ANNALES
POLONICI MATHEMATICI
105.2 (2012)

On Borel summable solutions of the multidimensional heat equation

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Abstract. We give a new characterisation of Borel summability of formal power series solutions to the n-dimensional heat equation in terms of holomorphic properties of the integral means of the Cauchy data. We also derive the Borel sum for the summable formal solutions.

1. Introduction. We consider the initial value problem for the complex n-dimensional heat equation

$$\partial_t u = \Delta u, \qquad u(0,z) = \varphi(z), \tag{1.1}$$

where $t \in \mathbb{C}$, $z \in \mathbb{C}^n$, $\Delta = \sum_{i=1}^n \partial_{z_i}^2$ is the complex Laplace operator and φ is holomorphic in a complex neighbourhood of the origin. The unique formal power series solution of (1.1) is given by

$$\hat{u}(t,z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{k!} t^k. \tag{1.2}$$

In dimension $n = 1$ the problem of convergence of the formal solution (1.2) was already solved by Kovalevskaya [7]. She showed that $\hat{u}$ is convergent if and only if the Cauchy data φ is an entire function of exponential order at most 2. In the multidimensional case Aronszajn et al. [1] solved the problem of convergence of $\hat{u}$ in terms of the growth of $\Delta^k\varphi(z)$ as $k \to \infty$. Another approach was given by Łysik [10]. He proved that $\hat{u}$ is convergent if and only if the integral mean of φ over the closed ball $B(x,r)$, or the sphere $S(x,r)$, as a function of the radius r extends to an entire function of exponential order at most 2.

If $\hat{u}$ diverges, it is natural to ask when it is Borel summable (see Definition 2.2). In the one-dimensional case the answer was given by Lutz et al. [8]. They proved that $\hat{u}$ is Borel summable in a direction d if and only

2010 *Mathematics Subject Classification*: 35K05, 35C10.
Key words and phrases: Borel summability, formal power series, mean-values, heat equation.

DOI: 10.4064/ap105-2-5 © Instytut Matematyczny PAN, 2012

if φ can be analytically continued to infinity in sectors bisected by $d/2$ and $\pi + d/2$, and the continuation is of exponential order at most 2. This result has been generalised in various ways. Balser [2] characterised the Borel summable solutions of (1.1) for the initial data in Gevrey classes. Balser and Loday-Richaud [5] studied summability properties of formal solutions of the inhomogeneous heat equation with variable coefficients. The result of Lutz et al. [8] was extended to quasi-homogeneous equations by Ichinobe [6] and to general linear partial differential equations in two variables with constant coefficients by Balser [4] and the author [12]. Moreover, Łysik [9] applied the result given by Lutz et al. [8] to study summability properties of solutions of the Burgers equation.

In the case of the multidimensional heat equation, the author [11] proved that $\hat{u}$ is Borel summable in a direction d if and only if the function

$$\Phi_n(t,z) = \begin{cases} \displaystyle\int_{S(0,1)} \varphi(z+tx)\,dS(x) & \text{if } n \text{ is odd},\\ \displaystyle\int_{B(0,1)} \frac{\varphi(z+tx)\,dx}{\sqrt{1-|x|^2}} & \text{if } n \text{ is even},\end{cases}$$

is analytic with respect to z in some complex disc centred at the origin and can be analytically continued to infinity with respect to t in sectors bisected by $d/2$ and $\pi + d/2$, and this continuation is of exponential order at most 2 as $t \to \infty$.

In the present paper we show that for an arbitrary dimension n, we may replace the function $\Phi_n(t,z)$ in the above characterisation by the holomorphic extension of the integral mean of φ over the closed ball $B(x,r)$ or the sphere $S(x,r)$. The result is based upon mean-value formulas for analytic functions (see [10, Theorem 3.1]). As an application, we use the procedure of Borel summability to find the Borel sum u of the formal solution $\hat{u}$. As a result we obtain the representation of the solution u of the problem (1.1) given by a complex version of the convolution of the initial data φ with the heat kernel.

2. Preliminaries. In the paper we use the following notation. The real closed ball (sphere, respectively) with centre at $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$ ($S(x,r)$, respectively). Moreover, the complex disc in $\mathbb{C}^n$ with centre at the origin and radius $r > 0$ is denoted by $D_r^n := \{z \in \mathbb{C}^n : |z| < r\}$. If the radius r is not essential, then we denote it briefly by D^n.

The *Pochhammer symbol* is defined for non-negative integers k and complex numbers a as $(a)_0 := 1$ and $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$.

A *sector in a direction* $d \in \mathbb{R}$ *with an opening* $\varepsilon > 0$ in the universal covering space $\widetilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(d,\varepsilon) := \{z \in \widetilde{\mathbb{C}} : z = re^{i\theta},\, d-\varepsilon/2 < \theta < d+\varepsilon/2,\, r>0\}.$$

If the value of the opening angle ε is not essential, then we write briefly S_d. We denote by $\hat{S}(d,\varepsilon)$ ($\hat{S}_d$, respectively) the set $S(d,\varepsilon) \cup D^1$ ($S_d \cup D^1$, respectively). Let $\mathcal{O}(G)$ denote the space of holomorphic functions on a domain $G \subseteq \mathbb{C}^n$.

Let us also recall some definitions and fundamental facts about Borel summability. For more details we refer the reader to [3].

DEFINITION 2.1. A function $u(t,z) \in \mathcal{O}(S(d,\varepsilon) \times D_r^n)$ is of *exponential growth of order at most* $s > 0$ *as* $t \to \infty$ *in* $S(d,\varepsilon)$ if and only if for every $r_1 \in (0,r)$ and every $\varepsilon_1 \in (0,\varepsilon)$ there exist $A, B < \infty$ such that

$$\max_{|z|\le r_1} |u(t,z)| \le Ae^{B|t|^s} \quad \text{for } t \in S(d,\varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^s(S(d,\varepsilon) \times D_r^n)$. We also write $\mathcal{O}^s(\hat{S}(d,\varepsilon) \times D^n)$ ($\mathcal{O}^s(\hat{S}_d \times D^n)$, respectively) for $\mathcal{O}^s(S(d,\varepsilon) \times D^n) \cap \mathcal{O}(\hat{S}(d,\varepsilon) \times D^n)$ ($\mathcal{O}^s(S_d \times D^n) \cap \mathcal{O}(\hat{S}_d \times D^n)$, respectively).

Analogously, a function $\varphi \in \mathcal{O}(S(d,\varepsilon))$ is of *exponential growth of order at most* $s > 0$ *as* $z \to \infty$ *in* $S(d,\varepsilon)$ if and only if for every $\varepsilon_1 \in (0,\varepsilon)$ there exist $A, B < \infty$ such that

$$|\varphi(z)| \le Ae^{B|z|^s} \quad \text{for } z \in S(d,\varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^s(S(d,\varepsilon))$. We also set $\mathcal{O}^s(\hat{S}_d) := \mathcal{O}^s(S_d) \cap \mathcal{O}(\hat{S}_d)$.

DEFINITION 2.2. Let $d \in \mathbb{R}$. A formal series

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{j!} t^j \quad \text{with } u_j \in \mathcal{O}(D^n) \tag{2.1}$$

is called *Borel summable in the direction d* if and only if its *Borel transform* $\hat{\mathcal{B}}\hat{u}$ satisfies

$$(\hat{\mathcal{B}}\hat{u})(s,z) := \sum_{j=0}^{\infty} \frac{u_j(z)}{(j!)^2} s^j \in \mathcal{O}^1(\hat{S}(d,\varepsilon) \times D^n) \quad \text{for some } \varepsilon > 0.$$

The *Borel sum* u^θ *of* $\hat{u}$ *in the direction* d is represented by the Laplace transform of $v(s,z) := (\hat{\mathcal{B}}\hat{u})(s,z)$,

$$u^\theta(t,z) := \frac{1}{t} \int_0^{\infty(\theta)} e^{-s/t} v(s,z)\, ds,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+ := \{re^{i\theta} : r \ge 0\}$ with $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$.

According to the general theory of moment summability (see [3, Section 6.5]), a formal series (2.1) is Borel summable in a direction d if and only if the same holds for the series

$$\sum_{j=0}^{\infty} u_j(z)\frac{j!}{(2j)!}t^j.$$

Consequently, we obtain a characterisation of Borel summability which is analogous to Definition 2.2 (see also [3, Theorem 38 and Section 11]).

PROPOSITION 2.3. *Let $d \in \mathbb{R}$. A formal series* (2.1) *is Borel summable in the direction d if and only if its modified Borel transform $\tilde{\mathcal{B}}\hat{u}$ satisfies*

$$(\tilde{\mathcal{B}}\hat{u})(s,z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{(2j)!}s^j \in \mathcal{O}^1(\hat{S}(d,\varepsilon)\times D^n) \quad \text{for some } \varepsilon > 0.$$

The Borel sum u^θ of $\hat{u}$ in the direction d is represented by the Ecalle acceleration operator acting on $\tilde{v}(s,z) := (\tilde{\mathcal{B}}\hat{u})(s,z)$ as follows:

$$u^\theta(t,z) = \frac{1}{\sqrt{t}} \int_0^{\infty(\theta)} \tilde{v}(s,z)C_2(\sqrt{s/t})\,d\sqrt{s}$$

with $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$. Here integration is taken over the ray $e^{i\theta}\mathbb{R}_+$ and C_2 is defined by

$$C_2(\zeta) := \frac{1}{2\pi i}\int_\gamma \frac{e^{u-\zeta\sqrt{u}}}{\sqrt{u}}\,du \tag{2.2}$$

with a path of integration γ as in the Hankel integral for the inverse gamma function (from ∞ along $\arg u = -\pi$ to some $u_0 < 0$, then along the circle $|u| = |u_0|$ to $\arg u = \pi$, and back to ∞ along this ray).

3. Integral means. In this section we recall the notion of integral means. To this end we take a continuous function φ on a domain $\Omega \subset \mathbb{R}^n$, $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$. We denote by $M(\varphi; r, x)$ and $N(\varphi; r, x)$ the integral means of φ over the closed ball $B(x,r)$ and the sphere $S(x,r)$, respectively, i.e.,

$$M(\varphi; r, x) = \fint_{B(x,r)} \varphi(y)\,dy := \frac{1}{\alpha(n)r^n}\int_{B(x,r)} \varphi(y)\,dy$$

$$N(\varphi; r, x) = \fint_{S(x,r)} \varphi(y)\,dS(y) := \frac{1}{n\alpha(n)r^{n-1}}\int_{S(x,r)} \varphi(y)\,dS(y),$$

where $\alpha(n) := \pi^{n/2}/\Gamma(1+n/2)$ is the volume of the n-dimensional unit ball $B(0,1)$. Moreover, since

$$M(\varphi;r,x) = \fint_{B(0,1)} \varphi(x+ry)\,dy \quad \text{and} \quad N(\varphi;r,x) = \fint_{S(0,1)} \varphi(x+ry)\,dS(y),$$

we may also consider $M(\varphi;t,z)$ and $N(\varphi;t,z)$ for complex variables $t \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Hence, according to mean-value properties for analytic functions we have

Proposition 3.1 ([10, Theorem 3.1]). *Let G be a domain in $\mathbb{C}^n$, $\varphi \in \mathcal{O}(G)$ and $z \in G$. Then $M(\varphi;t,z)$ and $N(\varphi;t,z)$ are holomorphic functions at the origin as functions of t, and for t small enough,*

$$M(\varphi;t,z) = \sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{4^k(n/2+1)_k k!} t^{2k}, \quad N(\varphi;t,z) = \sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{4^k(n/2)_k k!} t^{2k}. \tag{3.1}$$

Using the above proposition we find a relation between the two series $\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(2k)!} t^{2k}$ and $\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(k!)^2} t^{2k}$ and the integral means $M(\varphi;t,z)$ and $N(\varphi;t,z)$:

Lemma 3.2. *Assume that G is a domain in $\mathbb{C}^n$, $\varphi \in \mathcal{O}(G)$, $z \in G$ and t is small enough. Then*

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(2k)!} t^{2k} &= \frac{1}{n!!}\partial_t(t^{-1}\partial_t)^{(n-1)/2} t^n M(\varphi;t,z)\\ &= \frac{1}{(n-2)!!}\partial_t(t^{-1}\partial_t)^{(n-3)/2} t^{n-2} N(\varphi;t,z)\end{aligned}$$

for n odd (with $(-1)!! = 1$ and $(t^{-1}\partial_t)^{-1} = \partial_t^{-1} t$ for $n = 1$); and

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{\Delta^k\varphi(z)}{(k!)^2} t^{2k} &= \frac{1}{n!!}(t^{-1}\partial_t)^{n/2} t^n M(\varphi;2t,z)\\ &= \frac{1}{(n-2)!!}(t^{-1}\partial_t)^{(n-2)/2} t^{n-2} N(\varphi;2t,z)\end{aligned}$$

for n even.

Proof. First, note that

$$4^k\left(\frac{n}{2}+1\right)_k k! = (2k)!!\frac{(n+2k)!!}{n!!}, \qquad 4^k\left(\frac{n}{2}\right)_k k! = (2k)!!\frac{(n+2k-2)!!}{(n-2)!!}. \tag{3.2}$$

If n is odd, then by (3.1) and (3.2) we obtain

$$\frac{1}{n!!}\partial_t(t^{-1}\partial_t)^{(n-1)/2}t^nM(\varphi;t,z)=\frac{1}{n!!}\partial_t(t^{-1}\partial_t)^{(n-1)/2}\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{4^k(n/2+1)_kk!}t^{2k+n}$$
$$=\frac{1}{n!!}\sum_{k=0}^{\infty}\frac{(2k+n)(2k+n-2)\cdots(2k+1)\Delta^k\varphi(z)}{\frac{(2k)!!(2k+n)!!}{n!!}}t^{2k}=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}t^{2k}$$

and

$$\frac{1}{(n-2)!!}\partial_t(t^{-1}\partial_t)^{(n-3)/2}t^{n-2}N(\varphi;t,z)$$
$$=\frac{1}{(n-2)!!}\partial_t(t^{-1}\partial_t)^{(n-3)/2}\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{4^k(n/2+1)_kk!}t^{2k+n-2}$$
$$=\frac{1}{(n-2)!!}\sum_{k=0}^{\infty}\frac{(2k+n-2)(2k+n-4)\cdots(2k+1)\Delta^k\varphi(z)}{\frac{(2k)!!(2k+n-2)!!}{(n-2)!!}}t^{2k}$$
$$=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(2k)!}t^{2k},$$

which proves the first part of the lemma.

Analogously, if n is even, then by (3.1) and (3.2) we have

$$\frac{1}{n!!}(t^{-1}\partial_t)^{n/2}t^nM(\varphi;2t,z)=\frac{1}{n!!}(t^{-1}\partial_t)^{n/2}\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)4^k}{4^k(n/2+1)_kk!}t^{2k+n}$$
$$=\frac{1}{n!!}\sum_{k=0}^{\infty}\frac{(2k+n)(2k+n-2)\cdots(2k+2)\Delta^k\varphi(z)4^k}{\frac{(2k)!!(2k+n)!!}{n!!}}t^{2k}=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(k!)^2}t^{2k}$$

and

$$\frac{1}{(n-2)!!}(t^{-1}\partial_t)^{(n-2)/2}t^{n-2}N(\varphi;2t,z)$$
$$=\frac{1}{(n-2)!!}(t^{-1}\partial_t)^{(n-2)/2}\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)4^k}{4^k(n/2)_kk!}t^{2k+n-2}$$
$$=\frac{1}{(n-2)!!}\sum_{k=0}^{\infty}\frac{(2k+n-2)(2k+n-4)\cdots(2k+2)\Delta^k\varphi(z)4^k}{\frac{(2k)!!(2k+n-2)!!}{(n-2)!!}}t^{2k}$$
$$=\sum_{k=0}^{\infty}\frac{\Delta^k\varphi(z)}{(k!)^2}t^{2k},$$

which proves the second part of the lemma. ∎

4. Summability of formal solutions. Now we are ready to state the main result of the paper.

MAIN THEOREM 4.1. *Let $d \in \mathbb{R}$ and $\hat{u}$ be the formal solution* (1.2) *of the n-dimensional complex heat equation*

$$\partial_t u = \Delta u, \quad u(0,z) = \varphi(z) \in \mathcal{O}(D^n). \tag{4.1}$$

Then the following conditions are equivalent:

(i) *$\hat{u}$ is Borel summable in the direction d,*
(ii) *$M(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$,*
(iii) *$N(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$.*

Proof. We first assume that the dimension n is odd. Applying the modified Borel transform $\tilde{\mathcal{B}}$ to the formal solution $\hat{u}$ of (4.1) given by (1.2) and replacing s by t^2, we have

$$(\tilde{\mathcal{B}}\hat{u})(t^2, z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(2k)!} t^{2k}.$$

Moreover, by Proposition 2.3 the formal solution $\hat{u}$ is Borel summable in the direction d if and only if $(\tilde{\mathcal{B}}\hat{u})(t^2, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$. If we combine this with Lemma 3.2 and with the uniqueness of the analytic continuation of $\sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(2k)!} t^{2k}$ with respect to t, we conclude that $\hat{u}$ is Borel summable in the direction d if and only if

$$\frac{1}{n!!} \partial_t (t^{-1}\partial_t)^{(n-1)/2} t^n M(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n), \tag{4.2}$$

or equivalently, if and only if

$$\frac{1}{(n-2)!!} \partial_t (t^{-1}\partial_t)^{(n-3)/2} t^{n-2} N(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n). \tag{4.3}$$

Since the space $\mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$ is closed under differentiation ∂_t and multiplication by t, we see that (4.2) ((4.3), respectively) is equivalent to $M(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$ ($N(\varphi; t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n)$, respectively), which proves the theorem for n odd.

The proof for n even is similar. Namely, by Definition 2.2 the formal solution $\hat{u}$ is Borel summable in the direction d if and only if

$$(\hat{\mathcal{B}}\hat{u})(t^2, z) = \sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(k!)^2} t^{2k} \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n).$$

Hence, by Lemma 3.2 and the uniqueness of the analytic continuation of $\sum_{k=0}^{\infty} \frac{\Delta^k \varphi(z)}{(k!)^2} t^{2k}$ with respect to t, we conclude that $\hat{u}$ is Borel summable in the direction d if and only if

$$\frac{1}{n!!} (t^{-1}\partial_t)^{n/2} t^n M(\varphi; 2t, z) \in \mathcal{O}^2((\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}) \times D^n), \tag{4.4}$$

or equivalently, if and only if

$$\frac{1}{(n-2)!!}(t^{-1}\partial_t)^{(n-2)/2}t^{n-2}N(\varphi;2t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n). \tag{4.5}$$

As in the previous case, (4.4) ((4.5), respectively) is equivalent to $M(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$ $(N(\varphi;t,z)\in\mathcal{O}^2((\hat{S}_{d/2}\cup\hat{S}_{d/2+\pi})\times D^n)$, respectively), which completes the proof. ∎

Using the representation of the Borel transform $\hat{\mathcal{B}}$ and the modified Borel transform $\tilde{\mathcal{B}}$ of $\hat{u}$, we derive the Borel sum u for the Borel summable formal solution $\hat{u}$. To this end, we calculate the function C_2 defined by (2.2). Applying the power series expansion (see [3, p. 175]) of C_2, we have

$$C_2(\zeta)=\sum_{n=0}^{\infty}\frac{(-\zeta)^n}{n!\Gamma(1-(n+1)/2)}.$$

Since the gamma function $\Gamma(z)$ has simple poles at $z=0,-1,-2,\ldots$ and

$$\Gamma(-k+1/2)=\frac{(-1)^k k!4^k\sqrt{\pi}}{(2k)!}\quad\text{for } k\in\mathbb{N}_0,$$

we obtain

$$\begin{aligned}C_2(\zeta)&=\sum_{k=0}^{\infty}\frac{\zeta^{2k}}{(2k)!\Gamma(-k+1/2)}\\&=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\zeta^{2k}}{4^k k!}=\frac{1}{\sqrt{\pi}}e^{-\zeta^2/4}.\end{aligned} \tag{4.6}$$

Now we are ready to prove that the procedure of Borel summability gives us the solution u of the heat equation as the convolution of the initial data with the heat kernel.

Theorem 4.2. *Let $d\in\mathbb{R}$ and assume that the formal solution $\hat{u}$ of* (4.1) *is Borel summable in the direction d (i.e. there exists $\varepsilon>0$ such that $\hat{\mathcal{B}}\hat{u}(s,z)$ and $\tilde{\mathcal{B}}\hat{u}(s,z)$ belong to $\mathcal{O}^1(S(d,\varepsilon)\times D^n)$). Then the Borel sum of $\hat{u}$ in the direction d is given by*

$$u^\theta(t,z)=\frac{n\alpha(n)}{(4\pi t)^{n/2}}\int_0^{\infty(\theta/2)}e^{-\tau^2/(4t)}\tau^{n-1}N(\varphi;\tau,z)\,d\tau$$

for every $\theta\in(d-\varepsilon/2,d+\varepsilon/2)$. Moreover, if additionally $\varphi\in\mathcal{O}^2((S(d/2,\varepsilon/2)\cup S(d/2+\pi,\varepsilon/2))^n)$ then also

$$u^\theta(t,z)=\frac{1}{(4\pi t)^{n/2}}\int_{(e^{i\theta/2}\mathbb{R})^n}e^{-e^{i\theta}|x|^2/(4t)}\varphi(z+x)\,dx.$$

Proof. Let ε and θ be as in the statement. First, assume that n is odd. By Proposition 2.3, (4.6) and Lemma 3.2, we have

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{\sqrt{t}} \int_0^{\infty(\theta)} (\tilde{\mathcal{B}}\hat{u})(s,z) \frac{1}{\sqrt{\pi}} e^{-s/(4t)}\, d\sqrt{s} \\ &\overset{s=\tau^2}{=} \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} (\tilde{\mathcal{B}}\hat{u})(\tau^2,z) e^{-\tau^2/(4t)}\, d\tau \\ &= \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\tau^2/(4t)} \frac{1}{(n-2)!!} \partial_\tau (\tau^{-1}\partial_\tau)^{(n-3)/2} \tau^{n-2} N(\varphi;\tau,z)\, d\tau. \end{aligned}$$

Next, by $(1+(n-3)/2)$-fold integration by parts, we obtain

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{\sqrt{\pi t}} \int_0^{\infty(\theta/2)} \frac{\tau}{2t} e^{-\tau^2/(4t)} \frac{1}{(n-2)!!} (\tau^{-1}\partial_\tau)^{(n-3)/2} \tau^{n-2} N(\varphi;\tau,z)\, d\tau \\ &= \frac{1}{(n-2)!!(2t)^{(n-1)/2}\sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\tau^2/(4t)} \tau^{n-1} N(\varphi;\tau,z)\, d\tau. \end{aligned}$$

Finally, using the definition of the integral means over the sphere, we get

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{(n-2)!!(2t)^{(n-1)/2}\sqrt{\pi t}} \int_0^{\infty(\theta/2)} e^{-\tau^2/(4t)} \tau^{n-1} \fint_{S(0,1)} \varphi(z+\tau y)\, dS(y)\, d\tau \\ &\overset{\tau y = x}{=} \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} e^{-e^{i\theta}|x|^2/(4t)} \varphi(z+x)\, dx, \end{aligned}$$

since

$$\frac{1}{n\alpha(n)} = \frac{\Gamma(1+n/2)}{n\pi^{n/2}} = \frac{n!!\pi^{1/2}}{2^{(n+1)/2} n\pi^{n/2}} = \frac{(n-2)!!}{2^{(n+1)/2}\pi^{(n-1)/2}}.$$

Analogously, for n even, we apply Definition 2.2, (4.6) and Lemma 3.2 to calculate

$$\begin{aligned} u^\theta(t,z) &= \frac{1}{t} \int_0^{\infty(\theta)} e^{-s/t} (\mathcal{B}\hat{u})(s,z)\, ds \\ &\overset{s=\tau^2}{=} \frac{1}{t} \int_0^{\infty(\theta/2)} e^{-\tau^2/t} (\mathcal{B}\hat{u})(\tau^2,z) 2\tau\, d\tau \\ &= \frac{2}{t} \int_0^{\infty(\theta/2)} e^{-\tau^2/t} \tau \frac{1}{(n-2)!!} (\tau^{-1}\partial_\tau)^{(n-2)/2} \tau^{n-2} N(\varphi;2\tau,z)\, d\tau. \end{aligned}$$

By $(n-2)/2$-fold integration by parts and by the definition of the integral mean over the sphere, we have

$$u^\theta(t,z) = \frac{2^{n/2}}{t^{n/2}(n-2)!!} \int_0^{\infty(\theta/2)} e^{-\tau^2/t}\tau^{n-1} \fint_{S(0,1)} \varphi(z+2\tau y)\,dS(y)\,d\tau$$

$$\overset{2\tau=\sigma}{=} \frac{1}{(2t)^{n/2}(n-2)!!} \int_0^{\infty(\theta/2)} e^{-\sigma^2/(4t)}\sigma^{n-1} \fint_{S(0,1)} \varphi(z+\sigma y)\,dS(y)\,d\sigma$$

$$\overset{\sigma y=x}{=} \frac{1}{(4\pi t)^{n/2}} \int_{(e^{i\theta/2}\mathbb{R})^n} e^{-e^{i\theta}|x|^2/(4t)}\varphi(z+x)\,dx,$$

since

$$\frac{1}{n\alpha(n)} = \frac{\Gamma(1+n/2)}{n\pi^{n/2}} = \frac{n!!}{2^{n/2}n\pi^{n/2}} = \frac{(n-2)!!}{2^{n/2}\pi^{n/2}}. \blacksquare$$

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Received 21.11.2011
and in final form 12.3.2012 (2662)