, \quad \forall z \in \mathbb{C}^n,\ \forall t \in \mathbb{C}.$$

Thus it suffices to check that $V^*_{\pi(Z_r),Q}$ goes pointwise to $V^*_{E,Q}$ on E. For this, we first claim that given $\tilde{z}^0 = (t^0 z^0, t^0) \in Z$, where $z^0 = (z^0_1,\dots,z^0_n) \in Z$ and $|t_0| = \omega(z^0)$, for every $0 < r < 1$ we have

$$X_r(\tilde{z}^0) := \{(tz,t) : z \in E \cap \Delta(z^0, r/\alpha(z^0)), |t-t^0| < r/\alpha(z^0)\} \subset \tilde{\Delta}(\tilde{z}^0, r). \tag{2}$$

Indeed, if $|z_j - z^0_j| < r/\alpha(z^0)$ and $|t-t^0| < r/\alpha(z^0)$ then

$$|tz_j - t^0 z^0_j| \le |t^0|\,|z_j - z^0_j| + |z_j|\,|t-t^0| < \alpha(z^0)\frac{r}{\alpha(z^0)} = r.$$

This proves (2). Thus for every $\tilde{z}^0 \in Z$ we have

$$\nu(Z \cap X_r(\tilde{z}^0)) \le \nu(\bar{Z} \cap \tilde{\Delta}(\tilde{z}^0, r)), \quad \forall 0 < r < 1.$$

It follows that

$$Y_r := \{\tilde{z}^0 \in Z : \nu(Z \cap X_r(\tilde{z}^0)) \ge r^{T+2}\} \subset Z_r.$$

Hence

$$V^*_{E,Q} \le V^*_{\pi(Z_r),Q} \le V^*_{\pi(Y_r),Q}.$$

On the other hand, by definition of ν we have

$$\nu(Z \cap X_r(\tilde{z}^0)) = \int\limits_{E\cap\Delta(z^0, r/\alpha(z^0))} \varphi(z)\,d\mu(z), \tag{3}$$

where $\varphi(z) := m\{t : |t| = \omega(z),\ |t-t^0| < r/\alpha(z^0)\}$. We need a lower bound for $\varphi(z)$. After a rotation, we may assume $t^0 = \omega(z^0)$. We note the following elementary fact: if $t = e^{i\theta}\omega(z)$ then

$$|t-t^0| < r/\alpha(z^0) \ \Leftrightarrow\ |\omega(z) - \omega(z^0)| < r/\alpha(z^0),$$

and

$$\frac{\omega(z)^2 + \omega(z^0)^2 - r^2/\alpha(z^0)^2}{2\omega(z)\omega(z^0)} < \cos\theta.$$

Since $\cos\theta \geq 1 - \theta^2/2$ we infer that for $|\omega(z) - \omega(z^0)| < r/\alpha(z^0)$,

$$|\theta| \leq \left\{ \frac{\frac{r^2}{\alpha(z^0)^2} - (\omega(z) - \omega(z^0))^2}{\omega(z)\omega(z^0)} \right\}^{1/2} \Rightarrow \frac{\omega(z)^2 + \omega(z^0)^2 - \frac{r^2}{\alpha(z^0)^2}}{2\omega(z)\omega(z^0)} < \cos\theta.$$

This implies that for $z \in \Delta_\omega(z^0, r/\alpha(z^0))$ we have

$$\varphi(z) \geq \frac{\sqrt{3}}{2} \frac{r}{\alpha(z^0)\sqrt{\omega(z)\omega(z^0)}}.$$

Now we apply this lower bound for φ to get from (3) the estimate

$$\nu(Z \cap X_r(\tilde{z}^0)) \geq \frac{\sqrt{3}r}{2\alpha(z^0)\sqrt{\omega(z^0)}} \int_{E\cap\Delta_\omega(z^0, r/\alpha(z^0))} \frac{d\mu(z)}{\sqrt{\omega(z)}}.$$

Therefore

$$\left\{ \tilde{z}^0 \in Z : \frac{1}{\alpha(z^0)\sqrt{\omega(z^0)}} \int_{E\cap\Delta_\omega(z^0, r/\alpha(z^0))} \frac{d\mu(z)}{\sqrt{\omega(z)}} \geq \frac{2r^{T+1}}{\sqrt{3}} \right\} \subset Y_r.$$

So $E_r \subset \pi(Y_r)$ for $0 < r < \sqrt{3}/2$. By the assumptions we get $V^*_{\pi(Y_r),Q} \to V^*_{E,Q}$ on E. The proof is complete.

The following result is an almost immediate consequence of Theorem 3.3. In the case $\omega = 1$, it is just Bloom–Levenberg's theorem (see e.g. Theorem 2.4).

COROLLARY 3.4. *Let E be a compact, locally regular set in $\mathbb{C}^n$ and $\omega > 0$ be a continuous weight on E. Assume that there exists a constant $T > 0$ such that $V^*_{E_r,Q}$ converges pointwise to $V^*_{E,Q}$, where*

$$E_r := \{z^0 \in E : \mu(E \cap \Delta_\omega(z^0, r/T)) \geq r^T\}.$$

Then (E, μ, ω) has the Bernstein–Markov property.

Proof. Since E is compact, we have $\|\alpha\|_E < \infty$. Thus we can choose T large enough such that $\Delta_\omega(z^0, r/T) \subset \Delta_\omega(z^0, r/\alpha(z^0))$ for every $z^0 \in E$. Furthermore, by considering $T+1$ instead of T we have

$$\{z^0 \in E : \mu(E \cap \Delta_\omega(z^0, r/T)) \geq r^T\}$$
$$\subset \left\{ z^0 \in E : \frac{1}{\alpha(z^0)\sqrt{\omega(z^0)}} \int_{E\cap\Delta_\omega(z^0,r/\alpha(z^0))} \frac{d\mu(z)}{\sqrt{\omega(z)}} \geq r^T \right\}.$$

The proof is complete.

In the same spirit as in Theorem 3.1, we deal with another situation when E can be transformed to bounded sets by "invertible" maps. We need a piece of notation: for $1 \leq j \leq n$, denote by π_j the projection $\pi_j : \mathbb{C}^n \to \mathbb{C}$, $\pi_j(z) := z_j$.

PROPOSITION 3.5. *Let E be a closed, unbounded subset of $\mathbb{C}^n$. Assume that there exists $1 \le k \le n$ satisfying the following conditions:*

(a) *$\pi_j(E)$ is unbounded for every $1 \le j \le k$ and bounded for every $j > k$.*
(b) *$0 \notin \pi_j(E)$ for every $1 \le j \le k$.*
(c) *$\sup_E |z_1 \cdots z_k| \omega(z) := M < \infty$.*

Let $\tilde{E} := \{(1/z_1, \dots, 1/z_k, z_{k+1}, \dots, z_n) : (z_1, \dots, z_n) \in E\}$. Define the following function on $\bar{\tilde{E}}$:

$$\tilde{\omega}(z) := \begin{cases} \dfrac{\omega(1/z_1, \dots, 1/z_k, z_{k+1}, \dots, z_n)}{z_1 \cdots z_k}, & (z_1, \dots, z_n) \in \tilde{E}, \\ M, & z \in \bar{\tilde{E}} \setminus \tilde{E}. \end{cases}$$

Suppose that $(\bar{\tilde{E}}, \tilde{\mu}, \tilde{\omega})$ has the Bernstein–Markov property, where $\tilde{\mu}$ is the push-forward measure of μ under $\tilde{\omega}$, i.e., $\tilde{\mu}(A) := \mu(\tilde{\omega}^{-1}(A \cap \tilde{E}))$ for every $A \subset \bar{\tilde{E}}$. Then (E, μ, ω) also has the Bernstein–Markov property.

Proof. For a polynomial of P of degree d in $\mathbb{C}[z_1, \dots, z_n]$, we set

$$\tilde{P}(z_1, \cdots, z_n) := (z_1 \cdots z_k)^d P(1/z_1, \dots, 1/z_k, z_{k+1}, \dots, z_n).$$

Note that P is a polynomial of degree $\le (k+1)d$. Furthermore

$$\|\omega^d P\|_E = \|\tilde{\omega}^d \tilde{P}\|_{\tilde{E}}, \|\omega^d P\|_{L^2(E,\mu)} = \|\tilde{\omega}^d \tilde{P}\|_{L^2(\tilde{E},\tilde{\mu})} = \|\tilde{\omega}^d \tilde{P}\|_{L^2(\bar{\tilde{E}},\tilde{\mu})}. \tag{4}$$

Given $\varepsilon > 0$, by the assumption, we can find $C_\varepsilon > 0$ such that

$$\|\tilde{\omega}^d \tilde{P}\|_{\bar{\tilde{E}}} \le C_\varepsilon (1+\varepsilon)^{(k+1)d} \|\tilde{\omega}^d \tilde{P}\|_{L^2(\bar{\tilde{E}},\tilde{\mu})}. \tag{5}$$

Putting (4) and (5) together we obtain

$$\|\omega^d P\|_E \le C_\varepsilon (1+\varepsilon)^{(k+1)d} \|\omega^d P\|_{L^2(E,\mu)}.$$

The proof is complete.

The result below is a simple consequence of the above proposition.

COROLLARY 3.6. *Let $E \subset \mathbb{C}$ be a closed, unbounded, locally regular set such that $0 \notin E$, $\omega > 0$ a continuous admissible weight on E, and μ a determining measure for (E, ω). Suppose that there exists a continuous map $\gamma : [0,1) \to E$ satisfying $\lim_{t \to 1} |\gamma(t)| = \infty$. Then (E, μ, ω) has the Bernstein–Markov property.*

Proof. Set $\tilde{E} := \{1/z : z \in E\}$. Then $\tilde{E}$ is locally regular. We note that there exists $R > 0$ such that for every $r > R$ we have

$$\emptyset \ne \{|z| = r\} \cap \gamma[0,1) \subset \{|z| = r\} \cap \tilde{E}.$$

This implies that 0 is not a thin point for $\tilde{E}$. Now we define

$$\tilde{\omega}(z) := \begin{cases} \dfrac{1}{z}\omega\left(\dfrac{1}{z}\right), & z \in \tilde{E}, \\ \sup\limits_{\xi\in E} |\xi\omega(\xi)|, & z = 0. \end{cases}$$

Since $\tilde{E}$ is locally regular, not thin at 0 and $\tilde{\omega} > 0$ is continuous on $\tilde{E}$, by the choice of $\tilde{\omega}$, we have $V^*_{\bar{\tilde{E}},\tilde{Q}} \le \tilde{Q}$ on $\bar{\tilde{E}}$. Next, observe that $\bar{\tilde{E}} \setminus \tilde{E} = \{0\}$ is polar in $\mathbb{C}$, so

$$V^*_{\bar{\tilde{E}},\tilde{\omega}}(z) = V^*_{\tilde{E},\tilde{\omega}}(z) = V^*_{E,\omega}(1/z), \quad \forall z \neq 0.$$

By the assumption on μ we infer that the measure $\tilde{\mu}$ is $(\bar{\tilde{E}}, \tilde{\omega})$-determining. Thus $(\bar{\tilde{E}}, \tilde{\mu}, \tilde{\omega})$ has the Bernstein–Markov property. Applying Proposition 3.5 we reach the desired conclusion.

4. Bernstein–Markov property and weighted Green functions.

This section is devoted to some sufficient conditions for (E, μ, ω) to have the Bernstein–Markov property in terms of behaviour of weighted Green functions.

PROPOSITION 4.1. *Let ω be an admissible weight on E, and μ be a positive Borel measure on E. Assume that for every $\varepsilon > 0$, there exists a Borel, non-pluripolar subset E_ε of E satisfying the following conditions:*

(a) *$(E_\varepsilon, \mu, \omega)$ has the Bernstein–Markov property.*
(b) *$V_{E_\varepsilon,Q} \le Q + \log(1 + \varepsilon)$ on $E \setminus E_\varepsilon$.*

Then (E, μ, ω) also has the Bernstein–Markov property.

Proof. We define

$$\mathcal{F} := \{P \in \mathbb{C}[z_1, \cdots, z_n] : \|\omega^{\deg P} P\|_{L^2(E,\mu)} \le 1\}.$$

Given $\varepsilon > 0$, it suffices to prove that there exists $C_\varepsilon > 0$ such that

$$\|\omega^{\deg P} P\|_E \le C_\varepsilon (1 + \varepsilon)^{\deg P}, \quad \forall P \in \mathcal{F}.$$

Fix $P \in \mathcal{F}$ and choose $\varepsilon' > 0$ such that $(1 + \varepsilon')^2 < 1 + \varepsilon$. By assumption we can choose a closed subset $E_{\varepsilon'} \subset E$ satisfying (a)–(c). Thus, there exists $C_{\varepsilon'} > 0$ such that

$$\|\omega^{\deg P} P\|_{E_{\varepsilon'}} \le C_{\varepsilon'}(1 + \varepsilon')^{\deg P}\|\omega P\|_{E_{\varepsilon'}} \le C_{\varepsilon'}(1 + \varepsilon')^{\deg P}. \tag{6}$$

It follows that

$$\frac{1}{\deg P} \log \frac{|P|}{C_{\varepsilon'}} \le Q + \log(1 + \varepsilon') \quad \text{on } E_{\varepsilon'}.$$

This implies

$$\frac{1}{\deg P}\log\frac{|P|}{C_{\varepsilon'}} \le V_{E_{\varepsilon'},Q} + \log(1+\varepsilon') \quad \text{ on } \mathbb{C}^n.$$

Therefore

$$|\omega(z)^{\deg P}P(z)| \le C_{\varepsilon'}(\omega(z)e^{V_{E_{\varepsilon'},Q}(z)})^{\deg P}(1+\varepsilon')^{\deg P}, \quad \forall z \in \mathbb{C}^n.$$

In particular, in view of (b),

$$(7) \quad |\omega(z)^{\deg P}P(z)| \le C_{\varepsilon'}(1+\varepsilon')^{2\deg P} = C_{\varepsilon'}(1+\varepsilon)^{\deg P}, \quad \forall z \in E\setminus E_{\varepsilon'}.$$

Combining (6) and (7) we complete the proof.

Using the above result, we deal with the case where ω is assumed to satisfy the strong growth condition of Bloom and Levenberg.

Corollary 4.2. *Suppose that there exists a sequence $r_j \uparrow \infty$ such that $(E\cap B(0,r_j),\mu,\omega)$ has the Bernstein–Markov property for every j and*

$$\lim_{|z|\to\infty} |z|\omega(z) = 0.$$

Then (E,μ,ω) has the Bernstein–Markov property.

Proof. Fix $\varepsilon > 0$. We can choose $a > 0$ large enough such that $E\cap B(0,a)$ is non-pluripolar. Then for every $r > a$ we have

$$V^*_{E\cap B(0,r),Q}(z) \le V^*_{E\cap \bar{B}(0,a),Q}(z) \le \log^+|z| + M \quad \text{ on } \mathbb{C}^n,$$

where $M > 0$ is a constant independent of r. Note that, by assumption,

$$\lim_{|z|\to\infty} (Q(z) - \log|z|) = +\infty.$$

Therefore, we may choose j large enough such that

$$V_{E\cap B(0,r_j),Q}(z) \le Q(z) + \log(1+\varepsilon), \quad \forall |z| > r_j.$$

By Proposition 4.1 we get the desired conclusion.

We have the following simple consequence.

Corollary 4.3. *Let ω be an admissible weight on $\mathbb{C}^n$. Let $\omega' > 0$ be a usc function on $\mathbb{C}^n$ satisfying $\lim_{|z'|\to\infty}\omega'(z) = 0$. Then for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for every polynomial P of degree d,*

$$|\omega(z)^d P(z)| \le C_\varepsilon(1+\varepsilon)^d \frac{\|\omega'^d\|_{L^2(\mathbb{C}^n,d\lambda_{2n})}}{\omega'(z)^d}\|\omega^d P\|_{L^2(\mathbb{C}^n,d\lambda_{2n})}, \quad \forall z \in \mathbb{C}^n,$$

where $d\lambda_{2n}$ is the Lebesgue measure on $\mathbb{C}^n$.

Proof. We set $\omega'' := \omega\omega'$. Then ω'' is an admissible weight on $\mathbb{C}^n$ and

$$\lim_{|z|\to\infty} |z|\omega''(z) = 0.$$

By Theorem 2.2, the triple $(B(0,j), d\lambda_{2n}, \omega'')$ has the Bernstein–Markov property for every j. Thus, by Corollary 4.2, so does $(\mathbb{C}^n, d\lambda_{2n}, \omega'')$. Hence there exists a constant $C_\varepsilon > 0$ such that for every polynomial P of degree d we have

$$\begin{aligned}\|\omega''^d P\|_{\mathbb{C}^n} &\le C_\varepsilon(1+\varepsilon)^d \|\omega'' P\|_{L^2(\mathbb{C}^n, d\lambda_{2n})}\\ &\le C_\varepsilon(1+\varepsilon)^d \|\omega'^d\|_{L^2(\mathbb{C}^n, d\lambda_{2n})}\|\omega^d P\|_{L^2(\mathbb{C}^n, d\lambda_{2n})}.\end{aligned}$$

Here we use the Cauchy–Schwarz inequality in the last line. After rearranging, the desired estimate follows.

We now give a characterization for the regularity of the compact set $\bar{Z}$ defined in Theorem 3.1.

Proposition 4.4. *Let E be a closed, non-pluripolar subset in $\mathbb{C}^n$, and ω be an admissible weight on E. Then the following two assertions are equivalent:*

(a) $\overline{V^*_{E,Q}} \le \overline{Q}_*$ *on* $\overline{E}_{\mathbb{C}P^n}$.

(b) $\bar{Z}$ *is regular.*

Proof. (a)⇒(b). Since $V^*_{\bar{Z}} = V^*_Z$, it is enough to prove $V^*_{\bar{Z}} = 0$ on $\bar{Z}$. Take $(w,\xi) \in \overline{Z}$ and an arbitrary sequence $(t_k z_k, t_k) \to (w,\xi)$. According to Bloom's formula (1), it suffices to prove that

$$\varlimsup_{k\to\infty}\,[V^*_{E,Q}(z_k) + \log|t_k|] \le 0.$$

For this, choose a sequence $(t'_k z'_k, t'_k) \in Z$ such that $(t'_k z'_k, t'_k) \to (w,\xi)$. There are two cases to be considered.

Case 1: $\xi \ne 0$. We have $z'_k \to w/\xi$, $z_k \to w/\xi$, $t'_k \to \xi$, $t_k \to \xi$. It follows that

$$\begin{aligned}\varlimsup_{k\to\infty}\,[V^*_{E,Q}(z_k) + \log|t_k|] &= \varlimsup_{k\to\infty}\,[V^*_{E,Q}(z_k) + \log|t'_k|]\\ &= \varlimsup_{k\to\infty}\,[V^*_{E,Q}(z_k) - Q(z'_k)] \le V^*_{E,Q}(w/\xi) - Q_*(w/\xi) \le 0.\end{aligned}$$

Case 2: $\xi = 0$. If $w = 0$ we have $\lim_{k\to\infty}[\ln^+|z_k| + \ln|t_k|] = -\infty$. Since $V^*_{E,Q} \in \mathcal{L}^+$, there exists a constant C such that $V^*_{E,Q}(z) \le \log^+|z| + C$. It follows that

$$\varlimsup_{k\to\infty}\,[V^*_{E,Q}(z_k) + \log|t_k|] \le \varlimsup_{k\to\infty}\,[\log^+|z_k| + C + \log|t_k|] = -\infty.$$

Now we consider the case $w \ne 0$. We may assume that $w_1 \ne 0$. Write $z_k = (z_{k,1},\dots,z_{k,n})$ and $z'_k = (z'_{k,1},\dots,z'_{k,n})$. Since $(t'_k z'_k, t'_k) \to (w,0)$ and $(t_k z_k, t_k) \to (w,0)$ we obtain

$$\begin{aligned}&[1 : z'_{k,1} : \ldots : z'_{k,n}] \to [0 : w_1 : \ldots : w_n] \in \overline{E}_{\mathbb{C}P^n},\\ &[1 : z_{k,1} : \ldots : z_{k,n}] \to [0 : w_1 : \ldots : w_n],\end{aligned}$$

and

$$\lim_{k\to\infty}\,[\log|z'_k| + \log|t_k|] = \lim_{k\to\infty}\,[\log|z_k| + \log|t_k|] = \log|w|.$$

Thus we get the following string of estimates:

$$\begin{aligned}\varlimsup_{k\to\infty}[V^*_{E,Q}(z_k)+\log|t_k|] &\le \varlimsup_{k\to\infty}[\overline{V^*_{E,Q}}(z_k)+\tfrac{1}{2}\log(1+|z_k|^2)+\log|t_k|]\\ &\le \varlimsup_{k\to\infty}[\overline{V^*_{E,Q}}(z_k)+\log|w|]\\ &\le \varlimsup_{k\to\infty}[\overline{V^*_{E,Q}}(z_k)+\tfrac{1}{2}\log(1+|z'_k|^2)+\log|t'_k|]\\ &\le \varlimsup_{k\to\infty}[\overline{V^*_{E,Q}}(z_k)-\overline{Q}(z'_k)]\\ &\le \overline{V_{E,Q}}^*([0:w])-\overline{Q}_*([0:w])\le 0.\end{aligned}$$

(b)⇒(a). This can be proved by reversing the above reasoning. The details are omitted.

We come to the main result of the section.

THEOREM 4.5. *Let E be a closed, non-pluripolar subset of $\mathbb{C}^n$ and ω be an admissible weight on E. Assume that:*

(a) *$\omega(z)>0$ for every $z\in E$.*
(b) *$\overline{V^*_{E,Q}}\le\overline{Q}_*$ on $\overline{E}_{\mathbb{CP}^n}$.*
(c) *There exists a constant $T>0$ such that for every $\varepsilon>0$ and every Borel set X such that $\mu(X)=0$ and*

$$X\subset\left\{z\in E:\varliminf_{r\to0+}\frac{\mu(\{w\in E\cap\mathbb{B}(z,r):(1+\varepsilon)\omega(w)>\omega(z)\})}{r^T}=0\right\}$$

we have $V^*_{E,Q}=V^*_{E\setminus X,Q}$.

Then the triple (E,μ,ω) has the Bernstein–Markov property.

REMARKS. (i) We do not assume continuity of ω on E.

(ii) It follows from Proposition 4.4 that (b) is equivalent to regularity of the compact set $\bar{Z}$ defined in Theorem 3.1.

(iii) According to the auxiliary Lemma 4.6 below, if a set X as in (c) is pluripolar then $V^*_{E,Q}=V^*_{E\setminus X,Q}$.

LEMMA 4.6. *Let E be a non-pluripolar, Borel subset of $\mathbb{C}^n$ and ω be an admissible weight on E. Then:*

(a) *If $F_2\subset F_1$ and $F_1\setminus F_2$ is pluripolar then $V^*_{F_1,Q}=V^*_{F_2,Q}$.*
(b) *For every sequence $F_k\uparrow F$ such that F is not pluripolar we have $V^*_{F_k,Q}\downarrow V^*_{F,Q}$.*

The proof that follows is inspired by [CKL, pp. 265–266].

Proof. (a) By Siciak's theorem (Theorem 5.2.4 in [Kl]), we can find $\varphi\in\mathcal{L}$ such that $\varphi=-\infty$ on $F_1\setminus F_2$. By subtracting a large constant from φ we may achieve further that $\varphi\le Q$ on F_1. Let $u\in\mathcal{L}$ be such that $u\le Q$ on F_2.

Fix $\varepsilon > 0$. Then

$$(1-\varepsilon)u + \varepsilon\varphi \le Q \quad \text{ on } F_1.$$

This implies that

$$(1-\varepsilon)u(z) + \varepsilon\varphi(z) \le V^*_{F_1,Q}(z), \quad \forall z \in \mathbb{C}^n.$$

By letting $\varepsilon \to 0$ we obtain

$$u \le V^*_{F_1,Q} \quad \text{ on } \mathbb{C}^n \setminus \{z : \varphi(z) = -\infty\}.$$

It follows that $u \le V^*_{F_1,Q}$ everywhere on $\mathbb{C}^n$. Therefore $V^*_{F_2,Q} \le V^*_{F_1,Q}$. Since the reverse inequality is trivial, the proof is complete.

(b) Set $u := \lim_{k\to\infty} V^*_{F_k,Q}$. Clearly $u \in \mathcal{L}^+$ and $u \ge V^*_{F,Q}$. By Bedford–Taylor's theorem on negligible sets, we can find pluripolar sets X_k such that

$$V^*_{F_k,Q} = V_{F_k,Q} \le Q \quad \text{ on } F_k \setminus X_k.$$

Set $X := \bigcup X_k$. Then X is pluripolar and $u \le Q$ on $F \setminus X$. By (a) we have

$$u \le V^*_{F\setminus X,Q} = V^*_{F,Q}.$$

Thus $u = V^*_{F,Q}$. The proof is complete.

Proof of Theorem 4.5. In view of assumption (c), by replacing T by $T+1$, we may achieve that for every $\varepsilon > 0$ and a Borel set $X \subset E \setminus G$ such that $\mu(X) = 0$, where

$$G := \left\{ z \in E : \varliminf_{r\to 0+} \frac{\mu(\{w \in E \cap \mathbb{B}(z,r) : (1+\varepsilon)\omega(w) > \omega(z)\})}{r^T} \ge 2 \right\},$$

we have $V^*_{E,Q} = V^*_{E\setminus X,Q}$. Now assume that (E,μ,ω) does not have the Bernstein–Markov property. Then we can find $\varepsilon_0 > 0$ and a sequence of polynomials $\{P_j\}$ with $\deg P_j = d_j$ such that

$$\|\omega^{d_j} P_j\|_E \ge j^2 (1+\varepsilon_0)^{(T+2)d_j} \|\omega^{d_j} P_j\|_{L^2(E,\mu)}.$$

Define

$$Q_j := \frac{P_j}{j\|\omega^{d_j} P_j\|_{L^2(E,\mu)}}.$$

It follows that

$$\|\omega^{d_j} Q_j\|_E \ge j(1+\varepsilon_0)^{(T+2)d_j}, \quad \|\omega^{d_j} Q_j\|_{L^2(E,\mu)} = 1/j.$$

We define

$$F_k := \Big\{ z \in E : \sup_{j\ge 1} |\omega(z)^{d_j} Q_j(z)| \le k \Big\}.$$

Note that F_k is a sequence of Borel sets increasing to

$$F := \Big\{ z \in E : \sup_{j\ge 1} |\omega(z)^{d_j} Q_j(z)| < \infty \Big\}.$$

Since E is non-pluripolar, we infer $V^*_{E,Q} \in \mathcal{L}^+$. Next, we observe that for every $k \geq 1$,

$$\mu(E \setminus F) \leq \mu(E \setminus F_k) \leq \sum_{j\geq 1} \mu(z : |\omega^{d_j} Q_j| \geq k) \leq \frac{1}{k^2} \sum_{j\geq 1} \frac{1}{j^2}.$$

It follows that $\mu(E \setminus F) = 0$. Next, we define

$$G_k = \{z \in E \cap \mathbb{B}(0,k) : \mu(\{w \in E \cap \mathbb{B}(z,r) : (1+\varepsilon_0)\omega(w)/\omega(z) > 1\}) \geq r^T, \quad \forall 0 < r \leq 1/k\}.$$

Then $\{G_k\}$ is an increasing family of Borel sets satisfying $G \subset \bigcup_{k\geq 1} G_k$. Combining this inclusion with Lemma 4.6(b), from assumption (c) we obtain

$$V^*_{F_k \cup G_k, Q} \downarrow V^*_{F\cup G,Q} = V^*_{E,Q}.$$

It follows that

$$\overline{V^*_{F_k \cup G_k, Q}} \downarrow \overline{V^*_{E,Q}}.$$

Moreover, from assumption (b) and Dini's theorem, there exists k_0 such that

$$\overline{V^*_{F_{k_0} \cup G_{k_0}, Q}} \leq \overline{Q}_* + \log(1+\varepsilon_0)$$

on $\overline{E}_{\mathbb{CP}^n}$. In particular,

$$V^*_{F_{k_0} \cup G_{k_0}, Q}(z) \leq Q(z) + \log(1+\varepsilon_0), \quad \forall z \in E. \tag{8}$$

By the weighted Bernstein–Walsh inequality we get, for every polynomial P with $d = \deg P$,

$$\|\omega^d P\|_E \leq (1+\varepsilon_0)^d \|\omega^d P\|_{F_{k_0} \cup G_{k_0}}.$$

Since $\|\omega^{d_j} Q_j\|_{F_{k_0}} \leq k_0$, by the choice of Q_j we obtain, for every $j \geq k_0$,

$$j(1+\varepsilon_0)^{(T+2)d_j} \leq \|\omega^{d_j} Q_j\|_E \leq (1+\varepsilon_0)^{d_j} \|\omega^{d_j} Q_j\|_{G_{k_0}}.$$

It follows that for $j \geq k_0$,

$$\|\omega^{d_j} Q_j\|_{F_{k_0}} \leq \|\omega^{d_j} Q_j\|_{G_{k_0}}.$$

Next, we claim that there exist $\delta_0 > 0$ such that for every $z \in G_{k_0}$,

$$\sup_{\mathbb{B}(z,\delta_0)} V^*_{F_{k_0} \cup G_{k_0}, Q} \leq Q(z) + 2\log(1+\varepsilon_0). \tag{9}$$

Assume otherwise; then we can find sequences $\{z_j\} \subset G_{k_0}$ and $\{w_j\} \subset \mathbb{B}(z_j, 1/j)$ such that $z_j \to a \in E$ and

$$V^*_{F_{k_0} \cup G_{k_0}, Q}(w_j) > Q(z_j) + 2\log(1+\varepsilon_0).$$

Letting $j \to \infty$ we get $V^*_{F_{k_0} \cup G_{k_0}, Q}(a) \geq Q(a) + 2\log(1+\varepsilon_0)$. This is a contradiction to (8).

Given a polynomial Q_j and distinct points $z \in G_{k_0}, w \in \mathbb{B}(z, \delta_0/2)$, following the argument in [Bl1], we will estimate the quantity $|Q_j(z) - Q_j(w)|$.

Let $e = (e_1, \ldots, e_n) := (w - z)/|w - z|$ and

$$\tilde{Q}_j(t) = Q_j(z_1 + e_1 t, \ldots, z_n + e_n t).$$

Then $\tilde{Q}_j$ is a polynomial on $\mathbb{C}$. By the infinitesimal increment theorem we get

$$|Q_j(w) - Q_j(z)| = |\tilde{Q}_j(|w - z|) - \tilde{Q}_j(0)| \leq |w - z| \, \|\tilde{Q}_j'\|_{|t| \leq \delta_0/2}.$$

Moreover, by Cauchy's inequality for derivative, we have

$$\|\tilde{Q}_j'\|_{|t| \leq \delta_0/2} \leq \frac{2}{\delta_0} \|Q_j\|_{|t| \leq \delta_0} \leq \frac{2}{\delta_0} \|Q_j\|_{\mathbb{B}(z, \delta_0)}.$$

Putting these together we obtain

$$|Q_j(w) - Q_j(z)| \leq \frac{2}{\delta_0} |w - z| \, \|Q_j\|_{\mathbb{B}(z, \delta_0)}.$$

Using again the weighted Bernstein-Walsh inequality and (9), we get for every $z \in G_{k_0}$ and $j \geq k_0$ the estimates

$$\|Q_j\|_{\mathbb{B}(z,\delta_0)} \leq \|\omega^{d_j} Q_j\|_{F_{k_0} \cup G_{k_0}} e^{d_j \sup_{\mathbb{B}(z,\delta_0)} V_{F_{k_0} \cup G_{k_0}, Q}} \leq \frac{(1 + \varepsilon_0)^{2d_j}}{\omega(z)^{d_j}} \|\omega^{d_j} Q_j\|_{G_{k_0}}.$$

This implies that for every $z \in G_{k_0}$ and $w \in \mathbb{B}(z, \delta_0/2)$,

$$|Q_j(w) - Q_j(z)| \leq \frac{2(1 + \varepsilon_0)^{2d_j}}{\delta_0 \omega(z)^{d_j}} |w - z| \, \|\omega^{d_j} Q_j\|_{G_{k_0}}. \tag{10}$$

For $j \geq k_0$, we set $r_j := \delta_0/(8(1 + \varepsilon_0)^{2d_j})$. By shrinking δ_0, we may obtain $r_j < 1/k_0$. Choose $\xi_j \in G_{k_0}$ such that

$$\omega(\xi_j)^{d_j} |Q_j(\xi_j)| \geq \tfrac{1}{2} \|\omega^{d_j} Q_j\|_{G_{k_0}}. \tag{11}$$

For $j \geq 1$ we set

$$Z_j := \{w \in E \cap \mathbb{B}(\xi_j, r_j) : (1 + \varepsilon_0)\omega(w) > \omega(\xi_j)\}.$$

Then $\mu(Z_j) \geq r_j^T$ for every $j \geq 1$. Combining (10) and (11), we have

$$\begin{aligned}
\frac{1}{j^2} &= \int\limits_E \omega^{2d_j}(w) |Q_j(w)|^2 \, d\mu(w) \geq \int\limits_{Z_j} \omega(w)^{2d_j} |Q_j(w)|^2 \, d\mu(w) \\
&\geq \frac{\omega(\xi_j)^{2d_j}}{(1 + \varepsilon_0)^{2d_j}} \int\limits_{Z_j} \left[\frac{|Q_j(\xi_j)|^2}{2} - |Q_j(w) - Q_j(\xi_j)|^2 \right] d\mu(w) \\
&\geq \|\omega^{d_j} Q_j\|_{G_{k_0}}^2 \left(\frac{1}{8(1 + \varepsilon_0)^{2d_j}} - \frac{4}{\delta_0^2} r_j^2 (1 + \varepsilon_0)^{2d_j} \right) \mu(Z_j) \\
&= \|\omega^{d_j} Q_j\|_{G_{k_0}}^2 \frac{\mu(Z_j)}{16(1 + \varepsilon_0)^{2d_j}} \geq \|\omega^{d_j} Q_j\|_{G_{k_0}}^2 \frac{r_j^T}{16(1 + \varepsilon_0)^{2d_j}} \\
&= \|\omega^{d_j} Q_j\|_{G_{k_0}}^2 \frac{\delta_0^T}{16 \cdot 8^T (1 + \varepsilon_0)^{2(T+1)d_j}} \geq j^2 \frac{\delta_0^T}{16 \cdot 8^T}.
\end{aligned}$$

By letting $j \to \infty$ we get a contradiction. The proof is complete.

By specifying to the case of ω continuous, we have the following consequence of Theorem 4.5.

COROLLARY 4.7. *Let E be a closed, non-pluripolar subset of $\mathbb{C}^n$, ω be an admissible weight on E, and μ be a positive Borel measure on E. Assume that:*

(a) *$\overline{E}_{\mathbb{CP}^n}$ is locally regular.*
(b) *Q is continuous on E and the limit $\lim_{w\to z}[Q(w) - \frac{1}{2}\log(1+|w|^2)]$ exists for all $z \in \overline{E}_{\mathbb{CP}^n}\setminus E$.*
(c) *There exists a constant $T > 0$ such that for every Borel set X such that*
$$X \subset F := \left\{ z \in E : \varliminf_{r\to+0} \frac{\mu(E\cap\mathbb{B}(z,r))}{r^T} = 0 \right\}$$
and $\mu(X) = 0$ we have $V^*_{E,Q} = V^*_{E\setminus X,Q}$.

Then the triple (E,μ,ω) has the Bernstein–Markov property.

Proof. From (b) it follows that $\overline{Q}$ is continuous on $\overline{E}_{\mathbb{CP}^n}$. We have
$$\overline{V^*_{E,Q}} = (\sup\{\psi(z) : \psi \in \mathrm{PSH}(\mathbb{CP}^n,\theta),\ \psi \le \overline{Q} \text{ on } \overline{E}_{\mathbb{CP}^n}\})^*.$$
From (a) we get $\overline{V^*_{E,Q}} \le \overline{Q}$ on $\overline{E}_{\mathbb{CP}^n}$. Finally, since ω is *continuous* on E we deduce that for every $\varepsilon > 0$ and $z \in E$,
$$(1+\varepsilon)\omega(w) > \omega(z), \qquad \forall w \in E\cap\mathbb{B}(z,r)$$
for r small enough. This implies that μ satisfies condition (c) of Theorem 4.4. By invoking that result, we conclude the proof.

REMARK. Condition (c) holds if either μ is determining for (E,ω) or the set F is pluripolar.

In the result below, we deal with the case where E is *not* assumed to be regular. For simplicity, we only consider the case of $w = 1$ and E compact.

PROPOSITION 4.8. *Let E be a non-pluripolar compact subset of $\mathbb{C}^n$, and μ be a determining measure for E. Let $M := e^{\|V^*_E\|_E}$. Then for every $\lambda > M$, there exists a constant $C(\lambda) > 0$ such that for every polynomial P of degree d,*
$$\|P\|_E \le C(\lambda)\lambda^d \|P\|_{L^2(E,\mu)}.$$

Proof. By Bedford–Taylor's theorem on negligible sets, the set $F := \{z \in E : V^*_E > 0\}$ is pluripolar. Thus there exists a plurisubharmonic function φ on $\mathbb{C}^n$ such that
$$F \subset F' := \{z \in E : \varphi = -\infty\}.$$
Let $E_j := \{z \in E : \varphi(z) \ge -j\}$ and
$$\varepsilon := \lambda e^{-M} - 1, \qquad \varepsilon' := \sqrt{1+\varepsilon} - 1.$$

Observe that

$$V^*_{E_j} \downarrow V^*_{E\setminus F'} = V^*_E.$$

Thus we can find $j(\varepsilon)$ large enough such that

$$V^*_{E_{j(\varepsilon)}}(z) \le M + \log(1+\varepsilon'), \quad \forall z \in E. \tag{12}$$

We claim that there exists a constant $C > 0$ such that for every polynomial P of degree d we have

$$\|P\|_{E_{j(\varepsilon)}} \le C(1+\varepsilon')^d \|P\|_{L^2(E,\mu)}. \tag{13}$$

Suppose the above claim is false. Then there exists a sequence of polynomials $\{P_k\}$ with deg $P_k = d_k$ such that

$$\|P_k\|_{E_{j(\varepsilon)}} \ge k(1+\varepsilon')^{d_k}, \quad \|P_k\|_{L^2(E,\mu)} = 1/k. \tag{14}$$

By a measure-theoretic argument as in the proof of Theorem 4.4 we have $\mu(E \setminus E') = 0$, where

$$E' = \bigcup_{m\ge 1} E_m, \quad E_m := \{z \in E : \sup_{k\ge 1} |P_k(z)| \le m\}.$$

Since μ is determining we get $V^*_E = V^*_{E'}$. It follows that

$$V^*_{E_m} \downarrow V^*_{E'} = V^*_E = 0 \quad \text{on } E_{j(\varepsilon)}.$$

Thus, by Dini's theorem, we can find m_0 such that $V^*_{E_{m_0}} \le \log(1+\varepsilon')$ on $E_{j(\varepsilon)}$. This implies that

$$\frac{1}{d_k}\log\frac{|P_k(z)|}{m_0} \le \log(1+\varepsilon'), \quad \forall k, \forall z \in E_{j(\varepsilon)}.$$

By letting $k \to \infty$ we get a contradiction to (14).

Finally, we combine (12), (13) and use Bernstein-Walsh's inequality to obtain, for a polynomial P of degree d, a string of inequalities

$$\|P\|_E \le e^M(1+\varepsilon')^d\|P\|_{E(j(\varepsilon)} \le C[e^M(1+\varepsilon')^2]^d\|P\|_{L^2(E,\mu)} = C\lambda^d\|P\|_{L^2(E,\mu)}.$$

Here we use the choice of ε and ε' in the last identity. The proof is complete.

In the rest of this section, following the work of Bloom and Shiffman [BSh], we give an application of the weighted Bernstein–Walsh property to Bergman kernels. We assume that E is a closed, non-pluripolar subset of $\mathbb{C}^n$, μ is a *finite* positive Borel measure on E and $\omega > 0$ is a continuous admissible weight on E such that $\bar{Z}$ is regular in $\mathbb{C}^{n+1}$, where

$$Z := \{(tz, t) : z \in E, |t| = \omega(z)\}.$$

For $d \ge 1$, we let $\mathcal{P}_d$ be the linear space of homogeneous polynomials of degree $\le d$ in $\mathbb{C}^n$. Then $\mathcal{P}_d \subset L^2(E,\mu)$. We also note that $\dim \mathcal{P}_d = (n+d)!/(n!d!) := N_d$. Let $\{q_j^{(d)}\}_{1\le j\le N_d}$ be an orthonormal basis of $\mathcal{P}_d$ with respect to the weighted inner product

$$\langle p, q\rangle := \int_E \omega^{2d} p\bar{q}\, d\mu, \qquad \forall p, q \in \mathcal{P}_d.$$

Define

$$K_d^{\mu,\omega}(z,z) := \sum_{1\le j\le N_d} |q_j^{(d)}(z)|^2.$$

We cannot claim much originality for the next result.

THEOREM 4.9. *Assume that (E,μ,ω) has the Bernstein–Markov property. Then*

$$\lim_{d\to\infty} \frac{1}{2d} \log K_d^{\mu,\omega}(z,z) = V_{E,Q}(z)$$

locally uniformly on $\mathbb{C}^n$.

The proof of the above theorem is completely similar to that of Lemma 3.4 in [BSh] provided that the following extension of Lemma 3.2 of [BSh] is true.

LEMMA 4.10. *The sequence $\frac{1}{d}\log \Phi_{E,Q,d}$ converges to $V_{E,Q}$ locally uniformly on $\mathbb{C}^n$, where*

$$\Phi_{E,Q,d}(z) := \sup\{|P(z)| : \|\omega^{\deg P} P\|_E \le 1,\ P\in\mathcal{P}_d\}.$$

Proof. The above lemma was proved in [BSt] in a much more general setting. For the reader's convenience we give a more elementary proof. First, we apply a result of Siciak (see [Kl, Theorem 5.1.7]) to obtain

$$V_{\bar{Z}}(\tilde{z}) = \sup\left\{\frac{1}{\deg \tilde{P}} \log|\tilde{P}(\tilde{z})| : \|\tilde{P}\|_{\bar{Z}} \le 1,\ P\in\mathbb{C}[z_1,\dots,z_{n+1}]\right\}$$

for all $\tilde{z}\in\mathbb{C}^{n+1}$. Since $\bar{Z}$ is circular, by the argument in the proof of Proposition 3.2, we get

$$V_{\bar{Z}}(\tilde{z}) = \sup\left\{\frac{1}{d}\log|\tilde{P}(z)| : \|\tilde{P}\|_{\bar{Z}}\le 1,\ \tilde{P}\in\tilde{\mathcal{P}}_d\right\},$$

where $\tilde{\mathcal{P}}_d$ is the set of homogeneous polynomials of degree $\le d$ on $\mathbb{C}^{n+1}$. Let

$$\Psi_{\bar{Z},d}(\tilde{z}) := \sup\{|\tilde{P}(\tilde{z})| : \|\tilde{P}\|_{\bar{Z}}\le 1,\ \tilde{P}\in\tilde{\mathcal{P}}_d\}.$$

Since $\bar{Z}$ is regular, by the proof of Theorem 3.2 of [BSh] and Proposition 3.2(b) we have $\frac{1}{d}\log\Psi_{\bar{Z},d} \to V_{\bar{Z}} = V_Z$ locally uniformly on $\mathbb{C}^{n+1}$.

Now assume the conclusion of the lemma is false. Then, since $\frac{1}{d}\log\Phi_{E,Q,d} \le V_{E,Q}$ on $\mathbb{C}^n$ there exist $\varepsilon_0>0$, a sequence of points $z_k\to\xi\in\mathbb{C}^n$, and a sequence of positive integers $d_k\uparrow\infty$ such that

$$\Phi_{E,Q,d_k}(z_k) < e^{d_k(V_{E,Q}(z_k)-\varepsilon_0)}. \tag{15}$$

We choose $t_0 > 1$ such that

$$\log t_0 + V_{E,Q}(z_k) > 0, \qquad \forall k \geq 1.$$

By Bloom's formula (1) we have

$$V_Z(t_0 z_k, t_0) = \max\{V_{E,Q}(z_k) + \log t_0, 0\} = V_{E,Q}(z_k) + \log t_0.$$

Then we have

$$\frac{1}{d} \log \Psi_{\bar{Z},d} \to V_{E,Q}(z) + \log t_0 \quad \text{ uniformly on } K' := \{(t_0 z_k, t_0) : k \geq 1\}.$$

Thus for k large enough we have

$$\frac{1}{d_k} \log \Psi_{\bar{Z},d_k}(t_0 z_k, t_0) \geq V_{E,Q}(z_k) + \log t_0 - \varepsilon_0/2. \tag{16}$$

It follows from (15) that

$$\begin{aligned}\Psi_{\bar{Z},d_k}(t_0 z_k, t_0) &= \sup\{|t_0|^{\deg P}|P(z_k)| : \|\omega^{\deg P} P\|_E \leq 1,\ P \in \mathcal{P}_{d_k}\} \\ &\leq |t_0|^{d_k} e^{d_k(V_{E,Q}(z_k) - \varepsilon_0)}.\end{aligned}$$

We get a contradiction to (16) for k large enough. The proof is complete.

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