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Sufficient conditions for starlike and convex functions

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Abstract. For $n \geq 1$, let $\mathcal{A}$ denote the class of all analytic functions f in the unit disk Δ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For $\operatorname{Re}\alpha < 2$ and $\gamma > 0$ given, let $\mathcal{P}(\gamma, \alpha)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) - \alpha \frac{f(z)}{z} + \alpha - 1 \right| \leq \gamma, \quad z \in \Delta.$$

We find sufficient conditions for functions in $\mathcal{P}(\gamma, \alpha)$ to be starlike of order β. A generalization of this result along with some convolution results is also obtained.

1. Introduction and main results. Let $\mathcal{A}$ and $\mathcal{S}$ represent the classes of all normalized analytic functions f ($f(0) = f'(0) - 1 = 0$) and all univalent functions in the unit disk $\Delta = \{z : |z| < 1\}$, respectively. Also, let $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ represent the subclasses of $\mathcal{S}$ consisting of the starlike and convex functions of order β $(0 \leq \beta < 1)$, respectively. Analytically, for $0 \leq \beta < 1$, these are defined as follows:

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \beta,\ z \in \Delta \right\}$$

and $\mathcal{K}(\beta) = \{f \in \mathcal{S} : zf'(z) \in \mathcal{S}^*(\beta)\}$. Also, let

$$\mathcal{R}(\beta) = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \beta,\ z \in \Delta\}.$$

Consider the linear transformation A on $\mathcal{A}$ defined by

$$Ag(z) := G(z) = \int_0^1 \frac{g(tz)}{t}\, dt.$$

Here Ag is often referred to as the *Alexander transform* of g, and we have $zG'(z) = g(z)$. Since $zg' \in \mathcal{S}^*(\beta)$ if and only if $g \in \mathcal{K}(\beta)$, $Ag(z) := G(z)$ provides a one-to-one correspondence between $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$. At this point it is interesting to recall a well known result [4, Theorem 8.11] that there

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exists a function $g_0 \in \mathcal{S}$ such that $Ag_0(z) = \int_0^1 t^{-1} g_0(tz)\, dt \notin \mathcal{S}$. Note that if $G(z)$ is the Alexander transform of g, then G satisfies the differential equation

$$G'(z) + zG''(z) = g'(z).$$

The following weighted integral operator has been considered by a number of researchers (e.g. [7, 10, 11]):

$$V_\lambda(g)(z) = \int_0^1 \lambda(t)\, \frac{g(tz)}{t}\, dt, \qquad g \in \mathcal{A}, \tag{1.1}$$

where $\lambda(t)$ is a non-negative real-valued function on $[0,1]$ normalized by $\int_0^1 \lambda(t)\, dt = 1$. In the special case of $\lambda(t) \equiv 1$, $V_\lambda(g)(z)$ reduces to the Alexander transform of g. A natural problem is:

Problem 1.1. *Do there exist conditions on $\lambda(t)$ and a family $\mathcal{G} \subset \mathcal{A}$ so that $g \in \mathcal{G}$ implies that $V_\lambda(g)$ is in $\mathcal{S}$ or $\mathcal{S}^*$ or $\mathcal{K}$ or any other interesting subclass of $\mathcal{S}$?*

The above question is motivated by a general result due to Fournier and Ruscheweyh in [7] which has already been extended in a number of ways (see [10, 2, 1, 11]). For example, one has

Corollary 1.1 ([7]). *Let $\beta = 1 - 1/2(1 - \log 2) \approx -0.6294$ and let $g \in \mathcal{A}$ satisfy the condition $\mathrm{Re}\{e^{i\phi}(g'(z) - \beta)\} > 0$ in Δ for some $\phi \in \mathbb{R}$. Then the Alexander transform Ag is starlike, and the value of β is sharp.*

It is interesting to look at a perturbed version of Corollary 1.1. In this case,

$$\lambda(t) = (1/\alpha) t^{-1+1/\alpha}, \qquad \alpha > 0,$$

for which the corresponding $V_\lambda(g)(z) = f(z)$ in (1.1) is known as the *Bernardi integral operator* [3] and it satisfies the first order differential equation

$$(1-\alpha)\, \frac{f(z)}{z} + \alpha f'(z) = \frac{g(z)}{z},$$

which in turn implies that

$$f'(z) + \alpha z f''(z) = g'(z).$$

Thus, we have the following reformulated version of a result from [7].

Corollary 1.2. *Let $\alpha \geq 1/3$ and β be given by*

$$\frac{\beta}{1-\beta} = -\frac{1}{\alpha} \int_0^1 t^{-1+1/\alpha}\, \frac{1-t}{1+t}\, dt. \tag{1.2}$$

If $f \in \mathcal{A}$ satisfies the condition $\mathrm{Re}\{e^{i\phi}(f'(z) + \alpha z f''(z) - \beta)\} > 0$ in Δ for some $\phi \in \mathbb{R}$, then f is starlike, and the value of β is sharp.

We remark that the above two corollaries coincide in the case of $\alpha = 1$. Now, for $\alpha \in \mathbb{C}$ and $\gamma \geq 0$, define

$$\mathcal{P}(\gamma, \alpha) = \left\{ f \in \mathcal{A} : \left| f'(z) - \alpha \frac{f(z)}{z} + \alpha - 1 \right| \leq \gamma,\, z \in \Delta \right\}.$$

In a recent paper, Ponnusamy and Singh [12] obtained the following result, which is in fact a reformulation of their result via a simple transformation $\alpha \mapsto -\alpha/(1-\alpha)$.

Theorem 1.1 ([12, Corollary 2, p. 143]). *If $\alpha \leq 0$ and $f \in \mathcal{P}(\gamma, \alpha)$, then $f \in \mathcal{S}^*$ provided*

$$\gamma = \begin{cases} \dfrac{2-\alpha}{2(1-\alpha)} & \text{if } \alpha \leq \dfrac{1-\sqrt{3}}{2}, \\ \dfrac{(2-\alpha)\sqrt{1-\alpha^2}}{\sqrt{5-4\alpha}} & \text{if } \dfrac{1-\sqrt{3}}{2} \leq \alpha \leq 0. \end{cases}$$

Our main aim is to extend Theorem 1.1 and the results of [6]. We state our first result.

Theorem 1.2. *Let $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < 2$. Define*

$$\gamma_1(\alpha, \beta) = \sup\{\gamma \geq 0 : \mathcal{P}(\gamma, \alpha) \subset \mathcal{S}^*(\beta)\}.$$

Then

$$\gamma_1(\alpha, \beta) = |2-\alpha|(1-\beta) \inf_{T \in \mathbb{R}} \left\{ \frac{|1+iT|}{|2-\alpha| + |\alpha - \beta + (1-\beta)iT|} \right\}.$$

For real values of α, Theorem 1.2 takes the following simple form.

Corollary 1.3. *Let $\alpha < 2$ and*

$$\beta_1 = \frac{3\beta + 2 - \sqrt{9\beta^2 - 20\beta + 12}}{4}.$$

Then $\mathcal{P}(\gamma, \alpha) \subset \mathcal{S}^*(\beta)$ for $0 \leq \gamma \leq \gamma_1(\alpha, \beta)$, where

$$\gamma_1(\alpha, \beta) = \begin{cases} \dfrac{(2-\alpha)(1-\beta)}{2-\beta} & \text{if } \dfrac{1}{2-\beta} \leq \alpha < 2, \\ \dfrac{(2-\alpha)(1-\beta)}{2(1-\alpha)+\beta} & \text{if } \alpha \leq \beta_1, \\ \dfrac{(2-\alpha)\sqrt{(1-\alpha)(1+\alpha-2\beta)}}{\sqrt{5-4\alpha-2\beta(1-\alpha)}} & \text{if } \beta_1 \leq \alpha \leq \dfrac{1}{2-\beta}. \end{cases}$$

The result is sharp.

The proofs of Theorem 1.2 and Corollary 1.3 are presented in Section 2 and we adopt the approach of Fournier and Mocanu [6]. If we let $zf' \in \mathcal{P}(\gamma, \alpha)$, then Corollary 1.3 takes the following equivalent form.

Corollary 1.4. *Let* $\alpha < 2$, β_1 *and* $\gamma_1(\alpha, \beta)$ *be as in Corollary* 1.3. *If* $0 \le \gamma \le \gamma_1(\alpha, \beta)$ *and* $f \in \mathcal{A}$ *satisfies the condition*

$$|zf''(z) + (1-\alpha)f'(z) + \alpha - 1| \le \gamma \quad \text{for } z \in \Delta,$$

then $f \in \mathcal{K}(\beta)$. *The result is sharp.*

Remark. When $\beta = 0$, Theorem 1.2 and Corollary 1.3 give recent results obtained in [6, Theorem 1 and Corollary 1], which indeed extend Theorem 1.1. Moreover, for $\alpha = 1$, Corollary 1.4 yields the following:

(i) If $0 \le \gamma \le (1-\beta)/(2-\beta)$, then $|zf''(z)| \le \gamma$ implies $f \in \mathcal{K}(\beta)$. The result is sharp as seen by considering the function $f(z) = z + (\gamma/2)z^2$.

(ii) If

$$0 \le \gamma \le \begin{cases} \dfrac{2\sqrt{1-2\beta}}{\sqrt{5-2\beta}} & \text{if } 0 \le \beta < 1/4, \\ \dfrac{2(1-\beta)}{2+\beta} & \text{if } 1/4 \le \beta < 1, \end{cases}$$

then $|f'(z) - 1| \le \gamma$ implies that $f \in \mathcal{S}^*(\beta)$. The result is sharp. In particular, we have the following well known result [14]:

$$\{f \in \mathcal{A} : |f'(z) - 1| \le 2/\sqrt{5},\ z \in \Delta\} \subseteq \mathcal{S}^*$$

and $2/\sqrt{5}$ cannot be replaced by a larger number (see Fournier [5]).

Our next theorem gives sufficient conditions for the derivative of a function in $\mathcal{A}$ to have real part positive.

Theorem 1.3. *If* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re} \alpha < 2$ *and* $\gamma_2(\alpha, \beta) = \sup\{\gamma > 0 : \mathcal{P}(\gamma, \alpha) \subset \mathcal{R}(\beta)\}$, *then*

$$\gamma_2(\alpha, \beta) = \frac{(1-\beta)|2-\alpha|}{|2-\alpha| + |\alpha|}.$$

The result is sharp.

The proof of Theorem 1.3 is given in Section 2. For the proof of the sharpness parts of our results we need the following lemma due to Ruscheweyh [5].

Lemma 1.1. *Given* $\theta \in \mathbb{R}$, *there exists a sequence* $\{W_k\} \subset \mathcal{B}_0$ *of functions analytic in the closed unit disk* $\overline{\Delta}$ *such that* $W_k(1) = 1$ *and* $\lim_{k\to\infty} W_k(z) = e^{i\theta} z$ *uniformly on compact subsets of* $\overline{\Delta} \setminus \{1\}$. *Here* $\mathcal{B}_0 = \{f \in \mathcal{H} : |f(z)| \le 1, f(0) = 0\}$, *and* $\mathcal{H}$ *denotes the family of analytic functions in* Δ.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in the unit disk, let

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \qquad z \in \Delta.$$

Finally, we recall the well known fact observed by Ruscheweyh:

Lemma 1.2. *Define*

$$(\mathcal{S}^*(\beta))' := \left\{ \sum_{n=0}^{\infty} \frac{n+1-\beta+(1-\beta)iT}{(1-\beta)(1+iT)}\, z^{n+1} : T \in \mathbb{R} \right\}.$$

An analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to* $\mathcal{S}^*(\beta)$ *if and only if*

$$\frac{f(z)}{z} * \frac{h(z)}{z} \neq 0 \quad \text{in } z \in \Delta \text{ for all } h \in (\mathcal{S}^*(\beta))'.$$

For our final application, we require the following result.

Lemma 1.3. *If* p *is analytic in* Δ, $p(0) = 1$, *and* $\operatorname{Re} p(z) > 1/2$ *in* Δ *then for any function* F *analytic in* Δ, *the function* $p * F$ *takes values in the convex hull of the image of* Δ *under* F.

The conclusion of Lemma 1.3 readily follows by using the Herglotz representation for functions with positive real part.

2. Proof of main results.

The proofs of our theorems mainly rely on [6].

Proof of Theorem 1.2. Suppose that $f \in \mathcal{P}(\gamma, \alpha)$. Then, by definition, we have

$$f'(z) - \alpha\, \frac{f(z)}{z} + \alpha - 1 = \gamma w(z)$$

for some $w \in \mathcal{B}_0$. Let $w(z) = \sum_{n=1}^{\infty} a_n(w) z^n$, $z \in \Delta$. Then

$$f(z) = z + \gamma \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha}\, z^{n+1}. \tag{2.1}$$

To show that $f \in \mathcal{S}^*(\beta)$, by Lemma 1.2, it suffices to prove that

$$1 + \gamma \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} \left(\frac{n+1-\beta+(1-\beta)iT}{(1-\beta)(1+iT)} \right) z^n \neq 0 \quad \text{in } \Delta$$

which is clearly true if $\gamma M(\alpha,\beta) \le 1$, where $\gamma_1(\alpha,\beta)^{-1} = M(\alpha,\beta)$ with

$$M(\alpha,\beta) = \sup_{\substack{T\in\mathbb{R}\\ z\in\Delta\\ w\in\mathcal{B}_0}} \left| \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} \left(\frac{n+1-\beta+(1-\beta)iT}{(1-\beta)(1+iT)} \right) z^n \right|.$$

Define a functional I over $\mathcal{B}_0$ as

$$I(w) = \sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha}.$$

As observed in [6], the functional I is well defined and continuous on $\mathcal{B}_0$. Thus, by the compactness of the class $\mathcal{B}_0$, there exists a function B_α in $\mathcal{B}_0$

such that

$$|I(B_\alpha)| = \sup_{w\in\mathcal{B}_0} |I(w)|.$$

For a fixed $z \in \varDelta$, we have

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha}\left(\frac{n+1-\beta+(1-\beta)iT}{(1-\beta)(1+iT)}\right)z^n\right| \\
&\quad \le \frac{1}{(1-\beta)|1+iT|}\left|\sum_{n=1}^{\infty} a_n(w)z^n\right| + \left|\frac{\alpha-\beta+(1-\beta)iT}{(1-\beta)(1+iT)}\right|\left|\sum_{n=1}^{\infty}\frac{a_n(w)}{n+1-\alpha}z^n\right| \\
&\quad \le \frac{1}{(1-\beta)|1+iT|} + \left|\frac{\alpha-\beta+(1-\beta)iT}{(1-\beta)(1+iT)}\right| |I(B_\alpha)|,
\end{aligned}$$

which gives the following upper bound for $M(\alpha,\beta)$:

$$M(\alpha,\beta) \le \sup_{T\in\mathbb{R}}\left\{\frac{1}{(1-\beta)|1+iT|} + \left|\frac{\alpha-\beta+(1-\beta)iT}{(1-\beta)(1+iT)}\right| |I(B_\alpha)|\right\}.$$

It is now easy to prove that equality holds in the above inequality, that is, the bounds are sharp, which proves the sharpness of the result. Proceeding as in the proof of [6, Theorem 1], we deduce that there exist $T_\alpha \in \mathbb{R}\cup\{\infty\}$ and $\varrho \in \mathbb{R}$ with

$$\begin{aligned}
&\left|\frac{B_\alpha(1)}{(1-\beta)(1+iT_\alpha)} + \frac{\alpha-\beta+(1-\beta)iT_\alpha}{(1-\beta)(1+iT_\alpha)}\sum_{n=1}^{\infty}\frac{a_n(B_\alpha)}{n+1-\alpha}e^{i\varrho}\right| \\
&\qquad = \frac{1}{(1-\beta)|1+iT_\alpha|} + \left|\frac{\alpha-\beta+(1-\beta)iT_\alpha}{(1-\beta)(1+iT_\alpha)}\right| |I(B_\alpha)|, \qquad |B_\alpha(1)| = 1.
\end{aligned}$$

Observe that if $T_\alpha = \infty$, then the value of the above expression tends to 1 and therefore the corresponding $\gamma_1(\alpha,\beta)$ is $|2-\alpha|$. Now, by Lemma 1.1, there exists a sequence $\{W_k\}$ in $\mathcal{B}_0$ such that $W_k(1) = 1$ and $\lim_{k\to\infty} W_k(z) = e^{i\varrho}z$ in $\mathcal{B}_0$. Then $w(z) = B_\alpha(z)W_k(z)/z$ does the job. This proves the required result by observing that $|I(w)| \le 1/|2-\alpha|$. Indeed, we have equality in the last inequality, which is proved exactly as in the proof of Theorem 1 in [7]. ∎

Proof of Corollary 1.3. If α is real and $\alpha < 2$, then $\gamma_1(\alpha,\beta)$ defined in Theorem 1.2 becomes

$$\gamma_1(\alpha,\beta) = \inf_{T\in\mathbb{R}}\left[\frac{(1-\beta)(2-\alpha)\sqrt{1+T^2}}{2-\alpha+\sqrt{(\alpha-\beta)^2+(1-\beta)^2T^2}}\right].$$

It is clear that $\gamma_1(\alpha,\beta) = (1-\beta)(2-\alpha)\inf_{x\ge 0}\psi(x)$, where

$$\psi(x) = \frac{\sqrt{1+x}}{2-\alpha+\sqrt{(\alpha-\beta)^2+(1-\beta)^2x}},$$

and

$$\psi'(x) = \frac{N(x)}{2\sqrt{1+x}\sqrt{(\alpha-\beta)^2+(1-\beta)^2x}\,[2-\alpha+\sqrt{(\alpha-\beta)^2+(1-\beta)^2x}]^2},$$

where $N(x) = (2-\alpha)\sqrt{(\alpha-\beta)^2+(1-\beta)^2x} - (\alpha+1-2\beta)(1-\alpha)$.

Since $\alpha < 2$, $N'(x) \geq 0$ for all $x \geq 0$ so that $N(x)$ is increasing function of x.

CASE (i). Let $\alpha \geq 1/(2-\beta)$. This implies that $\alpha \geq \beta$ and we have

$$N(x) > N(0) = (2-\alpha)(\alpha-\beta) - (\alpha+1-2\beta)(1-\alpha) = \alpha(2-\beta) - 1,$$

so that $N(0) \geq 0$ as $\alpha \geq 1/(2-\beta)$. This in turn implies that $\psi(x)$ is an increasing function for all $x \geq 0$ when $\alpha \geq 1/(2-\beta)$. Therefore,

$$\inf_{x\geq 0} \psi(x) = \psi(0) = \frac{1}{2-\beta}.$$

CASE (ii). Let $\alpha \leq \beta_1 = [3\beta + 2 - \sqrt{9\beta^2 - 20\beta + 12}]/4$. Then a computation shows that $\beta > \alpha$ and

$$N(0) = 2\alpha^2 - \alpha(3\beta+2) + 4\beta - 1.$$

Therefore, $N(0) \geq 0$ if and only if $\alpha \leq \beta_1$. This implies that

$$\inf_{x\geq 0} \psi(x) = \psi(0) = \frac{1}{2(1-\alpha)+\beta}.$$

CASE (iii). In the case of $N(0) \leq 0$, since $N(x)$ is an increasing function and $N(x) > 0$ for large x, we see that there exists a unique x_0 such that $N(x_0) = 0$. Thus, $\psi(x)$ is decreasing for $x \in (0, x_0)$ and $\psi(x)$ is increasing for $x \in (x_0, \infty)$. Therefore, a minimum is attained at x_0. Now, solving for x_0 gives

$$N(x_0) = 0 \;\Leftrightarrow\; (\alpha-\beta)^2 + (1-\beta)^2 x_0 = \left(\frac{1-\alpha}{2-\alpha}\right)^2 (1+\alpha-2\beta)^2,$$

from which we can easily obtain the relation

$$1 + x_0 = \frac{(1-\alpha)(1+\alpha-2\beta)}{(1-\beta)^2}\left[\frac{5-4\alpha-2\beta(1-\alpha)}{(2-\alpha)^2}\right].$$

A simple computation shows that

$$\psi(x_0) = \frac{\sqrt{(1-\alpha)(1+\alpha-2\beta)(5-4\alpha-2\beta(1-\alpha))}}{(1-\beta)[(2-\alpha)^2 + (1-\alpha)|1+\alpha-2\beta|]}.$$

It is important to note that $\alpha \geq \beta_1$ implies $\alpha \geq 2\beta - 1$, so that

$$\psi(x_0) = \frac{\sqrt{(1-\alpha)(1+\alpha-2\beta)}}{(1-\beta)\sqrt{5-4\alpha-2\beta(1-\alpha)}},$$

and therefore $\inf_{x\geq 0} \psi(x) = \psi(x_0)$. ∎

Proof of Theorem 1.3. We know that $\operatorname{Re} f'(z) > \beta$ if and only if

$$\frac{f'(z)-\beta}{1-\beta} \neq -iT \quad \text{for all } T \in \mathbb{R}.$$

Let $f \in \mathcal{P}(\gamma, \alpha)$. Then by the above, $f \in \mathcal{R}(\beta)$ if and only if

$$\frac{\gamma}{(1-\beta)(1+iT)} \sum_{n=2}^{\infty} \frac{(n+1)a_n(w)}{n+1-\alpha}\, z^n \neq -1. \tag{2.2}$$

Let

$$M(\alpha,\beta) = \sup_{\substack{T\in\mathbb{R}\\ z\in\Delta\\ w\in\mathcal{B}_0}} \frac{1}{(1-\beta)|1+iT|}\left|\sum_{n=1}^{\infty} \frac{(n+1)a_n(w)}{n+1-\alpha}\, z^n\right|$$

so that $\gamma M(\alpha,\beta) \le 1$ is necessary for (2.2) to hold. Now, observe that

$$\begin{aligned} M(\alpha,\beta) &\le \frac{1}{1-\beta} \sup_{T\in\mathbb{R}, w\in\mathcal{B}_0} \left[\frac{1}{|1+iT|}\left|\sum_{n=1}^{\infty} a_n(w) z^n\right| \right. \\ &\qquad \left. + \frac{|\alpha|}{|1+iT|}\left|\sum_{n=1}^{\infty} \frac{a_n(w)}{n+1-\alpha} z^n\right|\right] \\ &\le \frac{1}{1-\beta} \sup_{T\in\mathbb{R}} \frac{1}{|1+iT|}\left[1 + \frac{|\alpha|}{|2-\alpha|}\right]. \end{aligned}$$

Hence, we have

$$\gamma_2(\alpha,\beta) \le (1-\beta) \inf_{T\in\mathbb{R}} \left(\frac{|1+iT|\,|2-\alpha|}{|\alpha|+|2-\alpha|}\right) = \frac{(1-\beta)|2-\alpha|}{|\alpha|+|2-\alpha|}.$$

Following the proof of Theorem 1.2 for sharpness, we can easily see that equality holds in the above relation. ∎

3. Another generalization and applications. As a motivation for our next result, we consider

$$\lambda(t) = \begin{cases} \dfrac{(a+1)(b+1)}{(b-a)}\, t^a(1-t^{b-a}) & \text{if } b \neq a, \\ (1+a)^2 t^a \log(1/t) & \text{if } b = a, \end{cases} \tag{3.1}$$

which means that (1.1) has the form

$$V_\lambda(g)(z) = \begin{cases} \dfrac{(a+1)(b+1)}{b-a} \displaystyle\int_0^1 t^{a-1}(1-t^{b-a}) g(tz)\, dt & \\ & \text{if } b \neq a,\ b > -1,\ a > -1. \\ (1+a)^2 \displaystyle\int_0^1 t^{a-1} \log(1/t) g(tz)\, dt & \text{if } b = a,\ a > -1. \end{cases}$$

We define $H_g(a,b;z) := V_\lambda(g)(z)$ when $\lambda(t)$ is as in (3.1). If we let

$$H(z) = H_g(a,b;z),$$

then the function H satisfies the second order differential equation

$$z^2H''(z) + (a+b+1)zH'(z) + abH(z) = (1+a)(1+b)g(z). \tag{3.2}$$

Clearly, letting $b \to \infty$ in $H_g(a,b;z)$ we get the Bernardi operator. Properties of $H_g(a,b;z)$ have been studied recently in [9] by the method of differential subordinations (see also [10, 2, 1]). In some cases the integral transform (1.1) leads to convolution involving classical special functions. For example, if we consider the differential equation (3.2) for complex values of a and b, then the solution to (3.2) in series form is given by

$$\begin{aligned} H_g(a,b;z) &= z + \sum_{n=2}^{\infty} a_n(g)\,\frac{(a+1)(b+1)}{(a+n)(b+n)}\,z^n \\ &= \left(z + \sum_{n=2}^{\infty} \frac{(a+1)(b+1)}{(a+n)(b+n)}\,z^n\right) * g(z). \end{aligned} \tag{3.3}$$

We observe the symmetry $H_g(a,b;z) = H_g(b,a;z)$, and note that the series form of H represents an analytic function on Δ if $\operatorname{Re} a > -2$ and $\operatorname{Re} b > -2$. Thus, an important question is to decide for which values of a and b the function $g \in \mathcal{A}$ satisfying the condition

$$\left|\frac{g(z)}{z} - 1\right| \le \gamma$$

has the property that H_g given by (3.3) is starlike of order β. Now, we are in a position to formulate our final result, which answers this question.

THEOREM 3.1. *Let $a, b \in \mathbb{C}$ with $\operatorname{Re} a > -2$, $\operatorname{Re} b > -2$, and let $f \in \mathcal{A}$ satisfy the condition*

$$\left|zf''(z) + (a+b+1)f'(z) + ab\,\frac{f(z)}{z} - (1+a)(1+b)\right| \le \gamma. \tag{3.4}$$

Then:

(i) *$f \in \mathcal{S}^*(\beta)$ whenever $a \neq b$ and $0 \le \gamma \le \gamma_3$, where*

$$\gamma_3 = \inf_{T\in\mathbb{R}} \left[\frac{|2+a|\,|2+b|\,|b-a|(1-\beta)\sqrt{1+T^2}}{|2+b|\,|a+\beta-(1-\beta)iT| + |2+a|\,|b+\beta-(1-\beta)iT|}\right].$$

(ii) *$f \in \mathcal{S}^*(\beta)$ whenever $a = b$ and $0 \le \gamma \le \gamma_4$, where*

$$\gamma_4 = \inf_{T\in\mathbb{R}} |a+2|^2(1-\beta)\left[\frac{\sqrt{1+T^2}}{|2+a| + \sqrt{(a+\beta)^2 + (1-\beta)^2T^2}}\right].$$

In particular, if a is real and $a > -2$, then

$$\gamma_4 = \begin{cases} \dfrac{(2+a)^2(1-\beta)}{2(1+a)+\beta} & \text{if } a \geq -\beta_1, \\ \dfrac{(2+a)^2\sqrt{(1+a)(1-a-2\beta)}}{\sqrt{5+4a-2\beta(1+a)}} & \text{if } -\beta_1 \geq a \geq \dfrac{-1}{2-\beta}, \\ \dfrac{(2+a)^2(1-\beta)}{2-\beta} & \text{if } \dfrac{-1}{2-\beta} \geq a \geq -2. \end{cases}$$

Proof. If $w(z) = \sum_{n=1}^{\infty} a_n(w)z^n \in \mathcal{B}_0$ and $f(z) = \sum_{n=1}^{\infty} a_n(f)z^n$ satisfies (3.4) then

$$z^2f''(z) + (a+b+1)zf'(z) + abf(z) - (1+a)(1+b)z = \gamma w(z),$$

from which one obtains

$$f(z) = z + \gamma \sum_{n=2}^{\infty} \frac{a_n(w)}{(n+a)(n+b)}\, z^n.$$

By Lemma 1.2, $f \in \mathcal{S}^*(\beta)$ if and only if

$$\frac{\gamma}{(1-\beta)(1+iT)} \sum_{n=1}^{\infty} \frac{(n+1-\beta+(1-\beta)iT)a_{n+1}(w)z^n}{(n+1+a)(n+1+b)} \neq -1. \tag{3.5}$$

(i) Let $a \neq b$. Then, by (3.5), $f \in \mathcal{S}^*(\beta)$ if and only if the series

$$\frac{\gamma/(b-a)}{(1+iT)} \sum_{n=1}^{\infty} \left[\frac{-(a+\beta)+(1-\beta)iT}{(n+1+a)(1-\beta)} - \frac{-(b+\beta)+(1-\beta)iT}{(n+1+b)(1-\beta)} \right] a_{n+1}(w)z^n$$

does not assume the value -1. Repeating the proof of Theorem 1.2, for $\mathrm{Re}\, a > -2$ and $\mathrm{Re}\, b > -2$, we see that the above holds for $0 \leq \gamma \leq \gamma_3$, where

$$\frac{1}{\gamma_3} = \sup_{T\in\mathbb{R}} \frac{1}{|b-a|\sqrt{1+T^2}} \left[\frac{|a+\beta-(1-\beta)iT|}{|2+a|(1-\beta)} + \frac{|b+\beta-(1-\beta)iT|}{|2+b|(1-\beta)} \right].$$

This in turn implies that

$$\gamma_3 = \inf_{T\in\mathbb{R}} \left[\frac{|2+a|\,|2+b|\,|b-a|(1-\beta)\sqrt{1+T^2}}{|2+b|\,|a+\beta-(1-\beta)iT| + |2+a|\,|b+\beta-(1-\beta)iT|} \right]$$

and the proof of the first part is complete.

(ii) Let $a = b$. Then

$$f(z) = z + \gamma \sum_{n=2}^{\infty} \frac{a_n(w)}{(n+a)^2}\, z^n$$

and so (3.5) may be rewritten as

$$\frac{\gamma}{(1-\beta)(1+iT)} \sum_{n=1}^{\infty} \left[\frac{1}{n+1+a} - \frac{(a+\beta)-(1-\beta)iT}{(n+1+a)^2} \right] a_{n+1}(w)z^n \neq -1.$$

As in part (i), this holds for $\operatorname{Re} a > -2$ if $0 \le \gamma \le \gamma_4$, where γ_4 is given by

$$\frac{1}{\gamma_4} = \sup_{T\in\mathbb{R}} \frac{1}{(1-\beta)\sqrt{1+T^2}}\left[\frac{1}{|2+a|} + \frac{|a+\beta-(1-\beta)iT|}{|2+a|^2}\right].$$

Finally, if a is real and $a > -2$ then, since the square-bracketed term in the expression for γ_4 is similar to the function $\psi(x)$ defined in the proof of Corollary 1.3, the desired conclusion is easy to obtain. So, we omit the details. ∎

In the case of $a \neq b$ with a and b real, if we replace a by $-\alpha$, γ by $\gamma|(a+1)(b+1)|$, and allow $b \to \infty$, then f satisfying (3.4) is equivalent to saying that $f \in \mathcal{P}(\gamma, \alpha)$. Thus, Theorem 3.1(i) extends Theorem 1.2 from two-parameter to three-parameter families.

Our final application is a consequence of Lemma 1.3 and Theorem 1.2.

THEOREM 3.2. *Let* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}\alpha < 2$, *and let* $\gamma_1 = \gamma_1(\alpha, \beta)$ *be as in Theorem* 1.2. *Suppose that* $f \in \mathcal{P}(\gamma_1, \alpha)$, $g \in \mathcal{A}$ *and* $h = f * g$.

(1) *If* $\operatorname{Re}(g(z)/z) > 1/2$ *in* Δ, *then* $h \in \mathcal{S}^*(\beta)$.
(2) *If* $\operatorname{Re} g'(z) > 1/2$ *in* Δ, *then* $h \in \mathcal{K}(\beta)$.

Proof. It is a simple exercise to see that

$$h'(z) - \alpha\frac{h(z)}{z} + \alpha - 1 = \left(f'(z) - \alpha\frac{f(z)}{z} + \alpha - 1\right) * \frac{g(z)}{z}$$

and

$$zh''(z) + (1-\alpha)h'(z) + \alpha - 1 = \left(f'(z) - \alpha\frac{f(z)}{z} + \alpha - 1\right) * g'(z).$$

The desired conclusion follows from Lemma 1.3. ∎

Using the theorems proved in [8], it is possible to state several results similar to Theorem 3.1, by requiring g to be an element of various classical subclasses of functions analytic in Δ.

We conclude the paper with a final remark. The general principle behind the problems discussed in this paper is essentially a simple differential subordination of the form

$$zp'(z) + \alpha p(z) \prec \alpha + \gamma z, \qquad z \in \Delta.$$

Consequences of this subordination have been studied in detail, along with a number of similar subordination results, for example by Ponnusamy and Singh [12, 13].

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