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Hodge type decomposition

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Abstract. In the space Λ^p of polynomial p-forms in $\mathbb{R}^n$ we introduce some special inner product. Let $\mathbf{H}^p$ be the space of polynomial p-forms which are both closed and co-closed. We prove in a purely algebraic way that Λ^p splits as the direct sum $d^\star(\Lambda^{p+1}) \oplus \delta^\star(\Lambda^{p-1}) \oplus \mathbf{H}^p$, where $d^\star$ (resp. $\delta^\star$) denotes the adjoint operator to d (resp. δ) with respect to that inner product.

1. Introduction and main result. To begin with, recall the classical Hodge decomposition theorem. Suppose (M^n, g) is an oriented Riemannian manifold. Introduce the following notation: $\Lambda^p(M)$ is the space of all smooth differential p-forms on M whereas d, δ and $\Delta = d\delta + \delta d$, are the differential, co-differential and the Laplace–Beltrami operator, respectively. Recall that for any $\omega \in \Lambda^p(M)$, $\delta\omega = (-1)^{n(p+1)+1} \star d \star \omega$, where $\star$ is the Hodge operator. Moreover, let $\mathbf{H}^p(M)$ denote the space of p-forms which are both closed and co-closed: $\mathbf{H}^p(M) = \{\omega \in \Lambda^p(M) : d\omega = \delta\omega = 0\}$.

If M is compact then the formula

$$(\omega|\eta)_M = \int_M \omega \wedge \star\eta$$

defines an inner product in $\Lambda^p(M)$. Then $d^\star = \delta$ and $\delta^\star = d$, i.e., d and δ are adjoint to each other. Take any $\omega \in \Lambda^p(M)$; then $(\omega|\Delta\omega)_M = (d\omega|d\omega)_M + (\delta\omega|\delta\omega)_M$. Therefore, ω is harmonic, i.e., $\Delta\omega = 0$, iff $\omega \in \mathbf{H}^p(M)$. The following is a classical result in analysis:

Theorem 1.1 (Hodge decomposition theorem). *On a compact oriented Riemannian manifold M, $\Lambda^p(M) = d\Lambda^{p-1}(M) \oplus^\perp \delta\Lambda^{p+1}(M) \oplus^\perp \mathbf{H}^p(M)$, or equivalently $\Lambda^p(M) = \delta^\star\Lambda^{p-1}(M) \oplus^\perp d^\star\Lambda^{p+1}(M) \oplus^\perp \mathbf{H}^p(M)$, where $\oplus^\perp$ denotes an orthogonal direct sum.*

For the proof of the Hodge decomposition theorem we refer to the book of F. Warner ([4, Chapter 6]). Notice that a historical survey of the devel-

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opment of the theory of elliptic operators is given in the beautiful article of L. Hörmander ([1]).

We treat $\mathbb{R}^n$ as a Riemannian manifold equipped with the canonical inner product. If f is a polynomial of the form $f = \sum_\alpha a_\alpha x^\alpha$ (here $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multi-index) we put

$$f(D) = \sum_\alpha a_\alpha D^\alpha, \quad D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}}.$$

Let $\mathcal{P}_k$ (resp. $\mathcal{H}_k$) denote the space of all homogeneous (resp. harmonic homogeneous) polynomials on $\mathbb{R}^n$ of degree k. Define the inner product $(\cdot,\cdot) = (\cdot,\cdot)_k$ in $\mathcal{P}_k$ as follows: $(f,g) = f(D)g$, for $f, g \in \mathcal{P}_k$ (cf. [3, p. 139]). Clearly, for any $f \in \mathcal{P}_k$, $g \in \mathcal{P}_l$ and $h \in \mathcal{P}_{k+l}$, $(gf,h)_{k+l} = (f, g(D)h)_k$. In particular [1], $(\Delta f, h) = (f, -r^2 h)$ where $r^2(x)$ is the square of the Euclidian norm of $x \in \mathbb{R}^n$. As a result we get the well-known identity ([3, Thm. 2.12])

$$\mathcal{P}_k = \mathcal{H}_k \oplus^{\perp} r^2 \mathcal{P}_{k-2}. \tag{1.1}$$

We may extend (using linearity) this inner product onto the space of all polynomials, putting $(f,g) = 0$ if $f \in \mathcal{P}_k$, $g \in \mathcal{P}_l$ and $k \neq l$.

Recall that any p-form ω in $\mathbb{R}^n$ has a unique expression

$$\omega = \frac{1}{p!} \sum_{i_1,\ldots,i_p=1}^{n} \omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where the functions $\omega_{i_1,\ldots,i_p}$, called *coefficients*, are skew-symmetric with respect to the indices. Denote by Λ^p the space of all p-forms in $\mathbb{R}^n$ that have polynomial coefficients. We put $\Lambda^p = \{0\}$ if $p < 0$. Let $\mathfrak{H}^p$ and $\mathbf{H}^p$ denote, respectively, the space of all harmonic polynomial p-forms and its subspace of all polynomial p-forms which are both closed and co-closed:

$$\mathfrak{H}^p = \{\omega \in \Lambda^p : \Delta\omega = (d\delta + \delta d)\omega = 0\}, \quad \mathbf{H}^p = \{\omega \in \Lambda^p : d\omega = \delta\omega = 0\}.$$

Consider the vector field ν and the 1-form $\nu^\star$ defined by

$$\nu_x = x^1 \frac{\partial}{\partial x^1} + \cdots + x^n \frac{\partial}{\partial x^n}, \quad \nu_x^\star = x^1 dx^1 + \cdots + x^n dx^n,$$

where $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. One sees that $\nu^\star \nu = r^2$. Let ε_ν (resp. ι_ν) denote the exterior (resp. interior) product, i.e., $\varepsilon_\nu \omega = \nu^\star \wedge \omega$ and $\iota_\nu \omega = \omega(\nu, \cdot, \ldots, \cdot)$. Manifestly, $\varepsilon_\nu^2 = 0$ and $\iota_\nu^2 = 0$. Since ι_ν is an anti-derivation, $\iota_\nu \varepsilon_\nu \omega = \nu^\star \nu \omega - \varepsilon_\nu \iota_\nu \omega$. Therefore, we get

$$(\iota_\nu \varepsilon_\nu + \varepsilon_\nu \iota_\nu)\omega = r^2 \omega. \tag{1.2}$$

One can easily check that

$$d\varepsilon_\nu = -\varepsilon_\nu d, \quad \delta\iota_\nu = -\iota_\nu \delta. \tag{1.3}$$

[1] Since we define the Laplace operator as $\Delta = d\delta + \delta d$, our Δ on smooth real-valued functions has the sign such that $\Delta u = -(\partial/(\partial x^1)^2 + \cdots + \partial/(\partial x^n)^2)u$ on $\mathbb{R}^n$.

Equip the space Λ^p with the inner product $(\cdot|\cdot)$ as follows: for any $\omega, \eta \in \Lambda^p$ we put

$$(\omega|\eta) = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} (\omega_{i_1,\dots,i_p}, \eta_{i_1,\dots,i_p}),$$

where $\omega_{i_1,\dots,i_p}$'s and $\eta_{i_1,\dots,i_p}$'s denote the coefficients of ω and η, respectively.

It turns out ([2, Thm. 2.2.1]) that the operators δ and $-\varepsilon_\nu$, and d and ι_ν, are adjoint to each other:

$$\delta^\star = -\varepsilon_\nu, \qquad d^\star = \iota_\nu. \tag{1.4}$$

If ω is a polynomial form such that $\iota_\nu\omega = \varepsilon_\nu\omega = 0$ then, by (1.2), $\omega = 0$. Since $\iota_\nu^2 = \varepsilon_\nu^2 = 0$, we see that $\varepsilon_\nu(\Lambda^{p-1}) \cap \iota_\nu(\Lambda^{p+1}) = \{0\}$. Moreover, by (1.4) the spaces $\mathbf{H}^p$ and $\varepsilon_\nu(\Lambda^{p-1}) \oplus \iota_\nu(\Lambda^{p+1})$ are mutually orthogonal.

The purpose of this paper is to prove, in a purely algebraic way, the following

THEOREM 1.2 (Hodge type decomposition). *For any* $0 \le p \le n$, *the space* Λ^p *splits as the direct sum* $\Lambda^p = \iota_\nu\Lambda^{p+1} \oplus \varepsilon_\nu\Lambda^{p-1} \oplus \mathbf{H}^p$, *or equivalently* $\Lambda^p = d^\star\Lambda^{p+1} \oplus \delta^\star\Lambda^{p-1} \oplus \mathbf{H}^p$.

2. Some preparations. Denote by Λ_k^p and $\mathfrak{H}_k^p$ the space of all forms in $\mathbb{R}^n$ that have coefficients from $\mathcal{P}_k$ and $\mathcal{H}_k$, respectively. It is convenient to put $\Lambda_k^p = \{0\}$ if either $p < 0$ or $k < 0$. Let $\mathbf{H}_k^p = \mathbf{H}^p \cap \Lambda_k^p = \{\omega \in \Lambda_k^p : d\omega = \delta\omega = 0\}$. Since for any differential p-form ω on $\mathbb{R}^n$,

$$(\Delta\omega)_{i_1,\dots,i_p} = \Delta\omega_{i_1,\dots,i_p},$$

we see that $\omega \in \mathfrak{H}^p$ iff all coefficients of ω are harmonic polynomials. In particular, $\mathfrak{H}_k^p = \mathfrak{H}^p \cap \Lambda_k^p$ and $\mathbf{H}_k^p = \mathfrak{H}_k^p \cap \mathbf{H}^p$. Moreover, the decomposition (1.1) translates immediately to Λ_k^p where we have

$$\Lambda_k^p = \mathfrak{H}_k^p \oplus^\perp r^2\Lambda_{k-2}^p. \tag{2.1}$$

In the proof of the main theorem we will need to use some formulæ from [2]. We list them below.

For any $\omega \in \Lambda_k^p$ we have ([2, Prop. 2.2.1])

$$\delta\varepsilon_\nu\omega = -\varepsilon_\nu\delta\omega - (n - p + k)\omega, \tag{2.2}$$

$$d\iota_\nu\omega = -\iota_\nu d\omega + (p + k)\omega. \tag{2.3}$$

Applying the second identity in (1.3) and (2.3) we see that for any polynomial form ω,

$$\Delta\iota_\nu\omega = \iota_\nu\Delta\omega + 2\delta\omega. \tag{2.4}$$

Define the spaces $\chi_{p,k}$, $\chi^0_{p,k}$ and the operator $I_{p.k}$ as follows:

$$\chi_{p,k} = \mathfrak{H}^p_k \cap \ker\delta, \qquad \chi^0_{p,k} = \chi_{p,k} \cap \ker\iota_\nu,$$

$$I_{p.k} = \varepsilon_\nu - c_k r^2 d, \quad \text{where} \quad c_k = \begin{cases} 1/(n+2k-4) & \text{if } k \ge 2,\ 0 < p \le n, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following decompositions ([2, (3.3.2), (3.3.3)]):

$$\chi_{p,k} = \chi^0_{p,k} \oplus^\perp d\chi^0_{p-1,k+1}, \tag{2.5}$$

$$\mathfrak{H}^p_k = \chi^0_{p,k} \oplus^\perp d\chi^0_{p-1,k+1} \oplus^\perp \varepsilon_\nu d\chi^0_{p-2,k} \oplus^\perp I_{p,k}(\chi^0_{p-1,k-1}). \tag{2.6}$$

Notice that in (2.6) some subspaces may degenerate sometimes. In particular, by (2.3) it follows easily ([2, (3.2.1)]) that

$$\chi^0_{p,0} = \{0\} \quad \text{if } p > 0. \tag{2.7}$$

Remark. The orthogonal decomposition (2.6) above is the key step in the proof of Hodge type decomposition. In fact, (2.6) is the very special case of the more general decomposition [2, Thm. 3.3.1] of the kernel of the operator $L = ad\delta + b\delta d$, $a, b > 0$.

The last part of this section has a technical character. From (1.3) and (2.4) it follows that $\iota_\nu \mathbf{H}^{p+1}_{k-1} \subset \chi^0_{p,k}$. On the other hand, if $p + k > 0$ then taking $\omega \in \chi^0_{p,k}$ we see that $\omega = \iota_\nu \eta$, where $\eta = (p+k)^{-1} d\omega$. Clearly, $\eta \in \mathbf{H}^{p+1}_{k-1}$. So we have

$$\iota_\nu \mathbf{H}^{p+1}_{k-1} = \chi^0_{p,k} \quad \text{if } p + k > 0. \tag{2.8}$$

If $k = p = 0$ then clearly $\mathbf{H}^0_0 = \mathbb{R}$ (the space of constant functions). Suppose that either $p > 0$ or $k > 0$. Let $\omega \in \mathbf{H}^p_k$. In the light of (2.5) and the relation $\mathbf{H}^p_k \subset \chi_{p,k}$ we may write $\omega = \omega' + \omega''$, where $\omega' \in \chi^0_{p,k}$ and $\omega'' \in d\chi_{p-1,k+1}$. Since ω and ω'' are closed, $d\omega' = 0$. Therefore, $0 = \iota_\nu d\omega' = -d\iota_\nu\omega' + (p+k)\omega' = (p+k)\omega'$. Since $p+k > 0$, $\omega' = 0$. Thus $\mathbf{H}^p_k \subset d\chi^0_{p-1,k+1}$. On the other hand, one easily checks that $d\chi^0_{p-1,k+1} \subset \mathbf{H}^p_k$. We have proved that

$$\mathbf{H}^p_k = \begin{cases} \mathbb{R} & \text{if } p = k = 0, \\ d\chi^0_{p-1,k+1} & \text{otherwise.} \end{cases} \tag{2.9}$$

Let us complete the section with the following observation: if either $p \ne 1$ or $k \ne 1$ then

$$I_{p,k}(\chi^0_{p-1,k-1}) \oplus r^2 \Lambda^p_{k-2} \subset \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2}). \tag{2.10}$$

Indeed, the inclusion $r^2\Lambda^p_{k-2} \subset \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2})$ follows from (1.2). Now we prove that $I_{p,k}(\chi^0_{p-1,k-1}) \subset \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2})$. If $p = 0$ or $k = 0$ then the inclusion is trivial. Suppose that $p, k > 0$. Take $\omega \in$

$I_{p,k}(\chi^0_{p-1,k-1})$, $\omega = I_{p,k}\eta$, $\eta \in \chi^0_{p-1,k-1}$. Using (1.2) and (2.3) one easily checks that $\omega = \iota_\nu\varepsilon_\nu\psi' + \varepsilon_\nu\iota_\nu\psi''$, where

$$\psi' = -c_k d\eta, \quad \psi'' = \frac{1-(p+k-2)c_k}{p+k-2}\, d\eta.$$

3. Proof of the Hodge type decomposition. To prove Theorem 1.2 it suffices to show that for any $p, k \geq 0$,

$$\Lambda^p_k = \iota_\nu \Lambda^{p+1}_{k-1} \oplus \varepsilon_\nu \Lambda^{p-1}_{k-1} \oplus \mathbf{H}^p_k. \tag{3.1}$$

To prove (3.1) we apply induction with respect to k. Before doing this, we check some very special case of (3.1) separately. Namely, we have

$$\Lambda^1_1 = \iota_\nu \Lambda^2_0 \oplus \varepsilon_\nu \Lambda^0_0 \oplus \mathbf{H}^1_1. \tag{3.2}$$

Proof of (3.2). Clearly, $\Lambda^1_1 = \mathfrak{H}^1_1$, $\Lambda^0_0 = \chi^0_{0,0} = \mathbb{R}$ and $\Lambda^2_0 = \mathfrak{H}^2_0$. Formula (2.6) implies that it suffices to show that $\chi^0_{1,1} = \iota_\nu \mathfrak{H}^2_0$. By (1.3), $\iota_\nu \mathfrak{H}^2_0 \subset \chi^0_{1,1}$. Let $\omega \in \chi^0_{1,1}$. Put $\eta = (1/2)d\omega$; then one can easily check that $\iota_\nu \eta = \omega$. Thus $\chi^0_{1,1} = \iota_\nu \mathfrak{H}^p_0$.

Proof of (3.1) by induction with respect to k. Suppose that $k = 0$. If $p = 0$ then (3.1) is a direct consequence of the equalities $\Lambda^0_0 = \mathfrak{H}^0_0 = \mathbf{H}^0_0 = \mathbb{R}$. If $p > 0$ then (3.1) follows from (2.9), (2.6), (2.7) and the equality $\Lambda^p_0 = \mathfrak{H}^p_0$.

Suppose that (3.1) holds for $k-1$, $k \geq 1$. Take any $p \geq 0$. We may assume that either $p \neq 1$ or $k \neq 1$ (see (3.2)). Using (1.4), (2.2) and (2.3) we find that $\iota_\nu\varepsilon_\nu(\Lambda^p_{k-2})$ and $\iota_\nu(\mathbf{H}^{p+1}_{k-1})$, and $\varepsilon_\nu\iota_\nu(\Lambda^p_{k-2})$ and $\varepsilon_\nu(\mathbf{H}^{p-1}_{k-1})$ are mutually orthogonal. Thus, by induction hypothesis we have

$$\begin{aligned} \varepsilon_\nu(\Lambda^{p-1}_{k-1}) &= \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu(\mathbf{H}^{p-1}_{k-1}), \\ \iota_\nu(\Lambda^{p+1}_{k-1}) &= \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \iota_\nu(\mathbf{H}^{p+1}_{k-1}). \end{aligned} \tag{3.3}$$

From (2.1), (2.6), (2.8), (2.9), (2.10) and (3.3) we get

$$\begin{aligned} \Lambda^p_k &\supset \varepsilon_\nu(\Lambda^{p-1}_{k-1}) \oplus \iota_\nu(\Lambda^{p+1}_{k-1}) \oplus \mathbf{H}^p_k \\ &\overset{(3.3)}{=} \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2}) \oplus \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu(\mathbf{H}^{p-1}_{k-1}) \oplus \iota_\nu(\mathbf{H}^{p+1}_{k-1}) \oplus \mathbf{H}^p_k \\ &\overset{(2.9),(2.8)}{=} \varepsilon_\nu\iota_\nu(\Lambda^p_{k-2}) \oplus \iota_\nu\varepsilon_\nu(\Lambda^p_{k-2}) \oplus \varepsilon_\nu d\chi_{p-2,k} \oplus \chi^0_{p,k} \oplus \mathbf{H}^p_k \\ &\overset{(2.10)}{\supset} r^2\Lambda^p_{k-2} \oplus I_{p,k}(\chi^0_{p-1,k-1}) \oplus \chi^0_{p,k} \oplus \varepsilon_\nu d\chi_{p-2,k} \oplus \mathbf{H}^p_k \\ &\overset{(2.6)}{=} r^2\Lambda^p_{k-2} \oplus \mathfrak{H}^p_k \\ &\overset{(2.1)}{=} \Lambda^p_k, \end{aligned}$$

which proves (3.1).

Induction completes the proof.

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