

Complex surfaces in \mathbb{C}^4 with recurrent shape operators

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Abstract. We study complex affine surfaces in \mathbb{C}^4 with the transversal bundle defined by Nomizu and Vrancken. We classify the surfaces that have recurrent shape operators and parallel transversal metric.

1. Preliminaries. The purpose of this paper is to characterize a class of complex surfaces holomorphically immersed in four-dimensional affine space \mathbb{C}^4 . An equiaffine structure for real surfaces in \mathbb{R}^4 was found by Nomizu and Vrancken ([NV]) and further investigations showed that the construction leads to natural geometric properties. In [W] a unique equiaffine transversal bundle is also obtained in the complex case. It has the same properties as the bundle constructed in [NV] and is holomorphic. There are three local pseudometrics in this bundle and we denote each of them by g^\perp . They differ only by a constant factor. A given g^\perp -null transversal frame $\{\xi_1, \xi_2\}$ determines shape operators S_1 and S_2 which are local endomorphisms of the tangent bundle. Let ∇ and ∇^\perp be the connections in the tangent and normal bundles, respectively.

We recall that the shape operators are *recurrent* if there exist 1-forms α_1 and α_2 such that $\nabla S_j = \alpha_j \otimes S_j$. We, however, consider a special case $\nabla S_j = (-1)^{j-1} df \otimes S_j$ because this condition does not depend on the frame $\{\xi_1, \xi_2\}$. We prove a classification theorem under the additional condition $\nabla^\perp g^\perp = 0$.

We now state the main theorem of the paper.

THEOREM 1.1. *Let M be a non-degenerate complex surface in \mathbb{C}^4 satisfying the conditions:*

- 1) *in a neighbourhood of an arbitrary point p of the surface, for every holomorphic null normal frame $\{\xi_1, \xi_2\}$ there is a local function f such*

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that

$$\nabla S_1 = df \otimes S_1, \quad \nabla S_2 = -df \otimes S_2,$$

2) $\nabla^\perp g^\perp = 0.$

Then there exists an open and dense set U in M which is locally affinely equivalent to one of the following surfaces:

(a) $x(u, v) = (u, v, u^2, v^2),$

(b) $x(u, v) = u \cdot (1, v, v^2, u^2),$

(c) $x(u, v) = (u^3 + 3u^2v, v^3 + 3uv^2, u - v, uv),$

(d) $x(u, v) = (\cosh u)^{2/3} \cdot \left(1, v, v^2, \int^u (\cosh s)^{-5/3} ds\right),$

(e) $x(u, v) = u^{-2} \cdot \left(1, u^5, v, \frac{1}{2}v^2\right).$

(f) $x(u, v) = (\sinh(u), \cosh(u), \sinh(v), \cosh(v)),$

(g) $x(u, v) = \left(v, \frac{1}{2}v^2, f_1(u), f_1(u) \int^u \frac{ds}{(f_1'(s))^2} - \int^u \frac{f_1(s) ds}{(f_1'(s))^2}\right),$

where f_1 is an arbitrary holomorphic function defined in a neighbourhood of zero and such that $f_1'(0) \neq 0$ and $f_1^{(3)}(0) \neq 0$. Conversely, all these surfaces satisfy conditions 1) and 2).

Let M be a two-dimensional complex submanifold of \mathbb{C}^4 . This means that there exists an immersion $f : M \rightarrow \mathbb{C}^4$ which is holomorphic in the sense that $f_*JX = Jf_*X$ for the complex structure J . Each tangent space T_xM has a natural structure of a complex vector space with the multiplication by i given by J . Let the vector fields, connections, bilinear and linear forms and functions used in the paper be of class $\mathcal{C}_\mathbb{R}^\infty$ unless otherwise stated (see [OI], [OII], compare also another approach in [A], [DVV]). Since our considerations are local and we can identify the complex structures on M and on \mathbb{C}^4 , we can also identify M , as a complex manifold, with its image in \mathbb{C}^4 .

Let σ denote a transversal complex plane bundle, that is, $\mathbb{C}^4 = \sigma_x \oplus T_xM$ over \mathbb{C} . Let ξ_1, ξ_2 be transversal vector fields that span σ locally over \mathbb{C} . Capital letters X, Y, Z etc. will denote tangent vector fields. If D denotes the standard affine connection on \mathbb{C}^4 , then we have

(1.1) $D_X Y = \nabla_X Y + h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2,$

(1.2) $D_X \xi_j = -S_j(X) + \tau_j^1(X)\xi_1 + \tau_j^2(X)\xi_2,$

for $j = 1, 2$, where the connection ∇ is determined by the condition $\nabla_X Y \in TM$ and the shape operators S_j by the condition $S_j(X) \in TM$. Then ∇ is a torsion-free linear connection compatible with J (and therefore called a complex connection). We can also see that both h^j are \mathbb{C} -valued and \mathbb{C} -bilinear 2-forms, called the second fundamental forms, S_j

are $(1, 1)$ - \mathbb{R} -linear tensors and τ_k^j are \mathbb{C} -valued, \mathbb{R} -linear 1-forms (see [KN], [OI], [OII]). We also define a complex-valued \mathbb{C} -linear skew-symmetric 2-form θ by $\theta(X, Y) = \text{Det}[X, Y, \xi_1, \xi_2]$, where Det denotes the usual determinant in \mathbb{C}^4 . Just as in real geometry ([NV]) we have the following equations of Gauss (1.3), Codazzi ((1.4)–(1.7)) and Ricci ((1.8)–(1.11)):

$$(1.3) \quad R(X, Y)Z = h^1(Y, Z)S_1X + h^2(Y, Z)S_2X \\ - h^1(X, Z)S_1Y - h^2(X, Z)S_2Y,$$

$$(1.4) \quad (\nabla_X h^1)(Y, Z) + \tau_1^1(X)h^1(Y, Z) + \tau_2^1(X)h^2(Y, Z) \\ \text{is symmetric in } X, Y \text{ and } Z,$$

$$(1.5) \quad (\nabla_X h^2)(Y, Z) + \tau_1^2(X)h^1(Y, Z) + \tau_2^2(X)h^2(Y, Z) \\ \text{is symmetric in } X, Y \text{ and } Z,$$

$$(1.6) \quad (\nabla_X S_1)Y - (\nabla_Y S_1)X \\ = -\tau_1^1(Y)S_1X + \tau_1^1(X)S_1Y - \tau_1^2(Y)S_2X + \tau_1^2(X)S_2Y,$$

$$(1.7) \quad (\nabla_X S_2)Y - (\nabla_Y S_2)X \\ = -\tau_2^1(Y)S_1X + \tau_2^1(X)S_1Y - \tau_2^2(Y)S_2X + \tau_2^2(X)S_2Y,$$

$$(1.8) \quad h^1(X, S_1Y) - h^1(Y, S_1X) \\ = d\tau_1^1(X, Y) + \tau_1^2(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^2(X),$$

$$(1.9) \quad h^2(X, S_1Y) - h^2(Y, S_1X) \\ = d\tau_1^2(X, Y) + \tau_1^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_1^1(X) \\ + \tau_1^2(Y)\tau_2^2(X) - \tau_2^2(Y)\tau_1^2(X),$$

$$(1.10) \quad h^2(X, S_2Y) - h^2(Y, S_2X) \\ = d\tau_2^2(X, Y) + \tau_2^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_2^1(X),$$

$$(1.11) \quad h^1(X, S_2Y) - h^1(Y, S_2X) \\ = d\tau_2^1(X, Y) + \tau_1^1(X)\tau_2^1(Y) - \tau_2^1(X)\tau_1^1(Y) \\ + \tau_2^1(X)\tau_2^2(Y) - \tau_2^2(X)\tau_2^1(Y)$$

Let $u = \{X_1, X_2\}$ be a local frame of class $\mathcal{C}_{\mathbb{R}}^{\infty}$ on a neighbourhood U of a point $p \in M$. Define a symmetric bilinear form G_u by

$$(1.12) \quad 2G_u(Y, Z) \\ = \text{Det}[X_1, X_2, D_Y X_1, D_Z X_2] + \text{Det}[X_1, X_2, D_Z X_1, D_Y X_2].$$

We call a surface *non-degenerate* if the form G_u is non-degenerate (which does not depend on the choice of u ; see [W]). From now on we will assume that the surface is non-degenerate. In a sufficiently small neighbourhood of each point we can define three branches of a \mathbb{C} -valued, \mathbb{C} -bilinear symmetric form

$$g_u(Y, Z) = G_u(Y, Z)(\det_u G_u)^{-1/3}.$$

The set of three branches of g_u is independent of the choice of u . In this way we get locally three complex-valued metrics which we denote by g . We call each of them an *affine metric* on M . From now on we fix an affine metric in a sufficiently small neighbourhood of a given point. We will work with *null frames* with respect to the affine metric g , that is, the frames $\{X_1, X_2\}$ satisfying $g(X_1, X_2) = 1$ and $g(X_j, X_j) = 0$ for $j = 1, 2$. The following theorem associates a unique transversal frame to each null tangent frame (see [NV], [W]).

THEOREM 1.2. *Let σ be a transversal plane bundle and $\{X_1, X_2\}$ be a null tangent frame. Then there exists a unique local transversal frame $\{\xi_1, \xi_2\}$ in σ such that*

$$(1.13) \quad \text{Det}[X_1, X_2, \xi_1, \xi_2] = -2,$$

and the second fundamental forms have the following matrices in the basis $\{X_1, X_2\}$:

$$(1.14) \quad h^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad h^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

From now on we will call the frame $\{\xi_1, \xi_2\}$ determined by the last theorem the transversal frame *associated* to the null tangent frame $\{X_1, X_2\}$. Notice that if we interchange the fields X_1 and X_2 , then the fields ξ_1, ξ_2 in the associated frame also interchange.

The following lemma gives the transformation rules for the associated frames.

LEMMA 1.3. *Let $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ be two null tangent frames. Let $\{\xi_1, \xi_2\}$ and $\{\eta_1, \eta_2\}$ be the respective associated transversal frames. Then there exists a \mathbb{C} -valued non-zero function γ such that*

$$(1.15) \quad Y_1 = \gamma X_1, \quad Y_2 = \gamma^{-1} X_2,$$

$$(1.16) \quad \eta_1 = \gamma^2 \xi_1, \quad \eta_2 = \gamma^{-2} \xi_2,$$

after possibly interchanging X_1 and X_2 , as well as ξ_1 and ξ_2 , if necessary.

Assume that an affine metric g is locally fixed. A complex-valued metric on an arbitrary transversal plane bundle σ is defined in the following way. Let $u = \{X_1, X_2\}$ be a null frame and $\{\xi_1, \xi_2\}$ the associated transversal frame in σ . We define a metric g_u^\perp on σ by

$$(1.17) \quad g_u^\perp(\xi_1, \xi_1) = g_u^\perp(\xi_2, \xi_2) = 0, \quad g_u^\perp(\xi_1, \xi_2) = -2$$

and extend it to a \mathbb{C} -bilinear symmetric form.

It turns out that g_u^\perp is independent of the tangent frame u . Therefore we will denote it by g^\perp . Lemma 1.3 shows that for every g^\perp -null frame $\{\xi_1, \xi_2\}$ satisfying (1.17) there is a unique tangent g -null frame $\{X_1, X_2\}$ such that

$\{\xi_1, \xi_2\}$ is associated to it. Therefore we will call both frames associated to each other.

The following theorem holds ([W], cf. [NV]):

THEOREM 1.5. *For every complex, non-degenerate surface M in \mathbb{C}^4 there is a unique equiaffine transversal bundle σ such that $\nabla^\perp g^\perp$ is symmetric, where g^\perp is an arbitrary transversal metric.*

COROLLARY 1.6. *A transversal bundle σ is the affine normal bundle if and only if the following equations hold:*

$$(1.18) \quad \begin{aligned} \nabla g(X_1, X_2, X_1) &= \nabla g(X_2, X_1, X_2) = 0, \\ \nabla g(X_1, X_1, X_1) &= \nabla g(X_2, X_2, X_2) = 0, \end{aligned}$$

where ∇ is the connection induced by σ , $\{X_1, X_2\}$ is a holomorphic null tangent frame and g is the affine metric on the surface.

COROLLARY 1.7. *Let σ be a transversal bundle, $\{X_1, X_2\}$ a holomorphic null tangent frame, and $\{\xi_1, \xi_2\}$ the associated transversal frame. Then σ is the affine normal bundle if and only if*

$$(1.19) \quad \tau_1^1 + \tau_2^2 = 0, \quad \tau_2^1(X_1) = \tau_1^2(X_2) = 0.$$

2. The proof of the main theorem. First we prove the following lemma:

LEMMA 2.1. *If the surface M satisfies assumption 1) of Theorem 1.1, then for every point of M there exists a neighbourhood and a null transversal frame $\{\xi_1, \xi_2\}$ such that the induced shape operators have the following properties:*

$$(2.1) \quad \nabla S_1 = 0, \quad \nabla S_2 = 0.$$

Proof. It suffices to define a new transversal frame $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ by (1.15) and (1.16) with $\gamma = e^{-\frac{1}{2}f}$.

In what follows we will use the transversal frame obtained in the previous lemma and the associated tangent frame $\{X_1, X_2\}$. Introduce local functions a_1 to a_8 by

$$(2.2) \quad \begin{aligned} \nabla_{X_1} X_1 &= a_1 X_1 + a_2 X_2, & \nabla_{X_1} X_2 &= a_3 X_1 + a_4 X_2, \\ \nabla_{X_2} X_1 &= a_5 X_1 + a_6 X_2, & \nabla_{X_2} X_2 &= a_7 X_1 + a_8 X_2. \end{aligned}$$

Using Corollary 1.6 we obtain

$$(2.3) \quad a_4 = -a_1, \quad a_5 = -a_8, \quad a_2 = 0, \quad a_7 = 0.$$

The fact that $\nabla^\perp g^\perp = 0$ implies that

$$(2.4) \quad \tau_2^1 = 0 \quad \text{and} \quad \tau_1^2 = 0$$

identically. If we define

$$(2.5) \quad a = \tau_1^1(X_1), \quad b = \tau_1^1(X_2),$$

we also have

$$(2.6) \quad -a = \tau_2^2(X_1), \quad -b = \tau_2^2(X_2),$$

using Corollary 1.7. The Codazzi equations (1.4) and (1.5) lead to

$$(2.7) \quad a_3 = -(b + 2a_8), \quad a_6 = a - 2a_1.$$

We introduce functions c_1 to c_4 as well as k_1 to k_4 such that

$$(2.8) \quad \begin{aligned} S_1X_1 &= c_1X_1 + c_2X_2, & S_1X_2 &= c_3X_1 + c_4X_2, \\ S_2X_1 &= k_1X_1 + k_2X_2, & S_2X_2 &= k_3X_1 + k_4X_2. \end{aligned}$$

In the next three lemmas we give some properties of these functions.

LEMMA 2.2. *The functions c_2 and k_3 vanish identically and $c_3 = k_2$.*

Proof. Since $\tau_1^2 = 0$, by (1.9) and (1.14) we have $0 = -h^2(X_2, S_1X_1) = -h^2(X_2, c_1X_1 + c_2X_2) = -c_2$. Moreover (1.11), (1.14) and $\tau_2^1 = 0$ imply that $0 = h^1(X_1, S_2X_2) = h^1(X_1, k_3X_1 + k_4X_2) = k_3$. Using Corollary 1.7 and adding (1.8) and (1.10) we obtain $0 = h^1(X_1, S_1X_2) - h^2(X_2, S_2X_1) = c_3 - k_2$, which completes the proof.

LEMMA 2.3. *The following equations hold:*

$$(2.9) \quad bc_1 = ac_3, \quad ac_4 = 0, \quad bk_1 = 0, \quad bk_2 = ak_4.$$

Proof. Since $\nabla S_j = 0$, the Codazzi equations (1.6) and (1.7) imply

$$\begin{aligned} 0 &= -bS_1X_1 + aS_1X_2 = (-bc_1 + ac_3)X_1 + (-bc_2 + ac_4)X_2, \\ 0 &= bS_2X_1 - aS_2X_2 = (bk_1 - ak_3)X_1 + (bk_2 - ak_4)X_2, \end{aligned}$$

and, by the previous lemma, we obtain (2.9).

LEMMA 2.4. *The coefficients c_j and k_j , where $j = 1, 2$, satisfy*

$$(2.10) \quad X_1(c_1) = 0,$$

$$(2.11) \quad X_1(c_3) + 2a_1c_3 - a_3(c_1 - c_4) = 0,$$

$$(2.12) \quad X_1(c_4) = 0,$$

$$(2.13) \quad X_2(c_1) - a_6c_3 = 0,$$

$$(2.14) \quad a_6(c_1 - c_4) = 0,$$

$$(2.15) \quad X_2(c_3) - 2a_8c_3 = 0,$$

$$(2.16) \quad X_2(c_4) + a_6c_3 = 0,$$

$$(2.17) \quad X_1(k_1) + a_3k_2 = 0,$$

$$(2.18) \quad X_1(k_2) - 2a_1k_2 = 0,$$

$$(2.19) \quad -a_3(k_4 - k_1) = 0,$$

$$(2.20) \quad X_1(k_4) - a_3k_2 = 0,$$

$$(2.21) \quad X_2(k_2) + 2a_8k_2 + a_6(k_1 - k_4) = 0,$$

$$(2.22) \quad X_2(k_1) = 0,$$

$$(2.23) \quad X_2(k_4) = 0.$$

Proof. The equations are consequences of the conditions $\nabla S_j = 0$, applied to all possible pairs of the vector fields X_k, X_l , for $j, k, l = 1, 2$.

Substituting the triples (X_1, X_2, X_1) and (X_1, X_2, X_2) in the Gauss equation (1.3) and using Lemma 2.2 we obtain the following lemma.

LEMMA 2.5. *The functions a_1, a_3, a_6 and a_8 satisfy*

$$(2.24) \quad X_1(a_8) + X_2(a_1) = c_3 - a_6a_8 - a_1a_3 - 2a_1a_8 + a_3a_6,$$

$$(2.25) \quad X_1(a_6) = -c_4 + a_6(a_1 - a_6),$$

$$(2.26) \quad X_2(a_3) = -k_1 + a_3(a_8 - a_3).$$

We now define an open and dense subset of M and we restrict our classification to this set. From the proof of Lemma 2.1 it follows that the frame defined there was defined up to constant non-zero factors. Thus the functions a_j are also determined up to constant factors. Therefore the following definitions make sense on the whole surface, although they use local functions. Set

$$(2.27) \quad U_1 = \{p \in M \mid a_3 \neq 0 \text{ or } a_6 \neq 0\}, \quad U_2 = U \setminus \overline{U_1}.$$

Then $U_1 \cup U_2$ is an open and dense subset of M . We consider two cases.

LEMMA 2.6. *Any $p \in U_1$ has a neighbourhood U_p in which $k_2 = c_3 = 0$ identically, the functions c_1, c_4, k_1 and k_4 are constant, $c_1 = c_4$ and $k_1 = k_4$.*

Proof. Let U_p be contained in the domain of the frame with respect to which the functions c_j and k_j are defined. From the definitions of U_1 and the holomorphy of the functions considered it follows that $c_1 = c_4$ or $k_1 = k_4$ on the whole U_p .

We first assume that $c_1 = c_4$. After adding (2.13) and (2.16) we have $X_2(c_1) = X_2(c_4) = 0$ on U_p . Taking into account (2.10) and (2.12) we conclude that c_1 and c_4 are constant. Now (2.13) implies that $c_3 = 0$, and Lemma 2.2 gives $k_2 = 0$. If we consider the system (2.19), (2.21), then by the assumptions we have $k_1 = k_4$. Using this fact and adding (2.17) and (2.20), we obtain $X_1(k_1) = X_1(k_2) = 0$. Taking into account (2.22) and (2.23) now shows that k_1 and k_4 are constant. This completes the proof for $c_1 = c_4$. The argument in the case when $k_1 = k_4$ is analogous.

LEMMA 2.7. *There exists an open and dense subset V of U_2 such that for every $p \in V$ there is a neighbourhood of p that is affinely equivalent to*

one of the following surfaces:

$$(2.28a) \quad x(u, v) = (u, v, u^2, v^2),$$

$$(2.28b) \quad x(u, v) = u \cdot (1, v, v^2, u^2),$$

$$(2.28c) \quad x(u, v) = (u^3 + 3u^2v, v^3 + 3uv^2, u - v, uv),$$

$$(2.28d) \quad x(u, v) = (\sinh(u), \cosh(u), \sinh(v), \cosh(v)),$$

$$(2.28e) \quad x(u, v) = \left(v, \frac{1}{2}v^2, f_1(u), f_1(u) \int \frac{ds}{(f_1'(s))^2} - \int \frac{f_1(s) ds}{(f_1'(s))^2} \right),$$

where f_1 is an arbitrary holomorphic function of one variable defined in a neighbourhood of zero such that $f_1' \neq 0$ and $f_1^{(3)}(0) \neq 0$.

Proof. By definition of U_2 , $a_3 = a_6 = 0$ identically. Using (2.10), (2.12), (2.13), (2.16), (2.17), (2.20), (2.22) and (2.23) we find that c_1, c_4, k_1 and k_4 are constant. By (2.25) and (2.26) we immediately obtain

$$c_4 = k_1 = 0.$$

We next use the Ricci equation (1.8) keeping the notation (2.5):

$$c_3 = X_1(b) - X_2(a) - \tau_1^1([X_1, X_2]).$$

The Poisson bracket $[X_1, X_2]$ is equal to $\nabla_{X_1}X_2 - \nabla_{X_2}X_1 = -a_1X_2 + a_8X_1$, and the last equation is equivalent to

$$(2.29) \quad c_3 = X_1(b) - X_2(a) + a_1b - a_8a.$$

Considering the equation (2.7), we rewrite (2.29) as

$$(2.30) \quad c_3 = -2X_1(a_8) - 2X_2(a_1) - 4a_1a_8.$$

Next we consider the equation (2.24), which now has the form

$$(2.31) \quad c_3 = X_1(a_8) + X_2(a_1) + 2a_1a_8.$$

Considering the system (2.30), (2.31), as well as Lemma 2.2, we obtain

$$(2.32) \quad c_3 = k_2 = 0.$$

We notice that the system (2.9) now has the form

$$(2.33) \quad bc_1 = 0, \quad ak_4 = 0.$$

We define the following open subsets of U_2 :

$$(2.34) \quad \begin{aligned} V_1 &= \{p \in U_2 \mid c_1k_4 \neq 0\}, \\ V_2 &= (U_2 \setminus \overline{V_1}) \cap \{c_1 \neq 0 \text{ or } k_1 \neq 0\}, \\ V_3 &= U_2 \setminus (\overline{V_1} \cup \overline{V_2}). \end{aligned}$$

Then $V_1 \cup V_2 \cup V_3$ is an open and dense subset of U_2 . We notice that in V_3 both shape operators S_1 and S_2 are zero, so it is an umbilical surface considered in [VW] in the real indefinite case. Since the argument and computations

are very similar in our case, we conclude that there is an open and dense subset W of V_3 which is affinely equivalent to one of the surfaces (2.28a-c).

Now, $V_1 \cup V_2 \cup W$ is also an open and dense subset of U_2 . Let $p \in V_1$. Then in a neighbourhood of p we have $a = b = 0$, by (2.33) and the definition of V_1 . This implies

$$(2.35) \quad a_1 = a_8 = 0.$$

In this case we have $\nabla_{X_j} X_k = 0$ for $j, k = 1, 2$, whence $[X_1, X_2] = 0$. There are local complex coordinates u and v such that for the immersion x the following equations hold:

$$x_u = X_1, \quad x_v = X_2.$$

By the Gauss formula (1.1) we obtain

$$(2.36) \quad x_{uu} = \xi_1, \quad x_{uv} = 0, \quad x_{vv} = \xi_2.$$

This implies that x_u depends only on u , and x_v only on v . By the Weingarten formula we immediately obtain

$$(2.37) \quad \begin{aligned} (\xi_1)_u &= -c_1 x_u, \\ (\xi_1)_v &= (\xi_2)_u = 0, \\ (\xi_2)_v &= -k_4 x_v. \end{aligned}$$

Combining (2.36) with (2.37) we obtain two differential equations:

$$(2.38) \quad x_{uuu} = -c_1 x_u, \quad x_{vvv} = -k_4 x_v.$$

Integrating them we have

$$\begin{aligned} x_u &= \cosh(\sqrt{-c_1}u)A + \sinh(\sqrt{-c_1}u)B, \\ x_v &= \cosh(\sqrt{-k_1}v)C + \sinh(\sqrt{-k_1}v)D, \end{aligned}$$

where A, B, C and D are constant vectors in \mathbb{C}^4 . Up to an affine transformation we obtain the parametric equation

$$x(u, v) = (\sinh(\sqrt{-c_1}u), \cosh(\sqrt{-c_1}u), \sinh(\sqrt{-k_1}v), \cosh(\sqrt{-k_1}v)).$$

If we treat x as a function of the variables $\sqrt{-c_1}u$ and $\sqrt{-c_1}v$, we finally get the surface (2.28d).

We consider the remaining case. Let $p \in V_2$. Let $c_1 \neq 0$ first. When we change the null frame $\{X_1, X_2\}$ putting a constant function γ such that $\gamma^2 = c_1$ in (1.15), the functions a and b defined in (2.5) do not change. Thus we can assume that $c_1 = 1$. By definition of V_1 , k_4 is equal to 0. The first equation of (2.33) gives $b = 0$, whence $a_8 = 0$. The equation (2.31) gives $X_2(a_1) = 0$. Consider the system of differential equations

$$(2.39) \quad X_1(\varrho) = a_1 \varrho, \quad X_2(\varrho) = 0$$

for an unknown function ϱ with $\varrho(p) \neq 0$. It is easy to see that the system satisfies the integrability conditions, so it has a solution. We define the vector

fields \tilde{X}_1 and \tilde{X}_2 by the following formulas:

$$(2.40) \quad \tilde{X}_1 = \frac{1}{\varrho} X_1, \quad \tilde{X}_2 = \varrho X_2.$$

We then have $\nabla_{\tilde{X}_1} \tilde{X}_2 = \varrho^{-1} X_1(\varrho) X_2 + \nabla_{X_1} X_2 = \varrho^{-1} a_1 \varrho X_2 - a_1 X_2 = 0$ and, similarly, $\nabla_{\tilde{X}_2} \tilde{X}_1 = 0$. Thus there exist local coordinates u and v in a neighbourhood of zero such that $x(0, 0) = p$ and

$$(2.41) \quad x_u = \frac{1}{\varrho} X_1, \quad x_v = \varrho X_2.$$

The frame $\{\tilde{X}_1, \tilde{X}_2\}$ is a null frame. The associated transversal frame $\{\tilde{\xi}_1, \tilde{\xi}_2\}$ is given by

$$(2.42) \quad \tilde{\xi}_1 = \frac{1}{\varrho^2} \xi_1, \quad \tilde{\xi}_2 = \varrho^2 \xi_2.$$

To use the Gauss formula (1.1), we compute

$$\nabla_{\tilde{X}_1} \tilde{X}_1 = \frac{1}{\varrho} \nabla_{X_1} \left(\frac{1}{\varrho} X_1 \right) = -\frac{1}{\varrho} \frac{X_1(\varrho)}{\varrho^2} X_1 + \frac{1}{\varrho^2} a_1 X_1 = 0.$$

Similarly, $\nabla_{\tilde{X}_1} \tilde{X}_2 = 0$ and $\nabla_{\tilde{X}_2} \tilde{X}_2 = 0$. Thus we have

$$(2.43) \quad x_{uu} = \tilde{\xi}_1, \quad x_{uv} = 0, \quad x_{vv} = \tilde{\xi}_2.$$

The Weingarten formula implies

$$(2.44) \quad \begin{aligned} (\tilde{\xi}_1)_u &= D_{\varrho^{-1} X_1} \left(\frac{1}{\varrho^2} \xi_1 \right) = \frac{1}{\varrho} \frac{-2X_1(\varrho)}{\varrho^3} \xi_1 + \frac{1}{\varrho^3} (-S_1 X_1 + \tau_1^1(X_1) \xi_1) \\ &= \frac{-2a_1}{\varrho^3} \xi_1 - \frac{1}{\varrho^3} X_1 + \frac{2a_1}{\varrho^3} \xi_1 = -\frac{1}{\varrho^2} \tilde{X}_1, \end{aligned}$$

$$(2.45) \quad \begin{aligned} (\tilde{\xi}_1)_v &= D_{\varrho X_2} \left(\frac{1}{\varrho^2} \xi_1 \right) = \varrho \frac{-2X_2(\varrho)}{\varrho^3} \xi_1 + \frac{1}{\varrho} (-S_1 X_2 + \tau_1^1(X_2) \xi_1) \\ &= \frac{2a_8}{\varrho} \xi_1 = 0, \end{aligned}$$

$$(2.46) \quad \begin{aligned} (\tilde{\xi}_2)_u &= D_{\varrho^{-1} X_1} (\varrho^2 \xi_2) = \frac{1}{\varrho} 2X_1(\varrho) \varrho \xi_2 + \varrho (-S_2 X_1 + \tau_2^2(X_1) \xi_2) \\ &= 2a_1 \varrho \xi_2 - 2\varrho a_1 \xi_2 = 0, \end{aligned}$$

$$(2.47) \quad (\tilde{\xi}_2)_v = D_{\varrho X_2} (\varrho^2 \xi_2) = \varrho \cdot 2X_2(\varrho) \varrho \xi_2 + \varrho (-S_2 X_2 + \tau_2^2(X_2) \xi_2) = 0.$$

The equations (2.46), (2.47) and (2.43) imply that x_{vv} is a constant vector. Hence

$$(2.48) \quad x_v = v \cdot A + B$$

for constant vectors $A, B \in \mathbb{C}^4$. By (2.42), (2.43), (2.45) and (2.46) we have

$$(2.49) \quad x_{uuu} = -\varrho^{-2} x_u.$$

We can treat (2.49) as a one-variable differential equation, homogeneous of order two. Its solution is of the form

$$(2.50) \quad x_u = g_1(u)C + g_2(u)E,$$

where C and E are elements of \mathbb{C}^4 , and the functions g_1 and g_2 are linearly independent solutions of the differential equation

$$(2.51) \quad y'' + \varrho^{-2}y = 0.$$

We can always choose g_1 and g_2 so that they are locally different from zero, which implies that the surface obtained by (2.48) and (2.49) is non-degenerate. (2.51) implies that their second derivatives are also different from zero. By the assumption, a_1 can be any function of u . Thus the function $\varrho(u)$ given by the equation $\varrho' = a_1$ is arbitrary and non-zero in a neighbourhood of zero. Let f_1 and f_2 be indefinite integrals of g_1 and g_2 , respectively. Since x_u and x_{uu} are linearly independent, the determinant $\begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}$ is different from zero.

On the other hand, f_1' and f_2' satisfy (2.51) and are non-zero, so

$$(2.52) \quad f_1''' f_2' - f_1' f_2''' = 0.$$

Using this equation, we compute

$$(f_1' f_2'' - f_2' f_1'')' = f_1'' f_2'' + f_1' f_2''' - f_1'' f_2'' - f_1''' f_2' = 0,$$

whence

$$(2.53) \quad r := f_1' f_2'' - f_2' f_1''$$

is a non-zero constant. We can see that each choice of f_1 with $f_1'(0) \neq 0$ and $f_1^{(3)} \neq 0$ leads to an equation of a desired surface. Fix such an f_1 . Then (2.51) defines the function ϱ explicitly. By (2.53), which we treat as a first order equation with respect to f_2' , we obtain this function and its integral up to a constant. It has the form

$$(2.54) \quad f_2(u) = r f_1(u) \int \frac{ds}{(f_1'(s))^2} - r \int \frac{f_1(s) ds}{(f_1'(s))^2}.$$

Then by (2.48) and (2.50) and after applying an affine transformation we get the parametric equation (g) from the main theorem. The last case, $c_1 = 0$ and $k_4 \neq 0$, can be reduced to the previous one by interchanging X_1 and X_2 .

Notice that the surfaces we have obtained are essentially different, that is, not affinely equivalent. In the case of the umbilical surfaces this follows from the explanation given in [V VW]. For the other surfaces it follows from the fact that the shape operators are different and cannot coincide after admissible transformations of the null frames.

Proof of Theorem 1.1. By (2.27) the set $U = U_1 \cup U_2$ is open and dense in M . If $p \in U_1$, then by Lemma 2.6 it follows that there is a neighbourhood of p which is an umbilical surface. Then from [VVW] (the real indefinite umbilical case) it follows that the neighbourhood is one of the surfaces (a) to (e). On the other hand, we can choose a transversal frame for which the shape operators satisfy $\nabla S_1 = 0$, $\nabla S_2 = 0$.

If $p \in U_2$, then Lemma 2.7 gives surfaces (a), (b), (c), (f) and (g). To prove the converse in this case it is enough to verify that the surfaces (f) and (g) satisfy assumption 1) of the theorem. For (f) we can see that the shape operators have constant coefficients with respect to the associated null tangent frame. This implies $\nabla S_1 = \nabla S_2 = 0$. For (g) we choose a frame $\{\tilde{X}_1, \tilde{X}_2\}$ such that $\nabla_{\tilde{X}_j} \tilde{X}_k = 0$ for all j, k . From (2.44)–(2.47), we see that only $\tilde{S}_1 \tilde{X}_1$ is non-zero and it is equal to $-\varrho^{-3} \tilde{X}_1$. The equations (2.1) are now satisfied with $f = -\varrho^{-3}$. This completes the proof of the theorem.

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