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On triple curves through a rational triple point of a surface

by M. R. Gonzalez-Dorrego (Madrid)

Abstract. Let k be an algebraically closed field of characteristic 0. Let C be an irreducible nonsingular curve in $\mathbb{P}^n$ such that $3C = S \cap F$, where S is a hypersurface and F is a surface in $\mathbb{P}^n$ and F has rational triple points. We classify the rational triple points through which such a curve C can pass (Theorem 1.8), and give an example (1.12). We only consider reduced and irreducible surfaces.

On curves passing through rational triple points of surfaces

Definition 1.1. Let F be a reduced surface and P a point of F. Let (F,P) be a surface singularity (that is, the spectrum of an equicharacteristic local noetherian complete ring of Krull dimension 2, without zero divisors, whose closed point P is singular). Let $\pi : \widetilde{F} \to F$ be the minimal desingularization of F at P. The *genus* of a normal singularity P is defined to be $\dim_k (R^1\pi_*\mathcal{O}_{\widetilde{F}})_P$. If the genus is 0, the singularity is said to be *rational*. A rational singularity, P, such that the multiplicity of the maximal ideal of the local ring $\mathcal{O}_{F,P}$ is 3, is called a *rational triple point*. We are going to use configurations of dots and x ($\bullet^2 = -2$, $x^2 = -3$) as vertices of the dual graph of the minimal desingularization of the singularity; each vertex corresponds to a curve and each arc to an intersection [1, p. 135]. We list the following *singularity types* of P:

(1) X_{ijk}, $i,j,k \geq 1$; i denotes the number of dots $\bullet$ to the left of x, j the number of $\bullet$'s above x, and k the number of $\bullet$'s to the right of x.

$$\begin{array}{ccccccccc} & & & & \bullet & & & & \\ & & & & \bullet & & & & \\ \bullet & \cdots & \bullet & & x & & \bullet & \cdots & \bullet \end{array}$$

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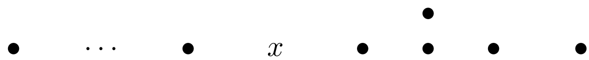
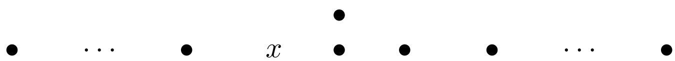
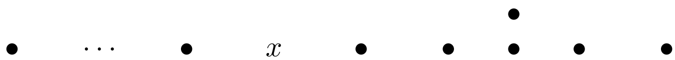
(2) Y_i, $i \geq 1$; i denotes the number of $\bullet$'s to the left of x.



(3) R_{ij}, $i, j \geq 1$; i denotes the number of $\bullet$'s to the left of x and j the number of $\bullet$ to the right of the vertical edge of the graph.

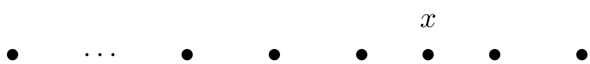


(4) T_i, $i \geq 1$; i denotes the number of $\bullet$'s to the left of x.

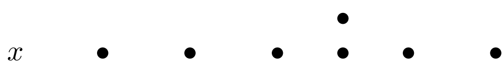


(5) U_{ij}, $i \geq 1$, $j \geq 2$; i denotes the number of $\bullet$'s to the left of x, and j the number of $\bullet$'s between x and the vertical edge of the graph.

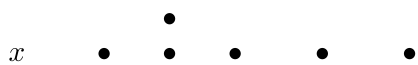
(6) V_i, $i \geq 1$; i denotes the number of $\bullet$'s to the left of the vertical edge of the graph.



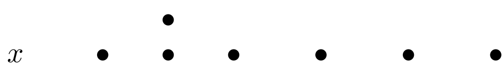
(7) W_2.



(8) W_3.



(9) W_4.



1.2. *Notation.* Let F be a reduced surface and P a point of F. Let (F, P) be a surface singularity (that is, the spectrum of an equicharacteristic complete local ring of Krull dimension 2 whose closed point P is singular). Let $\mathrm{Reg}(F)$ denote the regular locus of (F, P). Let $\mathcal{L}$ be the set of smooth curves Γ on (F, P) whose generic point lies on $\mathrm{Reg}(F)$. Let $\pi : \widetilde{F} \to F$ be the minimal desingularization of F at P. Let $\Phi_{\widetilde{F}} : \mathcal{L} \to \pi^{-1}(P)$ be the map of sets which sends $\Gamma \in \mathcal{L}$ to the exceptional point of its strict transform $\Gamma_{\widetilde{F}}$ on $\widetilde{F}$ (see [2]).

DEFINITION 1.3.

(1) The *maximal cycle* is the cycle $Z_{\widetilde{F}} = \sum m_i E_i$, defined by the divisorial part of $\mathcal{M}\mathcal{O}_{\widetilde{F}}$, where $\mathcal{M}$ is the maximal ideal $\operatorname{Max} \mathcal{O}_{F,P}$ of $\mathcal{O}_{F,P}$; the E_i are the irreducible components of dimension 1 of the exceptional fiber $\pi^{-1}(P)$ and the m_i are nonnegative integers. A component E_j such that $m_j = 1$ is called a *reduced component* of the cycle.

(2) Consider positive cycles $Z = \sum r_i E_i$, $r_i \geq 0$, such that

$$(Z.E_i) \leq 0 \quad \text{for all } i.$$

The unique componentwise smallest cycle Z satisfying this condition is called the *fundamental cycle* of $\widetilde{F}$.

PROPOSITION 1.4 (see [2, 1.2]). *Let (F, P) be a complete surface singularity. For any irreducible component E of $\pi^{-1}(P)$, let ord_E denote the divisorial valuation of the function field of (F, P) given by the filtration of $\mathcal{O}_{\widetilde{F},E}$ by the powers of its maximal ideal. The components E such that*

$$\mathcal{L}_E := \{\Gamma \in \mathcal{L} \mid \Phi_{\widetilde{F}}(\Gamma) \in E\} \neq \emptyset$$

are those for which $\operatorname{ord}_E(\mathcal{M}\mathcal{O}_{\widetilde{F}}) = 1$. The set $\mathcal{L}$ is the disjoint union of the $\mathcal{L}_E$.

LEMMA 1.5 (see [2, 1.14]). *The families of smooth curves on a normal surface singularity are in one-to-one correspondence with the reduced components of the maximal cycle of its minimal desingularization π.*

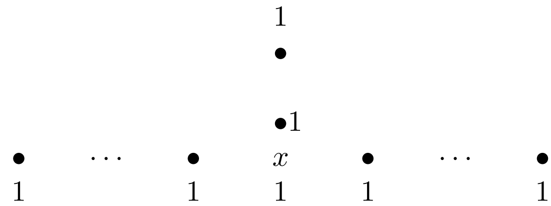
NOTE 1.5.1. For a rational surface singularity, the maximal cycle of π and the fundamental cycle of its weighted dual graph coincide [1].

COROLLARY 1.6. *If an irreducible nonsingular curve C passes through a rational singularity P of a surface F, then its strict transform must intersect transversally only one reduced component of the fundamental cycle.*

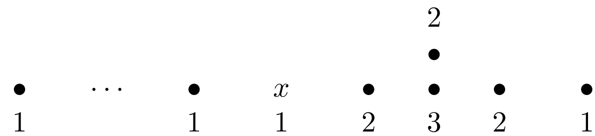
Proof. By Lemma 1.5 and Note 1.5.1 the families of nonsingular curves on a rational surface singularity are in one-to-one correspondence with the reduced components of the fundamental cycle of its minimal desingularization. By Proposition 1.4, $C \in \mathcal{L}_E$ where E is an irreducible component of $\pi^{-1}(P)$ such that $\operatorname{ord}_E(\mathcal{M}\mathcal{O}_{\widetilde{F}}) = 1$; thus its strict transform must intersect E transversally, and can intersect no other irreducible exceptional curves because the set of nonsingular curves C on (F, P) whose generic point lies on $\operatorname{Reg}(F)$, $\mathcal{L}$, is a disjoint union of $\mathcal{L}_E$ by 1.4.

1.6.1. *Fundamental cycles for rational triple singularities.* We exhibit the fundamental cycle for each singularity type.

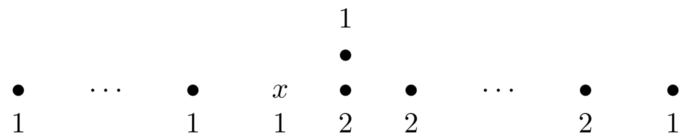
(1) *Case* X_{ijk}, $i, j, k \geq 1$. The fundamental cycle is



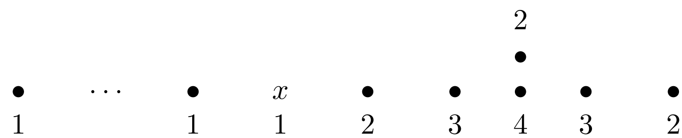
(2) *Case* Y_i, $i \geq 1$. The fundamental cycle is



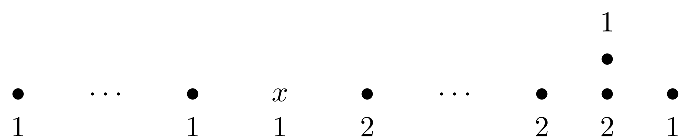
(3) *Case* R_{ij}, $i, j \geq 1$. The fundamental cycle is



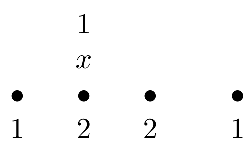
(4) *Case* T_i, $i \geq 1$. The fundamental cycle is



(5) *Case* U_{ij}, $i \geq 1$, $j \geq 2$. The fundamental cycle is

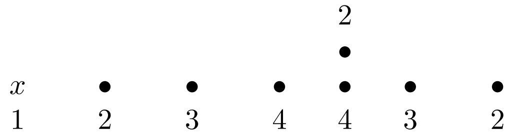


(6) *Case* V_1. The fundamental cycle is

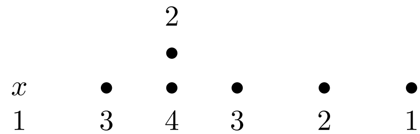


(7) *Case* V_i, $i \geq 2$. The fundamental cycle is

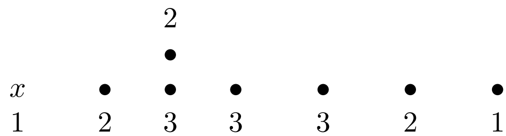
(8) *Case* W_2. The fundamental cycle is



(9) *Case* W_3. The fundamental cycle is



(10) *Case* W_4. The fundamental cycle is



1.7.0. Let C be an irreducible nonsingular curve with $3C = S \cap F$, where S is a hypersurface and F is a surface with rational triple points. Suppose that C passes through a rational triple point P of F. Let $\widetilde{F}$ be the minimal desingularization of F at P, $\pi : \widetilde{F} \to F$. Let E_k, $1 \leq k \leq n$, be the irreducible components of the exceptional divisor. The total transform $\pi^*(3C)$ equals $\sum_{j=1}^n \beta_j E_j + 3E$, where E is the strict transform of C, $\beta_j \in \mathbb{N}$.

Lemma 1.7. *Let E and E_j, $1 \leq j \leq n$, be as in* 1.7.0. *The square of the exceptional cycle of C is*

$$\Big(\sum_{j=1}^{n} \beta_j E_j\Big)^2 = -3E.\Big(\sum_{j=1}^{n} \beta_j E_j\Big) = -3\beta_l$$

where l is the unique natural number such that $E_l \cap E \neq \emptyset$.

Proof. Since $\sum_{j=1}^n \beta_j E_j + 3E$ is a Cartier divisor, it has intersection 0 with E_j for all j. Thus, $(\sum_{j=1}^n \beta_j E_j + 3E)(\sum_{j=1}^n \beta_j E_j) = 0$.

Theorem 1.8. *Let C be as in* 1.7.0. *The square of the exceptional cycle of C is -6 if C passes through one singularity of type X_{iii}, $i \geq 1$, or of type R_{11}, or of type U_{1j}, $j \geq 2$, or of type W_2, or of type W_3. If C passes through one singularity of type V_i, $i \in \mathbb{N}$, the square of the exceptional cycle is -9; moreover if $i = 3b + 2$ for $b \in \mathbb{Z}^+$, the square of the exceptional cycle is $-3b - 9$, and if $i = 3a - 1$, $a \in \mathbb{N}$, it is $-3a - 3$. The curve C cannot pass through any other rational triple point of the surface.*

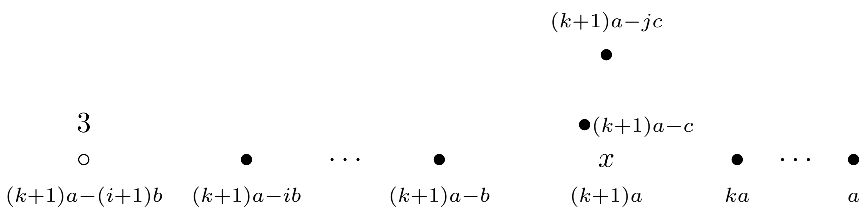
Proof. By Corollary 1.6, if an irreducible nonsingular curve C passes through a rational singularity of F, then its strict transform must intersect transversally only one reduced component of the fundamental cycle. We consider $3C$ and consider its total transform which must have intersection 0 with each exceptional divisor.

The numbers in the diagrams below under the dots are the multiplicities of the E_i's, i.e. the β_i's. The number assigned to the small circle is the multiplicity of E in the cycle $\pi^*(3C)$.

(1) *Case* $X_{ijk}, i, j,\ k \geq 1$. Consider its fundamental cycle. Let a be the multiplicity with which E_1 appears in the total transform.

If we could find, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, a cycle

(1.8.1)



then C would pass through a rational singularity of type $X_{ijk},\ i, j, k \geq 1$. Since

$$(k+1)a-(i+1)b=3,$$
$$(k+1)a-b+ka+(k+1)a-c+(k+1)a(-3)=0,$$

we have $a = \frac{3+(i+1)b}{k+1}$, $a = -b-c$. On the other hand,

$$2((k+1)a-jc) = (k+1)a-(j-1)c$$

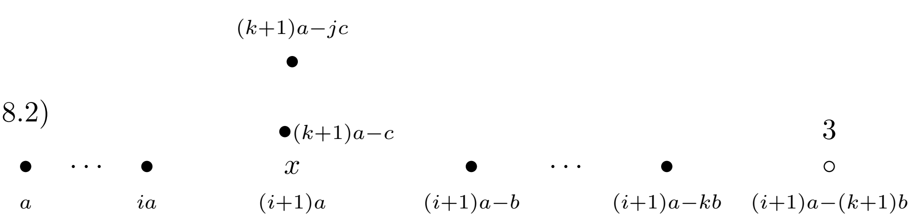
implies that $a = \frac{c(j+1)}{k+1}$, so $c > 0$. Thus, $b = \frac{c(j+1)-3}{i+1}$. Therefore,

$$c = \frac{3(k+1)}{(j+1)(i+1)+(i+j+2)(k+1)} \notin \mathbb{Z}.$$

The cycle (1.8.1) cannot occur.

The cycle

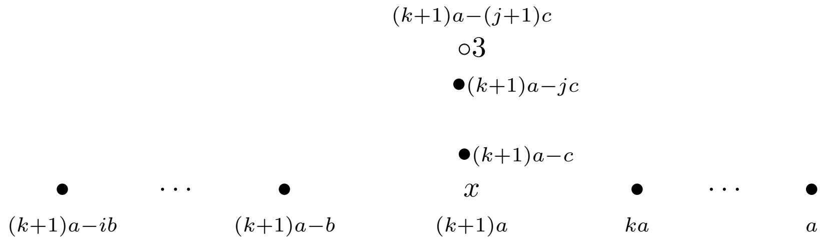
(1.8.2)



is symmetric to (1.8.1) (k and i are interchanged).

Let us see whether we can find, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, a cycle of the form

(1.8.3)



From the equations

$$\begin{aligned}&(k+1)a - b + (k+1)a - c + ka + (k+1)a(-3) = 0,\\&(k+1)a - (j+1)c = 3,\end{aligned}$$

we obtain

$$a = -b - c, \qquad c = -\frac{3 + (k+1)b}{k + j + 2}.$$

Since

$$2((k+1)a - ib) = (k+1)a - (i-1)b,$$

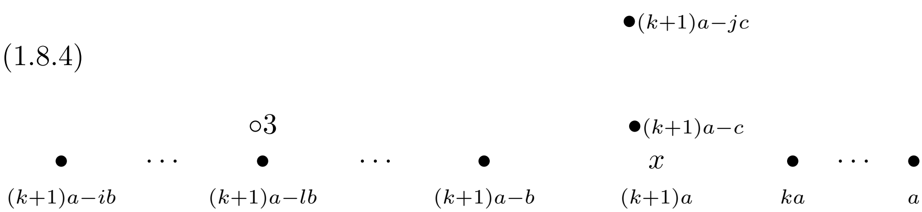
we get $a = \frac{(i+1)b}{k+1}$; so $b > 0$. Thus,

$$b = \frac{3k + 3}{k(i + j + 2) + ij + 2i + 2j + 3} \notin \mathbb{Z}.$$

Therefore, the cycle (1.8.3) cannot occur.

Let l be the unique natural number, $1 < l < i$, such that $E_l \cap E \neq \emptyset$. Let us see whether we can find, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, a cycle of the form

(1.8.4)



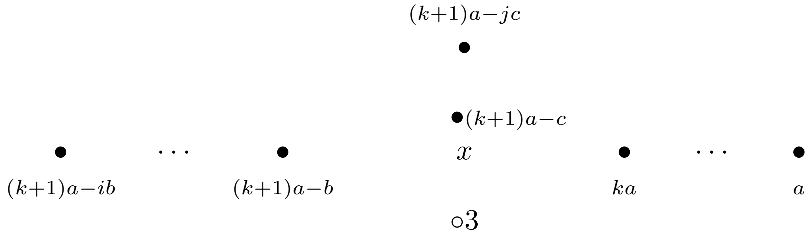
The equation

$$(k+1)a - (l+1)b + (k+1)a - (l-1)b + 3 + ((k+1)a - lb)(-2) = 0$$

leads to a contradiction. Thus, the cycle (1.8.4) cannot occur.

Let us see whether we can find, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, a cycle of the form

(1.8.5)



From the equations

$$\begin{aligned} &(k+1)a-b+3+(k+1)a-c+ka+(k+1)a(-3)=0,\\ &2((k+1)a-jc)=(k+1)a-(j-1)c,\\ &2((k+1)a-ib)=(k+1)a-(i-1)b, \end{aligned}$$

we obtain

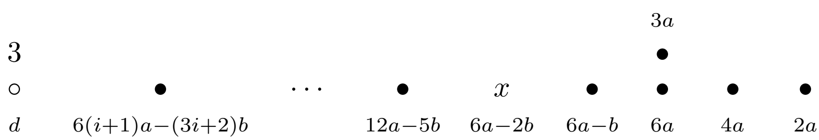
$$a=-b-c+3, \qquad a=\frac{c(j+1)}{(k+1)}, \qquad a=\frac{b(i+1)}{k+1};$$

so, $b>0$, $c>0$. Since $a=-b-c+3$, we get $b=1$ and $c=1$, which implies that $a=1$. Thus, $i=j=k$; so C can pass through a singularity of type X_{iii}.

We have $(\sum_{j=1}^{i} \beta_j E_j)^2=-6$. In this case, β_l of Lemma 1.7 is 2.

We have shown that C can pass through a singularity of type X_{iii}, $i \geq 1$, but not through an X_{ijk} singularity with $i \neq j$ or $j \neq k$ or $i \neq k$.

(2) *Case* Y_i, $i \geq 1$. Consider its fundamental cycle. If we could find, for $2a, 3a \in \mathbb{N}$, $b \in \mathbb{Z}$, $d=6(i+2)a-(3(i+1)+2)b$, a cycle



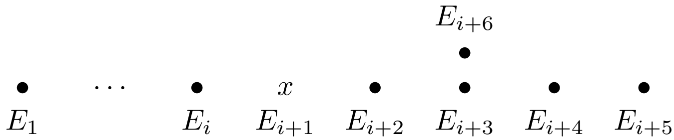
then C would pass through a rational singularity of type Y_i, $i \geq 1$. From the equations

$$6a-b+4a+3a+6a(-2)=0, \qquad 6(i+2)a-(3(i+1)+2)b=3,$$

we obtain $a=b=\frac{3}{3i+7} \notin \mathbb{Z}$ for $i \geq 1$.

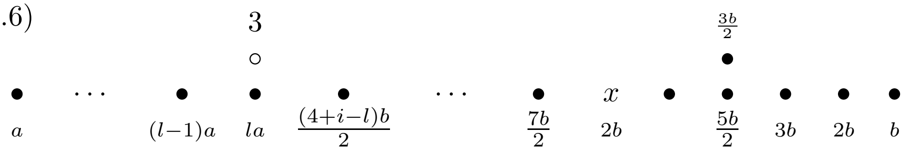
We consider $3C$ and its total transform $\pi^*(3C)$ which must have intersection 0 with each exceptional divisor. Let l be the unique natural number, $1<l\leq i$, such that $E_l \cap E \neq \emptyset$. Number the exceptional curves E_t as

follows:



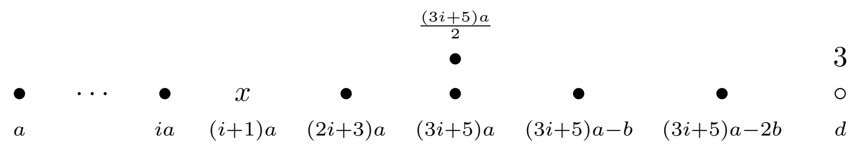
Let a and b denote the respective multiplicities with which E_1 and E_{i+5} appear in $\pi^*(3C)$. Then the equation $\pi^*(3C).E_t = 0$ for all t and the fact that E meets only E_l and no other exceptional curve implies that the cycle $\pi^*(3C)$ has the form, for $a \in \mathbb{N}$, $b \in 2\mathbb{N}$,

(1.8.6)



Now, $\pi^*(3C).E_l = 0$ reads $2la = (l-1)a + 3 + (4+i-l)b/2$. Putting this together with the equation $la = (7+i-l)b/2$, we obtain $(l+1)a = 3 + la - 3b/2$ or $a = 3 - 3b/2$. Since no strictly positive integers a and b can satisfy the last equality, we arrive at a contradiction.

Let us see whether we can find, for $a, (3i+5)a/2 \in \mathbb{N}$, $b \in \mathbb{Z}$, $d = (3i+5)a - 3b$, a cycle of the form



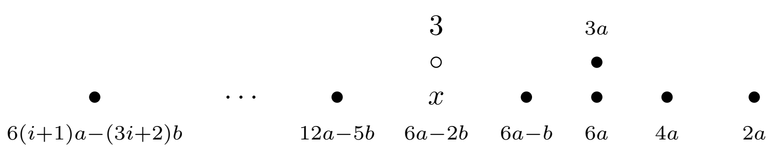
From the equations

$$(3i+5)a - b + (2i+3)a + \frac{(3i+5)a}{2} + (3i+5)a(-2) = 0,$$

$$(3i+5)a - 3b = 3,$$

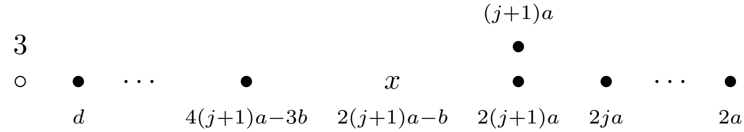
we obtain $(i+1)a = 2b$, $a = \frac{6}{3i+7} \notin \mathbb{N}$ for $i \geq 1$.

For $2a, 3a \in \mathbb{N}$, $b \in \mathbb{Z}$, consider a cycle



The equation $6a-b+12a-5b+3+(6a-2b)(-3) = 0$ leads to a contradiction. Thus, C cannot pass through a Y_i singularity for $i \geq 1$.

(3) *Case* R_{ij}, $i, j \geq 1$. Consider its fundamental cycle. If we could find, for $2a, (j+1)a \in \mathbb{N}$, $b \in \mathbb{Z}$, $d = 2(i+1)(j+1)a - (2i+1)b$, a cycle

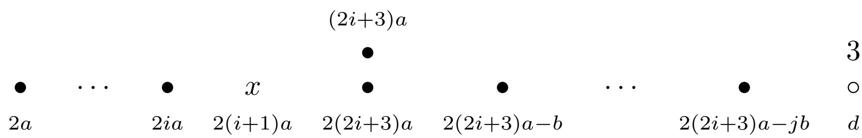
(1.8.7) 

then C would pass through a rational singularity of type R_{ij}, $i, j \geq 1$. From the equations

$$2(j+1)a - b + (j+1)a + 2ja + 2(j+1)a(-2) = 0,$$
$$2(i+2)(j+1)a - (2(i+1)+1)b = 3,$$

we obtain $(j-1)a = b$, $a = \frac{3}{4i+j+7}$, $2a \notin \mathbb{N}$ for $i, j \geq 1$.

Let $2a \in \mathbb{N}$, $d = 2(2i+3)a - (j+1)b$. If we could find a cycle

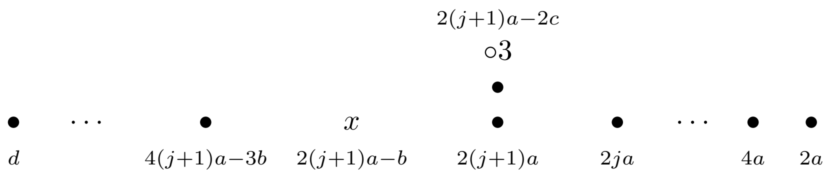


then C would pass through a rational singularity of type R_{ij}, $i, j \geq 1$. From the equations

$$2(i+1)a + 2(2i+3)a - b + (2i+3)a = 4(2i+3)a,$$
$$2(2i+3)a - (j+1)b = 3,$$

we obtain $a = -b$, $a = \frac{3}{4i+j+7}$, $2a \notin \mathbb{N}$ for $i, j \geq 1$.

Let us see whether we can find, for $2a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $d = 2(i+1)(j+1)a - (2i+1)b$, a cycle of the form

(1.8.8) 

From the equations

$$2(j+1)a - 2c = 3,$$
$$2(j+1)a - b + 2(j+1)a - c + 2ja + 2(j+1)a(-2) = 0,$$

we obtain

$$c = \frac{2(j+1)a - 3}{2}, \qquad b = \frac{2(j-1)a + 3}{2}.$$

Since

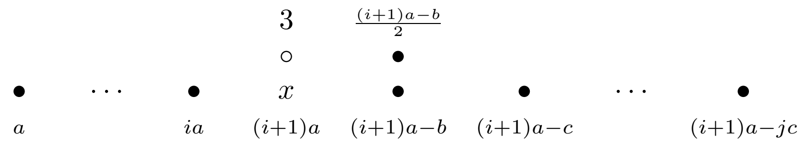
$$2(2(i+1)(j+1)a - (2i+1)b) = 2i(j+1)a - (2i-1)b,$$

we get $a = \frac{3}{8i+6j+10}$; so $2a \notin \mathbb{N}$ for $i, j \geq 1$.

Let l be the unique natural number, $1 < l \leq i$, such that there is no cycle with $E_l \cap E \neq \emptyset$ (see (1.8.6)).

Consider, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $\frac{(i+1)a-b}{2} \in \mathbb{N}$, a cycle

(1.8.9)



From the equation

$$(i+1)a - b + 3 + ia + (i+1)a(-3) = 0,$$

we obtain $b = 3 - (i+2)a$. From

$$(i+1)a + \frac{(i+1)a-b}{2} + (i+1)a - c + ((i+1)a-b)(-2) = 0,$$

we obtain $\frac{9-(2i+5)a}{2} = c$. On the other hand,

$$2((i+1)a - jc) = (i+1)a - (j-1)c,$$

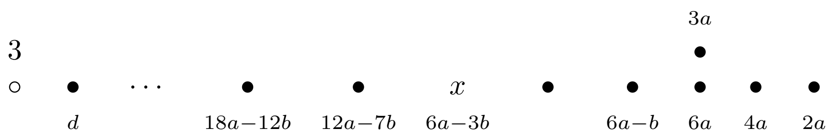
so $c = \frac{9(i+1)}{2(i+1)(j+2)+3(j+1)} \geq 1$; since $a \in \mathbb{N}$ and $a = \frac{9-2c}{2i+5}$, we get $c = 1$. Consequently, $i = 1$, $a = 1$, $b = 0$, $j = 1$.

We have $(\sum_{j=1}^{5} \beta_j E_j)^2 = (E_1 + 2E_2 + 2E_3 + E_4 + E_5)^2 = -6$.

In this case, β_l of Lemma 1.7 is 2.

Hence, C can pass through a rational singularity of type R_{11}, but not through a rational singularity of type R_{ij} with i or $j > 1$.

(4) *Case* T_i, $i \geq 1$. Consider its fundamental cycle. If we could find, for $2a, 3a \in \mathbb{N}$, $b \in \mathbb{Z}$, $d = (i+1)6a - (5i+2)b$, a cycle



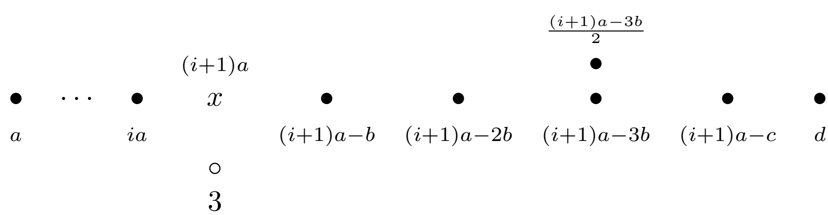
then C would pass through a rational singularity of type T_i, $i \geq 1$. From the equations

$$6a - b + 4a + 3a + 6a(-2) = 0, \qquad (i+2)6a - (5i+7)b = 3,$$

we see that $a = b$ and $i = -2$, which is absurd.

Let l be the unique natural number, $1 < l \leq i$, such that there is no cycle such that $E_l \cap E \neq \emptyset$.

Consider, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $\frac{(i+1)a-3b}{2} \in \mathbb{N}$, $d = (i+1)a - 2c$, a cycle



From the equations

$$ia + (i+1)a - b + 3 + (i+1)a(-3) = 0,$$
$$(i+1)a - 2b + \frac{(i+1)a - 3b}{2} + (i+1)a - c + ((i+1)a - 3b)(-2) = 0,$$

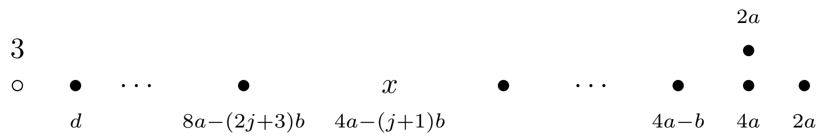
we obtain $b = 3 - (i+2)a$ and $c = \frac{15-(4i+9)a}{2}$. On the other hand,

$$2((i+1)a - 2c) = (i+1)a - c;$$

so $a = \frac{45}{14i+29} \notin \mathbb{N}$ for $i \geq 1$.

Hence, no C can pass through a T_i singularity for $i \geq 1$.

(5) *Case* U_{ij}, $i \geq 1$, $j \geq 2$. Consider its fundamental cycle. If we could find, for $2a \in \mathbb{N}$, $b \in \mathbb{Z}$, $d = 4(i+1)a - ((i+1)j + (2i+1))b$, a cycle

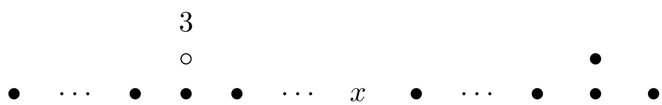


then C would pass through a rational singularity of type U_{ij}, $i \geq 1$, $j \geq 2$. From the equations

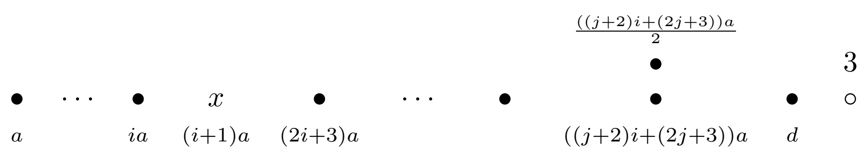
$$4a - b + 2a + 2a + 4a(-2) = 0,$$
$$4(i+2)a - ((i+2)j + (2i+3))b = 3,$$

we obtain $b = 0$, $a = \frac{3}{4(i+2)}$, $2a \notin \mathbb{N}$ for $i \geq 1$.

As in (1.8.6), we cannot find a cycle



Let us see whether we can find, for a, $\frac{((j+2)i+(2j+3))a}{2} \in \mathbb{N}$, $b \in \mathbb{Z}$, $d = ((j+2)i + (2j+3))a - b$, a cycle

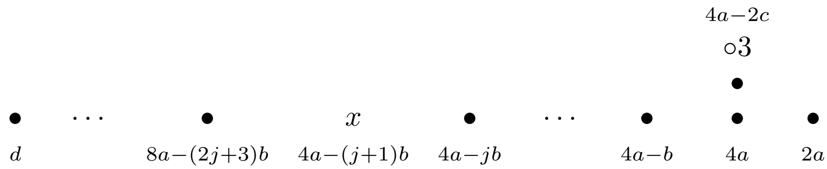


From the equations

$$((j+1)i + (2j+1))a + \frac{((j+2)i + (2j+3))a}{2} + ((j+2)i + (2j+3))a - b + ((j+2)i + (2j+3))a(-2) = 0,$$
$$((j+2)i + (2j+3))a - 2b = 3,$$

we obtain $a = \frac{3}{2(i+2)} \notin \mathbb{N}$ for $i \geq 1$.

Consider, for $2a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $d = 4(i+1)a - ((i+1)j + (2i+1))b$ a cycle

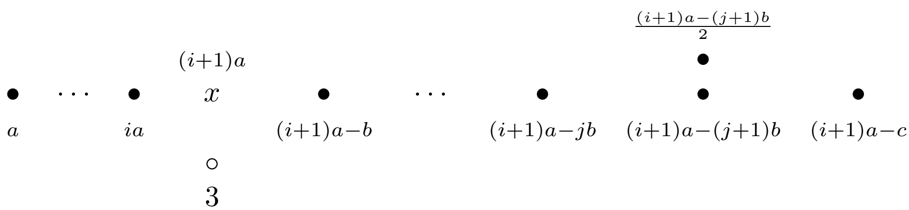


From the equations

$$4a - b + 4a - c + 2a + 4a(-2) = 0, \qquad 4a - 2c = 3,$$

we obtain $b + c = 2a$, $c = \frac{4a-3}{2}$; so $b = \frac{3}{2} \notin \mathbb{Z}$.

Consider, for $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, $\frac{(i+1)a-(j+1)b}{2} \in \mathbb{N}$, a cycle



From the equations

$$ia + (i+1)a - b + 3 + (i+1)a(-3) = 0,$$

$$(i+1)a - jb + \frac{(i+1)a - (j+1)b}{2} + (i+1)a - c + ((i+1)a - (j+1)b)(-2) = 0,$$

we obtain

$$b = 3 - (i+2)a, \qquad c = \frac{(i+1)a + (j+3)b}{2},$$

so $c = \frac{9+3j-((i+2)(j+2)+1)a}{2}$. Since

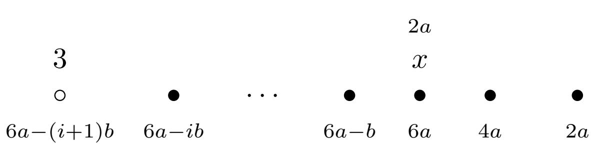
$$2((i+1)a - c) = (i+1)a - (j+1)b,$$

we get $c - 2b = c$, so $b = 0$; thus $a = \frac{3}{i+2} \in \mathbb{N}$ implies that $i = 1$ and $a = 1$; so $c = 1$ for $j \geq 1$.

Hence, C can pass through a rational singularity of type U_{1j} for $j \geq 2$, but there are no C passing through an U_{ij} singularity for $i > 1$, $j \geq 2$.

(6) *Case* V_i, $i \geq 1$. Consider its fundamental cycle. For $2a \in \mathbb{N}$, $b \in \mathbb{Z}$, consider a cycle

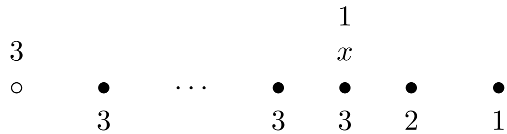
(1.8.10)



From the equations

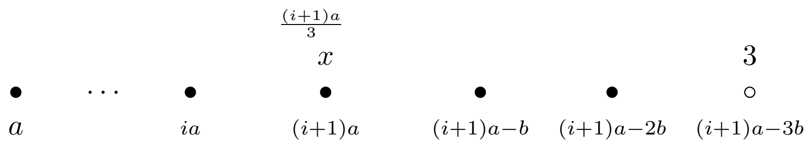
$$6a - b + 2a + 4a + 6a(-2) = 0,$$
$$6a - (i+1)b = 3,$$

we obtain $b = 0$, $2a = 1$; we have the cycle



For $i \in \mathbb{N}$, the singularity V_i can occur.

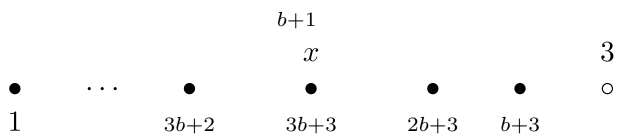
For $a \in \mathbb{N}$, $b \in \mathbb{Z}$, $\frac{(i+1)a}{3} \in \mathbb{N}$, consider a cycle



Since

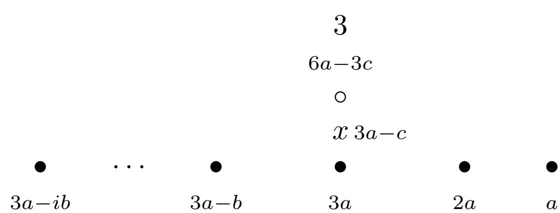
$$ia + \frac{(i+1)a}{3} + (i+1)a - b + (-2)(i+1)a = 0,$$
$$(i+1)a - 3b = 3,$$

we obtain $a = 1$ and $b = \frac{i-2}{3}$. In this case, the total transform of $3C$, for $b \in \mathbb{Z}^+$, is



For $b \in \mathbb{Z}^+$, the singularity V_{3b+2} can occur.

For $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, consider a cycle

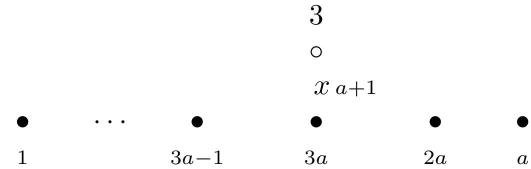


Since $6a - 3c = 3$, we have $c = 2a - 1$. From the equation

$$3a - b + 3a - c + 2a + 3a(-2) = 0,$$

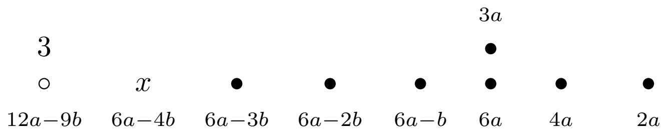
we obtain $2a = b + c$; so $b = 1$. From $2(3a - ib) = 3a - (i-1)b$, we get $a = \frac{i+1}{3}$. For $a \geq 1$, the singularity V_{3a-1} can occur.

In this case, the total transform of $3C$ is



So C can pass through a rational singularity of type V_{3a-1}, $a \geq 1$.

(7) *Case* W_2. Consider its fundamental cycle. If we could find, for $2a, 3a \in \mathbb{N}$, $b \in \mathbb{Z}$, a cycle

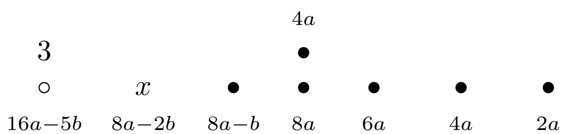


then C would pass through a rational singularity of type W_2. Let us see if it is possible to find a and b satisfying the equations $12a - 9b = 3$ and $6a - b + 4a + 3a + 6a(-2) = 0$. From the second equation we obtain $a = b$. Thus $a = 1$.

We have $(\sum_{j=1}^{8} \beta_j E_j)^2 = -6$. In this case, β_l of Lemma 1.7 is 2.

Hence, C can pass through a rational singularity of type W_2.

(8) *Case* W_3. Consider its fundamental cycle. If we could find, for $2a \in \mathbb{N}$, $b \in \mathbb{Z}$, a cycle



then C would pass through a rational singularity of type W_3. From the equations

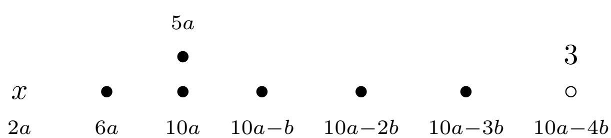
$$8a - b + 6a + 4a + 8a(-2) = 0, \qquad 16a - 5b = 3,$$

we obtain $2a = b$ and $a = \frac{1}{2}$, so $2a = 1$.

We have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$. In this case, β_l of Lemma 1.7 is 2.

Hence, C can pass through a rational singularity of type W_3.

Let us see whether we can find, for $2a, 5a \in \mathbb{N}$, $b \in \mathbb{Z}$, a cycle

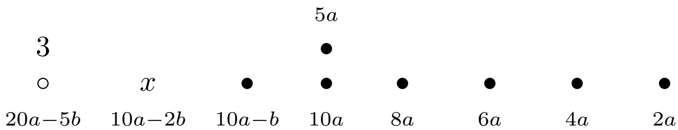


From the equations

$$6a + 5a + 10a - b + 10a(-2) = 0, \qquad 10a - 4b = 3,$$

we obtain $a = b = \frac{1}{2}$, so $2a = 1$ but $5a \notin \mathbb{N}$. Thus, this possibility cannot occur.

(9) *Case* W_4. If C passed through a rational singularity of type W_4, we would be able to find, for $2a, 5a \in \mathbb{N}$, $b \in \mathbb{Z}$, a cycle

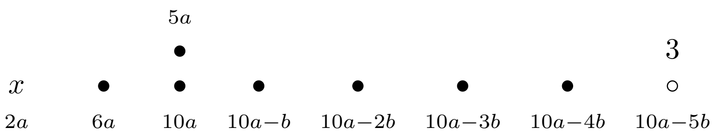


From the equations

$$10a - b + 8a + 5a + 10a(-2) = 0, \qquad 20a - 5b = 3,$$

we obtain $3a = b$, $a = \frac{3}{5}$, $b \notin \mathbb{Z}$.

Let us see whether we can find, for $2a, 5a \in \mathbb{N}$, $b \in \mathbb{Z}$, a cycle



From the equations

$$6a + 5a + 10a - b + 10a(-2) = 0, \qquad 10a - 5b = 3,$$

we obtain $a = b = \frac{3}{5}$, $b \notin \mathbb{Z}$. Thus, C cannot pass through a W_4 singularity.

Definition 1.9. Let R be a regular 2-dimensional local noetherian ring. A *sandwiched singularity* is the singularity of a blowing-up of $\operatorname{Spec} R$ along a complete ideal [4, p. 432].

Note 1.10. A normal surface singularity is minimal if and only if it is rational with reduced fundamental cycle [3, 4.4.10]

Proposition 1.11 ([4, Proposition 2.4]). *Every minimal singularity is sandwiched.*

Example 1.12. A singularity of type X_{111} is a sandwiched singularity because it is minimal (1.11). It is minimal because it is a rational singularity with reduced fundamental cycle (1.10). We consider the blowing-up of $\operatorname{Spec} k[x,y]$ along the complete ideal $(yx^4, x^6, y^2 + yx^2)$. It has a unique singularity with local coordinates (x, y, t', u'), where

$$t' = \frac{yx^4}{y^2 + yx^2}, \qquad u' = \frac{x^6}{y^2 + yx^2}.$$

This blowing-up is a surface F defined in $\mathbb{A}^4_k$ by the relations

$$y(t' + u') = x^4, \qquad u'y = t'x^2, \qquad u'x^2 = t'^2 + t'u'.$$

Let I be the ideal generated in $k[x,y,t',u']$ by the relations above. Then $k[x,y,t',u']/I$ is a free module over $k[y,u']$ generated by $(1,x,x^2,x^3,t',t'x)$; it is the affine coordinate ring of the surface F with a rational triple point X_{111} at the origin. In $k[x,y,t',u']$ we consider the ideal J generated by the irreducible polynomial $f(x,y,t',u') = 7t' - u' + y - 4x^2$. This polynomial is obtained as follows. Let us look at the dual graph (1.8.6). Then $f(x,y,t',u')$ equals $\frac{(y-x^2)^3}{y(y+x^2)}$ modulo I. The quotient $k[x,y,t',u']/J$ is the affine coordinate ring of a hypersurface S. After projectivization, the intersection of S and F is a multiplicity-three structure on a curve C passing through the singularity X_{111} of F.

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Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: mrosario.gonzalez@uam.es

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