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Characterizations of analytic functions associated with functions of bounded variation

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Abstract. We define certain classes of functions associated with functions of bounded variation. Some characterizations of those classes are given.

1. Introduction. We denote by $\mathcal{A}$ the class of functions which are *analytic* in $\mathcal{U} := \mathcal{U}_1$, where $\mathcal{U}_r := \{z \in \mathbb{C} : |z| < r\}$, and let $\mathcal{A}_p$ $(p \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$ denote the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

Let $a \in \mathbb{C}$, $a \neq 1$, $0 < \beta \leq 1$, $k \geq 2$ and let M_k denote the class of real-valued functions m of bounded variation on $[0, 2\pi]$ which satisfy the conditions

$$\int_0^{2\pi} dm(t) = 2, \qquad \int_0^{2\pi} |dm(t)| \leq k. \tag{2}$$

It is clear that M_2 is the class of nondecreasing functions on $[0, 2\pi]$ satisfying (2) or equivalently $\int_0^{2\pi} dm(t) = 2$.

We denote by $\mathcal{P}_k(a, \beta)$ the class of functions $q \in \widetilde{A}_0 := \{q \in \mathcal{A}_0 : 0 \notin q(\mathcal{U})\}$ for which there exists $m \in M_k$ such that

$$q(z) = a + \frac{1-a}{2} \int_0^{2\pi} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}} \right)^{\beta} dm(t) \quad (z \in \mathcal{U}). \tag{3}$$

Here and throughout we assume that all powers denote principal determi-

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nations. Moreover, let us denote

$$S^*_{k,p}(a,\beta) := \left\{ f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \in \mathcal{P}_k(a,\beta) \right\},$$

$$S^c_{k,p}(a,\beta) := \left\{ f \in \mathcal{A}_p : \frac{1}{p} + \frac{zf''(z)}{pf'(z)} \in \mathcal{P}_k(a,\beta) \right\},$$

$$\mathcal{S}^*_{k,p}(a) := \mathcal{S}^*_{k,p}(a,1), \qquad \mathcal{S}^c_{k,p}(a) := \mathcal{S}^c_{k,p}(a,1), \qquad \mathcal{P}_k(a) := \mathcal{P}_k(a,1).$$

These classes of functions have recently been intensively investigated (see for example [1–3, 5–15]). We record that they were introduced by:

- Paatero [12], Pinchuk [14] for $p = \beta = 1, a = 0$,
- Padmanabhan and Parvatham [13] for $p = \beta = 1$, $0 \le a < 1$,
- Moulis [7] for $p = \beta = 1$, $a = 1 - e^{-i\alpha}(1-\rho)\cos\alpha$.

In particular, $V_k := \mathcal{S}^c_{k,1}(0,1)$ is called the class of functions of *bounded boundary rotation*. The classes $\mathcal{P} := \mathcal{P}_2(0)$, $\mathcal{S}^* := \mathcal{S}^*_{2,1}(0,1)$, $\mathcal{S}^c := \mathcal{S}^c_{2,1}(0,1)$ and $\mathcal{S}^*_\beta := \mathcal{S}^*_{2,1}(0,\beta)$ are the well-known classes of Carathéodory functions, starlike functions, convex functions and strongly starlike functions of order β, respectively.

The main object of the paper is to obtain some characterizations of the classes of functions defined above.

2. Main results. Let $\mathcal{H}(\mathcal{U}_r)$ and $\mathcal{SH}(\mathcal{U}_r)$ denote the classes of harmonic and subharmonic functions in $\mathcal{U}_r$, respectively. Moreover, let us denote

$$h_{a,\beta}(z) := (1-a)\left(\frac{1+z}{1-z}\right)^\beta + a, \qquad h_a := h_{a,1} \qquad (z \in \mathcal{U}) \tag{4}$$

and

$$\mathcal{B}_k(a,\beta) := \left\{ \left(\frac{k}{4} + \frac{1}{2}\right) q_1 - \left(\frac{k}{4} - \frac{1}{2}\right) q_2 : q_1, q_2 \prec h_{a,\beta} \right\}.$$

From the result of Hallenbeck and MacGregor [4, p. 50] we have the following lemma.

Lemma 1. *$q \prec h_{a,\beta}$ if and only if there exists $m \in M_2$ such that*

$$q(z) = a + \frac{1-a}{2} \int_0^{2\pi} \left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^\beta dm(t) \qquad (z \in \mathcal{U}).$$

Theorem 1.

$$\mathcal{B}_\lambda(a,\beta) \subset \mathcal{B}_k(a,\beta) \qquad (2 \le \lambda < k).$$

Proof. Let $q \in \mathcal{B}_\lambda(a,\beta)$. Then there exist $q_1, q_2 \prec h_{a,\beta}$ such that $q = (\lambda/4 + 1/2)q_1 - (\lambda/4 - 1/2)q_2$. Thus, we obtain

$$q = \left(\frac{k}{4} + \frac{1}{2}\right) q_1 - \left(\frac{k}{4} - \frac{1}{2}\right) \widetilde{q}_2 \qquad \left(\widetilde{q}_2 = \frac{k-\lambda}{k-2} q_1 + \frac{\lambda - 2}{k-2} q_2\right).$$

Since $h_{a,\beta}$ is a convex function in $\mathcal{U}$, we have $\widetilde{q}_2 \prec h_{a,\beta}$ and consequently $q \in \mathcal{B}_k(a,\beta)$. ∎

THEOREM 2. *The class $\mathcal{B}_k(a,\beta)$ is convex.*

Proof. Let $q, r \in \mathcal{B}_k(a,\beta)$, $\alpha \in [0,1]$ and $\mu := k/4 + 1/2$. Then there exist $q_j, r_j \prec h_{a,\beta}$ $(j = 1,2)$ such that

$$q = \mu q_1 + (1-\mu)q_2, \qquad r = \mu r_1 + (1-\mu) r_2.$$

It follows that

$$\alpha q + (1-\alpha)r = \mu[\alpha q_1 - (\alpha - 1) r_1] + (1-\mu)[\alpha q_2 + (1-\alpha) r_2].$$

Since $\alpha q_j + (1-\alpha) r_j \prec h_{a,\beta}$ $(j = 1,2)$, we conclude that $\alpha q + (1-\alpha) r \in \mathcal{B}_k(a,\beta)$. Hence, the class $\mathcal{B}_k(a,\beta)$ is convex. ∎

THEOREM 3.

$$\mathcal{P}_k(a,\beta) = \mathcal{B}_k(a,\beta).$$

Proof. Let $q \in \mathcal{P}_k(a,\beta)$. Then there exists $m \in M_k$ such that q is of the form (3). If $m \in M_2$, then by Lemma 1 and Theorem 1 we have $q \in \mathcal{B}_2(a,\beta) \subset \mathcal{B}_k(a,\beta)$. Let now $m \in M_k \setminus M_2$. Since m is a function of bounded variation, by the Jordan theorem there exist real-valued functions μ_1, μ_2 which are nondecreasing and nonconstant on $[0, 2\pi]$ such that

$$m = \mu_1 - \mu_2, \qquad \int_0^{2\pi} |dm| = \int_0^{2\pi} d\mu_1 + \int_0^{2\pi} d\mu_2. \tag{5}$$

Thus, putting

$$\alpha_j := \frac{\mu_j(2\pi) - \mu_j(0)}{2}, \qquad m_j := \frac{1}{\alpha_j}\mu_j \qquad (j = 1,2)$$

we have $m_1, m_2 \in M_2$ and

$$m = \alpha_1 m_1 - \alpha_2 m_2. \tag{6}$$

Combining (5) and (6) we obtain

$$2\alpha_1 - 2\alpha_2 = \int_0^{2\pi} dm(t) = 2, \qquad 2\alpha_1 + 2\alpha_2 = \int_0^{2\pi} |dm(t)| \le k,$$

and consequently

$$\alpha_1 = \lambda/4 + 1/2, \qquad \alpha_2 = \lambda/4 - 1/2 \qquad \Big(\lambda = \int_0^{2\pi} |dm| \le k\Big).$$

Therefore, by (3) and (6) we get

$$q = (\lambda/4 + 1/2) q_1 - (\lambda/4 - 1/2) q_2, \tag{7}$$

where

$$q_j(z) = a + \frac{1-a}{2}\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^{\beta} dm_j(t) \qquad (z \in \mathcal{U},\ j = 1,2). \tag{8}$$

Hence, by Lemma 1 we have $q_1, q_2 \prec h_{a,\beta}$ and so $q \in \mathcal{B}_\lambda(a,\beta) \subset \mathcal{B}_k(a,\beta)$.

Conversely, let $q \in \mathcal{A}_0$ be a function of the form (7) for some $q_1, q_2 \prec h_{a,\beta}$. Thus, by Lemma 1 we have (8) for some $m_1, m_2 \in M_2$. Therefore, by (7) we have (3) with $m = (k/2+1)m_1 - (k/2-1)m_2$. Since

$$\int_0^{2\pi} dm(t) = (k/2+1)\int_0^{2\pi} dm_1 - (k/2-1)\int_0^{2\pi} dm_2 = 2,$$

$$\int_0^{2\pi} |dm(t)| \le (k/2+1)\int_0^{2\pi} dm_1 + (k/2-1)\int_0^{2\pi} dm_2 = k,$$

we have $m \in M_k$ and consequently $q \in \mathcal{P}_k(a,\beta)$. ∎

Lemma 2. *Let $q \in \mathcal{A}_0$. Then $q \in \mathcal{P}_2(a)$ if and only if*

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{q(re^{it})-a}{1-a}\right| dt = 2\pi \qquad (0 < r < 1). \tag{9}$$

Proof. Let $q \in \mathcal{A}_0$. Then, by the properties of subordination we get

$$q \in \mathcal{P}_2(a) \;\Leftrightarrow\; q(z) \prec \frac{1+(1-2a)z}{1-z} \;\Leftrightarrow\; \frac{q(z)-a}{1-a} \prec \frac{1+z}{1-z} \tag{10}$$

$$\Leftrightarrow\; \operatorname{Re}\frac{q(z)-a}{1-a} > 0 \qquad (z \in \mathcal{U}).$$

Moreover, we have

$$\int_0^{2\pi} \operatorname{Re}\frac{q(re^{it})-a}{1-a}\, dt = \operatorname{Re}\int_{|z|=r} \frac{1}{iz}\,\frac{q(z)-a}{1-a}\, dz$$

$$= 2\pi\frac{q(0)-a}{1-a} = 2\pi \qquad (0 < r < 1).$$

Thus, condition (9) is equivalent to

$$\operatorname{Re}\frac{q(z)-a}{1-a} > 0 \qquad (z \in \mathcal{U}),$$

and by (10) we obtain the required equivalence. ∎

Theorem 4. *Let $q \in \mathcal{A}_0$. Then $q \in \mathcal{P}_k(a)$ if and only if*

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{q(re^{it})-a}{1-a}\right| dt \le k\pi \qquad (0 < r < 1). \tag{11}$$

Proof. By Lemma 2, we can assume $k > 2$. Let $q \in \mathcal{P}_k(a)$. Then there exist $q_1, q_2 \prec h_a(z)$ such that

$$q = (k/4 + 1/2)q_1 - (k/4 - 1/2)q_2.$$

Hence, by Lemma 2 we have

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{q(re^{it}) - a}{1-a} \right| dt \le \left(\frac{k}{4} + \frac{1}{2} \right) \int_0^{2\pi} \left| \operatorname{Re} \frac{q_1(re^{it}) - a}{1-a} \right| dt$$
$$+ \left(\frac{k}{4} - \frac{1}{2} \right) \int_0^{2\pi} \left| \operatorname{Re} \frac{q_2(re^{it}) - a}{1-a} \right| dt = k\pi \quad (0<r<1).$$

To obtain a contradiction, suppose that $q \in \mathcal{A}_0$ satisfies (11). If we put

$$F(z) := \operatorname{Re} \frac{q(z)-a}{1-a}, \qquad F^+(z) := \max\{F(z), 0\} \ge 0,$$
$$F^-(z) := \max\{-F(z), 0\} \ge 0 \quad (z \in \mathcal{U}),$$
$$V_r^\tau(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|re^{it} - z|^2} F^\tau(re^{it})\, dt \ge 0 \quad (|z| \le r < 1,\ \tau \in \{+,-\}),$$

then the functions F^τ, V_r^τ $(\tau \in \{+,-\})$ are nonconstant and

$$\begin{aligned} &F \in \mathcal{H}(\mathcal{U}), \quad V_r^+, V_r^- \in \mathcal{H}(\mathcal{U}_r), \quad F^+, F^- \in \mathcal{SH}(\mathcal{U}), \\ &F = F^+ - F^-, \quad |F| = F^+ + F^-,\ V_r^\tau(z) = F^\tau(z) \\ &\qquad\qquad (|z| = r,\ \tau \in \{+,-\}). \end{aligned} \tag{12}$$

Thus, we have

$$\max\{F^\tau(z), V_r^\tau(z)\} = V_r^\tau(z) \quad (|z| \le r,\ r \in (0,1),\ \tau \in \{+,-\}),$$
$$\max\{V_r^\tau(z) : |z| \le r\} = \max\{F^\tau(z) : |z| = r\}$$
$$\le \max\{F^\tau(z) : |z| \le R\} \quad (r \le R < 1,\ \tau \in \{+,-\}).$$

Therefore, the functions

$$U_r^\tau(z) := \begin{cases} V_r^\tau(z) & |z| < r, \\ F^\tau(z) & r \le |z| < 1, \end{cases} \qquad (r \in (0,1),\ \tau \in \{+,-\})$$

are continuous subharmonic functions in $\mathcal{U}$ and the families $\{U_r^+ : r \in (0,1)\}$, $\{U_r^- : r \in (0,1)\}$ are locally uniformly bounded. Hence, if we define

$$U^\tau(z) := \sup\{U_r^\tau(z) : r \in (0,1)\} = \lim_{n\to\infty} U_{1-1/n}^\tau(z) \quad (z \in \mathcal{U},\ \tau \in \{+,-\}),$$

then

$$U^\tau \in \mathcal{SH}(\mathcal{U}), U_r^\tau, U^\tau \in \mathcal{H}(\mathcal{U}_r) \qquad (r \in (0,1),\ \tau \in \{+,-\})$$

and so $U^+, U^- \in \mathcal{H}(\mathcal{U})$, $U^+, U^- > 0$. Moreover, by (12) we get

$$\begin{aligned} F(z) &= U_r^+(z) - U_r^-(z) \quad (|z| \le r,\ r \in (0,1)), \\ |F(z)| &= U_r^+(z) + U_r^-(z) \quad (|z| = r,\ r \in (0,1)). \end{aligned} \tag{13}$$

Therefore, we have

$$F(z)=\alpha_1 U_1(z)-\alpha_2 U_2(z) \quad (z\in\mathcal{U}), \tag{14}$$

where

$$U_1:=\frac{1}{\alpha_1}U^+,\qquad U_2:=\frac{1}{\alpha_2}U^- \quad (\alpha_1=U^+(0),\, \alpha_2=U^-(0))$$

are positive harmonic functions in $\mathcal{U}$. Moreover, by (13) we obtain

$$\lim_{r\to1^-}\int_0^{2\pi}|F(re^{it})|\,dt=\alpha_1\lim_{r\to1^-}\int_0^{2\pi}U_1(re^{it})\,dt+\alpha_2\lim_{r\to1^-}\int_0^{2\pi}U_2(re^{it})\,dt. \tag{15}$$

Now, we consider functions $q_1,q_2\in\mathcal{A}_0$ such that

$$\operatorname{Re}\frac{q_j(z)-a}{1-a}=U_j(z)>0 \quad (z\in\mathcal{U},\, j=1,2).$$

Then $q_1,q_2\prec h_\alpha$, and by (14) we have

$$\frac{q(z)-a}{1-a}=\alpha_1\frac{q_1(z)-a}{1-a}-\alpha_2\frac{q_2(z)-a}{1-a} \quad (z\in\mathcal{U}),$$

or simply

$$q=\alpha_1 q_1-\alpha_2 q_2. \tag{16}$$

Hence, $\alpha_1-\alpha_2=1$. Moreover, by (15) and Lemma 2 we have $2\alpha_1+2\alpha_2=\lambda$, where

$$\lambda:=\frac{1}{\pi}\lim_{r\to1^-}\int_0^{2\pi}|F(re^{it})|\,dt$$

and $2\le\lambda\le k$, by (11). Thus,

$$\alpha_1=\lambda/4+1/2,\qquad \alpha_2=\lambda/4-1/2,\qquad 2\le\lambda\le k.$$

Therefore, by (16) and Theorem 1 we have $q\in\mathcal{P}_\lambda(a)\subset\mathcal{P}_k(a)$, which completes the proof. ∎

Let us mention some consequences of Theorems 1-4.

Corollary 1. *The class $\mathcal{P}_k(a,\beta)$ is convex and*

$$\mathcal{P}_k(a,\beta)\subset\mathcal{P}_\lambda(a,\beta),\ \mathcal{S}^*_k(a,\beta)\subset\mathcal{S}^*_\lambda(a,\beta),\ \mathcal{S}^c_k(a,\beta)\subset\mathcal{S}^c_\lambda(a,\beta) \quad (2\le k<\lambda).$$

Corollary 2. *Let $q\in\mathcal{A}_0$, $0\le\rho<1$, $|\alpha|<\pi/2$. Then the following conditions are equivalent:*

(i) $q\in P_k^\alpha(\rho):=\mathcal{P}_k(1-e^{-i\alpha}(1-\rho)\cos\alpha)$.

(ii) *q is of the form (7) for some* $q_1,q_2\prec\frac{\cos\alpha}{e^{i\alpha}}\frac{2(1-\rho)z}{1-z}+1$.

(iii) $\int_0^{2\pi}\left|\operatorname{Re}\left(e^{i\alpha}\frac{q(re^{it})-\rho}{1-\rho}\right)\right|dt\le k\pi\cos\alpha$ $(0<r<1)$.

Corollary 3. *Let $q\in\mathcal{A}_0$, $0\le\rho<1$. Then the following conditions are equivalent:*

(i) $q \in \mathcal{P}_k(\rho)$.
(ii) q *is of the form* (7) *for some* $q_1, q_2 \prec \frac{1+(1-2\rho)z}{1-z}$.
(iii) $\int_0^{2\pi} \left|\operatorname{Re} \frac{q(re^{it})-\rho}{1-\rho}\right| dt \le k\pi \ (0 < r < 1)$.

Corollary 4. *Let* $q \in \mathcal{A}_0$. *Then the following conditions are equivalent:*

(i) $q \in \mathcal{P}_k := \mathcal{P}_k(0)$.
(ii) q *is of the form* (7) *for some* $q_1, q_2 \in \mathcal{P}$.
(iii) $\int_0^{2\pi} |\operatorname{Re} q(re^{it})|\, dt \le k\pi \ (0 < r < 1)$.

Remark 1. The implication (iii)⇒(i) in Corollary 3 was obtained in [13]. The conditions (iii) in Corollary 2 and Corollary 3 give definitions of the classes $\mathcal{P}_k(\rho)$ and $P_k^\alpha(\rho)$ introduced by Padmanabhan and Parvatham [13] and Moulis [7], respectively.

Theorem 5. *Let* $f \in \mathcal{A}_p$. *Then* $f \in \mathcal{S}^*_{k,p}(a,\beta)$ *if and only if there exists* $m \in M_k$ *such that*

$$f(z) = z^p \exp\left\{\int_0^{2\pi}\int_0^z \frac{p(1-a)}{2u}\left[\left(\frac{1+ue^{-it}}{1-ue^{-it}}\right)^\beta - 1\right] du\, dm(t)\right\} \quad (z \in \mathcal{U}). \tag{17}$$

Proof. From the definitions of the classes $\mathcal{S}^*_{k,p}(a,\beta)$ and $\mathcal{P}_k(a,\beta)$ we find that $f \in \mathcal{S}^*_{k,p}(a,\beta)$ if and only if there exists $m \in M_k$ such that

$$\frac{zf'(z)}{pf(z)} = a + \frac{1-a}{2}\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^\beta dm(t) \quad (z \in \mathcal{U}),$$

or equivalently

$$z\left(\log\frac{f(z)}{z^p}\right)' = \frac{p(1-a)}{2}\int_0^{2\pi}\left[\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^\beta - 1\right] dm(t) \quad (z \in \mathcal{U}). \tag{18}$$

Easy computations show that the conditions (17) and (18) are equivalent. ∎

From Theorem 5 we obtain the following two corollaries.

Corollary 5. *Let* $f \in \mathcal{A}_p$. *Then* $f \in \mathcal{S}^*_{k,p}(a)$ *if and only if there exists* $m \in M_k$ *such that*

$$f(z) = z^p \exp\left\{p(a-1)\int_0^{2\pi} \log(1-ze^{-it})\, dm(t)\right\} \quad (z \in \mathcal{U}).$$

Corollary 6. *Let* $b \in \mathbb{C}$, $b \ne 1$. *Then*

$$f \in \mathcal{S}^*_{k,p}(a,\beta) \ \Leftrightarrow \ f^p \in \mathcal{S}^*_{k,1}(a,\beta),$$
$$f \in \mathcal{S}^*_{k,p}(a,\beta) \ \Leftrightarrow \ z^p[z^{-p}f(z)]^{\frac{1-b}{1-a}} \in \mathcal{S}^*_{k,p}(b,\beta).$$

THEOREM 6. *$f \in \mathcal{S}^*_{k,p}(a,\beta)$ if and only if there exist $f_1, f_2 \in \mathcal{S}^*_{2,p}(a,\beta)$ such that*

$$f = f_1^{k/4+1/2} / f_2^{k/4-1/2}. \tag{19}$$

Proof. $f \in \mathcal{S}^*_{k,p}(a,\beta)$ if and only if f is of the form (17) for some $m = (k/4+1/2)m_1 - (k/4-1/2)m_2 \in M_k$. Thus, equivalently there exist $f_1, f_2 \in \mathcal{S}^*_{2,p}(a,\beta)$, where

$$f_j(z) = z^p \exp\left\{ \int_0^{2\pi} \int_0^z \frac{p(1-a)}{2u} \left[\left(\frac{1+ue^{-it}}{1-ue^{-it}} \right)^\beta - 1 \right] du\, dm_j(t) \right\} \quad (z \in \mathcal{U},\ j = 1,2),$$

such that (19) holds. ∎

It is clear that

$$f \in \mathcal{S}^c_{k,p}(a,\beta) \;\Leftrightarrow\; \frac{z}{p} f'(z) \in \mathcal{S}^*_{k,p}(a,\beta).$$

Therefore, by Theorems 5–6 and Corollary 5 we obtain the corollaries listed below.

COROLLARY 7. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{S}^c_{k,p}(a,\beta)$ if and only if there exists $m \in M_k$ such that*

$$f'(z) = pz^{p-1} \exp\left\{ \int_0^{2\pi} \int_0^z \frac{p(1-a)}{2u} \left[\left(\frac{1+ue^{-it}}{1-ue^{-it}} \right)^\beta - 1 \right] du\, dm(t) \right\} \quad (z \in \mathcal{U}).$$

COROLLARY 8. *Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{S}^c_{k,p}(a)$ if and only if there exists $m \in M_k$ such that*

$$f'(z) = pz^{p-1} \exp\left\{ p(a-1) \int_0^{2\pi} \log(1 - ze^{-it})\, dm(t) \right\} \quad (z \in \mathcal{U}).$$

COROLLARY 9. *$f \in \mathcal{S}^c_{k,p}(a,\beta)$ if and only if there exist $f_1, f_2 \in \mathcal{S}^c_{2,p}(a,\beta)$ such that*

$$f' = (f_1')^{k/4+1/2} / (f_2')^{k/4-1/2}.$$

REMARK 2. Putting $p = 1$ and $a = 1 - e^{-i\alpha}(1-\rho)\cos\alpha$ in Corollary 5 and Corollary 8 we obtain the results of Moulis [7]. Moreover, putting $\alpha = 0$ we obtain the results of Padmanabhan and Parvatham [13]. Also, Corollary 5 and Corollary 8 for $p = 1$ and $a = 0$ give the definitions of the classes $U_k = \mathcal{S}^*_{k,1}(0)$, $V_k = \mathcal{S}^c_{k,1}(0)$, introduced by Pinchuk [14].

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