ANNALES
POLONICI MATHEMATICI
LXXIX.2 (2002)

Infinitely many solutions for a semilinear elliptic equation in $\mathbb{R}^N$ via a perturbation method

by Marino Badiale (Torino)

Abstract. We introduce a method to treat a semilinear elliptic equation in $\mathbb{R}^N$ (see equation (1) below). This method is of a perturbative nature. It permits us to skip the problem of lack of compactness of $\mathbb{R}^N$ but requires an oscillatory behavior of the potential b.

1. Introduction. The existence of positive solutions of elliptic equations on $\mathbb{R}^N$ like

$$\begin{cases} -\Delta u + u = b(x)|u|^{p-1}u, \quad 1 < p < (N+2)/(N-2), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

has been extensively investigated by variational methods. One looks for solutions of (1) as critical points of the energy functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u|^2 + u^2]\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} b(x) u^{p+1}\, dx. \tag{2}$$

Under reasonable assumptions (see e.g. (b_1) below) J satisfies the geometric properties of the Mountain Pass Theorem, while the compactness Palais–Smale condition (PS in short) does not hold, in general, because of the unboundedness of the domain and the non-compactness of Sobolev embeddings. To overcome this lack of compactness the structure of the Palais–Smale sequences of J has been deeply studied in recent years, starting with the celebrated papers of P. L. Lions on the *concentration-compactness* principle (see [15]–[18]). The nature of the obstruction to compactness seems now clear enough: see for example [8], [13] and the references therein. See also [5] for a generalization to quasilinear equations.

A standard hypothesis for (1) is the following:

(b_1) $\quad b \geq b_0 > 0, \quad b \in C(\mathbb{R}^N), \quad b(x) \to b_\infty > 0 \quad \text{as } |x| \to \infty,$

2000 *Mathematics Subject Classification*: Primary 35J65.

Key words and phrases: semilinear equation, oscillatory potential, critical point.

The author was supported by M.U.R.S.T., "Variational and Nonlinear Differential Equations".

It is worth noticing that, to our knowledge, there is no result on existence of a solution for (1) under hypothesis (b_1). However, thanks to the analysis of loss of compactness discussed above, many different existence results for (1) have been obtained. For example it is not difficult to prove the existence of a solution if (b_1) holds and furthermore $b \geq b_\infty$, and in general one can show that (PS) holds at any level $c < c_\infty$, the Mountain Pass critical level of

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla u|^2 + u^2]\,dx - \frac{b_\infty}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx. \tag{3}$$

Among many others we quote, as examples, the papers [6], [7], [11], [12] and we refer to the book of J. Chabrowski quoted above for an extensive bibliography.

We just recall that the main results achieved in [6, 7] require that b satisfies (b_1) as well as

$$b(x) \geq b_\infty - Ce^{-\delta|x|} \quad \forall x \in \mathbb{R}^N \ (C, \delta > 0). \tag{b$_2$}$$

In this case the critical level can be greater than c_∞ and is found by exploiting algebraic topology tools.

The main aim of this paper is to face a class of problems that cannot be handled by the preceding results. We deal with a class of functions b that satisfy (b_1) but oscillate in a suitable way at infinity. No condition on the decay of b at infinity is required. For such b's we are able to show that (1) has infinitely many solutions. See Theorems 3.2, 3.3, 4.1 and the example at the end of Section 3. In this example we build a function b which does not satisfy (b_2), which can even be negative somewhere, and such that (1) has infinitely many solutions.

Our approach is different from the papers cited above. Instead of trying to overcome the lack of (PS) we adapt a perturbation method discussed in [2–4]. Roughly, our arguments are based on the following steps:

1) If $z \in H^1(\mathbb{R}^N)$ is a radial positive solution of

$$-\Delta u + u = b_\infty |u|^{p-1}u, \tag{4}$$

then the manifold $Z = \{z_\theta = z(\cdot + \theta) : \theta \in \mathbb{R}^N\}$ consists of critical points of J_∞.

2) For a large R we construct, near $Z_R = \{z_\theta = z(\cdot + \theta) : |\theta| > R\}$, a perturbed manifold $\widetilde{Z}_R$, locally diffeomorphic to Z_R, which is a natural constraint for J: by this we mean that the critical points of J constrained on $\widetilde{Z}_R$ are global critical points, that is,

$$u \in Z_R \text{ and } \nabla J_{|\widetilde{Z}_R}(u) = 0 \quad \text{implies} \quad \nabla J(u) = 0.$$

In this way we reduce the infinite-dimensional problem of finding critical points of J to a finite-dimensional one.

3) Through an asymptotic expansion, the critical points of $J_{|\widetilde{Z}_R}$ correspond to the ones of a suitable functional Γ on Z_R. Such a functional inherits the oscillation behavior of b and this permits us to find infinitely many critical points of J.

This paper is organized as follows: in Section 2 we give the abstract setting for our treatment of problem (1); then we apply this setting to get infinitely many solutions when b oscillates and $a = b - b_\infty$ has constant sign (Section 3) or changes sign (Section 4).

NOTATIONS. We give a list of the main notations we will use.

• For $u \in \mathbb{R}$, u^+ is the positive part, u^- the negative part.

• We write $\|u\|$ for the usual norm in the Sobolev space $H^1(\mathbb{R}^N)$, $(u|v)$ for the inner product. For $u \in L^p(\mathbb{R}^N)$, $|u|_p$ is the usual L^p norm.

• $2^* = 2N/(N-2)$ is the critical exponent for Sobolev embedding.

• For any $P \in \mathbb{R}^N$ and $r > 0$ we denote by $B(P,r)$ the open ball in $\mathbb{R}^N$ with center P and radius r, while $B_r = B(0,r)$.

• $\{e_1, \ldots, e_N\}$ is the usual vector basis of $\mathbb{R}^N$.

• $o(1)$ is any asymptotically vanishing quantity.

• If E is a Banach space we define $\mathcal{L}(E)$ to be the space of bounded linear operators from E to E, with norm $\|T\|_{\mathcal{L}} = \sup_{\|u\|_E=1} \|Tu\|_E$.

• If E is a Hilbert space and $f \in C^1(E,E)$, we denote by f' or ∇f the gradient of f.

• We use C to indicate any fixed positive constant.

The author thanks A. Ambrosetti for useful discussions.

2. The abstract setting. In this section we give the general setting in which we study problem (1). We want to reduce problem (1) to the problem of looking for critical points of a functional defined in Z_R. For this we construct a "perturbed manifold", near Z_R, and we show that the critical points of J, constrained on this manifold, are true critical points, hence solutions of (1). Then we will look for critical points of this constrained functional by making use of suitable asymptotic expansions. In Sections 3, 4 we will apply these general results to obtain existence of solutions.

Let $\mathcal{K} = (-\Delta + 1)^{-1} : H^{-1} \to H^1$ be the usual isomorphism. We can write (1) as

$$u = \mathcal{K}(b|u|^{p-1}u).$$

We define z to be the unique positive radial solution of the equation

$$\begin{cases} -\Delta u + u = b_\infty |u|^{p-1}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{5}$$

which we can write as $u = \mathcal{K}(b_\infty |u|^{p-1}u)$, so that z satisfies

$$z = \mathcal{K} b_\infty z^p.$$

For the existence, uniqueness and other properties of z see [9], [10], [14]. We recall in particular the asymptotic decay of z: there is $\eta > 0$ such that

$$z(x)\exp(|x|)|x|^{(N-1)/2} \to \eta \quad \text{as } |x| \to \infty. \tag{6}$$

Let $z_\theta(x) = z(x-\theta)$ be a translation of z. Notice that the partial derivatives $D_i z_\theta$ satisfy the linearized equation

$$\begin{cases} -\Delta u + u = p b_\infty z_\theta^{p-1} u, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{7}$$

It is well known (see [19] and the references therein) that the subspace $\mathbb{K}_\theta = \operatorname{span}\{D_i z_\theta : i = 1, \ldots, N\}$ gives all solutions of (7).

As mentioned in the introduction, we now want to build a manifold $\widetilde{Z}_R$, near Z_R and locally diffeomorphic to it, which is a natural constraint for J. We will write

$$\widetilde{Z}_R = \{z_\theta + w_\theta : |\theta| > R\},$$

for a suitable $w_\theta \in H^1(\mathbb{R}^N)$, and we want to find w_θ. For this, consider the function $F_\theta : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ defined by

$$F_\theta(w) = w + \mathcal{K} b_\infty z_\theta^p - \mathcal{K} b[|z_\theta + w|^{p-1}(z_\theta + w)]. \tag{8}$$

Let $\mathbb{K}_\theta^\perp$ be the orthogonal complement of $\mathbb{K}_\theta$ in H^1, and P_θ the orthogonal projection of H^1 on $\mathbb{K}_\theta^\perp$. Now, for any θ outside a compact set, we want to find $w_\theta \in \mathbb{K}_\theta^\perp$ such that $F_\theta(w_\theta) \in \mathbb{K}_\theta$, that is, we want to find solutions of the following problem:

$$w \in \mathbb{K}_\theta^\perp, \quad P_\theta F_\theta(w) = 0. \tag{9}$$

The next theorem gives our main result concerning the solutions of (9), while Theorem 2.2 says that the manifold $\widetilde{Z}_R$ that we construct with such w_θ's is indeed a natural constraint for J.

Theorem 2.1. *There are a positive number R and a function $w : \mathbb{R}^N \setminus B_R \to H^1(\mathbb{R}^N)$ such that, for all $\theta \in \mathbb{R}^N \setminus B_R$, $w(\theta)$ satisfies* (9). *Furthermore, w is a C^1 function of θ and $\|w(\theta)\|$ and $\|D_{\theta_j} w(\theta)\|$ go to zero as $|\theta| \to \infty$.*

We will prove Theorem 2.1 in a sequence of lemmas. First, we rewrite (9) as a fixed point equation. For this, write

$$F_\theta(w) = F_\theta(0) + F_\theta'(0)w + N_\theta(w),$$

and notice that, setting

$$a(x) = b(x) - b_\infty,$$

we have

$$F_\theta(0) = -\mathcal{K} a z_\theta^p, \quad F_\theta'(0)w = w - p\mathcal{K} b z_\theta^{p-1} w,$$

which implies

$$N_\theta(w) = -\mathcal{K}b[|z_\theta + w|^{p-1}(z_\theta + w) - z_\theta^p - pz_\theta^{p-1}w].$$

We can write (9) as

$$w \in \mathbb{K}_\theta^\perp, \quad P_\theta F_\theta'(0)w = -P_\theta F_\theta(0) - P_\theta N_\theta(w). \tag{10}$$

Define

$$L_\theta = P_\theta F_\theta'(0) \in \mathcal{L}(\mathbb{K}_\theta^\perp).$$

In the following lemma we prove that L_θ is invertible.

LEMMA 2.1. *There is $R > 0$ such that, for all θ with $|\theta| \geq R$, L_θ is invertible.*

Proof. We prove the following property, which implies the invertibility of L_θ: there is $\gamma > 0$ such that for all $|\theta|$ large enough and for all $\varphi \in \mathbb{K}_\theta^\perp$ with $\|\varphi\| = 1$,

$$\|L_\theta\varphi\| \geq \gamma. \tag{11}$$

To prove (11) we write $L_\theta = L_\theta^1 + L_\theta^2$ where

$$L_\theta^1\varphi = P_\theta(\varphi - pb_\infty\mathcal{K}z_\theta^{p-1}\varphi), \quad L_\theta^2\varphi = -pP_\theta\mathcal{K}[az_\theta^{p-1}\varphi].$$

As mentioned above, $\mathbb{K}_\theta$ is the kernel of the operator $\varphi \mapsto \varphi - pb_\infty\mathcal{K}z_\theta^{p-1}\varphi$, so $L_\theta^1 : \mathbb{K}_\theta^\perp \to \mathbb{K}_\theta^\perp$ is invertible and it is easy to see that there is $\gamma_1 > 0$ such that for all $|\theta|$ we have $\|L_\theta^1\varphi\| \geq \gamma_1$ if $\varphi \in \mathbb{K}_\theta^\perp$ and $\|\varphi\| = 1$. Hence, the assertion follows from the following claim:

$$\|L_\theta^2\|_{\mathcal{L}} \to 0 \quad \text{as } |\theta| \to \infty.$$

To prove the claim, we notice that

$$\begin{aligned}
\|L_\theta^2\|_{\mathcal{L}} = \sup_{\|v\|=1} \|L_\theta^2 v\| &= \sup_{\|v\|,\|w\|=1} p \int_{\mathbb{R}^N} az_\theta^{p-1}vw\,dx \\
&\leq \sup_{\|v\|,\|w\|=1} p\Big(\int_{\mathbb{R}^N} |a|^{N/2}z_\theta^{(p-1)N/2}\,dx\Big)^{2/N}|v|_{2^*}|w|_{2^*} \\
&\leq C\Big(\int_{\mathbb{R}^N} |a|^{N/2}z_\theta^{(p-1)N/2}\,dx\Big)^{2/N}.
\end{aligned}$$

It is easy to see that $\int_{\mathbb{R}^N} |a|^{N/2}z_\theta^{(p-1)N/2}\,dx \to 0$ as $|\theta| \to \infty$, hence the claim is proved, and so is the lemma. ∎

Thanks to the previous lemma we can write (10) in the following way:

$$w \in \mathbb{K}_\theta^\perp, \quad w = -L_\theta^{-1}[P_\theta F_\theta(0) + P_\theta N_\theta(w)]. \tag{12}$$

Define $G_\theta : \mathbb{K}_\theta^\perp \to \mathbb{K}_\theta^\perp$ by $G_\theta(w) = -L_\theta^{-1}[P_\theta F_\theta(0) + P_\theta N_\theta(w)]$. To solve (12) means to find a fixed point of G_θ. In the next lemma we prove that

G_θ, when restricted to little balls, is a contraction. For any $\delta > 0$ define $B_\delta^\perp = \{\varphi \in \mathbb{K}_\theta^\perp : \|\varphi\| < \delta\}$.

LEMMA 2.2. *For any δ small enough there is $R > 0$ such that for all $|\theta| > R$, $G_\theta(B_\delta^\perp) \subset B_\delta^\perp$ and $G_{\theta|B_\delta^\perp}$ is a contraction.*

Proof. By some lengthy computations that we skip, it is not difficult to get the following statements:

(i) $F_\theta(0) = -\mathcal{K}az_\theta^p \to 0$ as $|\theta| \to \infty$.
(ii) $\|N_\theta(w)\| = o(\|w\|)$ as $\|w\| \to 0$ uniformly with respect to θ.
(iii) $\|N_\theta(w_1) - N_\theta(w_2)\| \le o(1)\|w_1 - w_2\|$ uniformly with respect to θ.

From (i)–(iii), the assertion follows easily. ∎

As $G_{\theta|B_\delta^\perp}$ is a contraction, it has a unique fixed point, which we call w_θ. So w_θ is the solution of (9) we were looking for. It is defined for $|\theta| > R$, where R is given by the previous lemma. We now want to prove the other properties stated in Theorem 2.1. We first obtain some information on the asymptotic behavior of w_θ.

LEMMA 2.3. *The following inequality holds*:

$$\|w_\theta\| \le C\Big(\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big)^{p/(p+1)}.$$

In particular $\|w_\theta\| \to 0$ *as* $|\theta| \to \infty$.

Proof. Let $G_\theta^{(k)}$ be the kth iterate of G_θ. As w_θ is the fixed point of the contraction G_θ, we have (setting $G_\theta^{(0)}(0) = 0$)

$$\|w_\theta\| = \|\lim_k G_\theta^{(k)}(0)\| = \Big\|\sum_{k=1}^{\infty}[G_\theta^{(k)}(0) - G_\theta^{(k-1)}(0)]\Big\|$$
$$\le \sum_{k=1}^{\infty} \lambda^{k-1}\|G_\theta(0) - 0\| \le C\|G_\theta(0)\| \le C\|az_\theta^p\|_{H^{-1}},$$

where λ is a contraction constant for G_θ. It is easy to see that

$$\|az_\theta^p\|_{H^{-1}} \le C\Big(\int_{\mathbb{R}^N} |a|^{(p+1)/p} z_\theta^{p+1}\,dx\Big)^{p/(p+1)} \tag{13}$$
$$\le C\Big(\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big)^{p/(p+1)}. \ ∎$$

Let us rewrite (9) in the following form:

$$F_\theta(w_\theta) = \sum_{i=1}^{N} \alpha_\theta^i D_i z_\theta, \tag{14}$$

where $\alpha_\theta = (\alpha_\theta^1, \dots, \alpha_\theta^N) \in \mathbb{R}^N$. We now want to prove that w_θ, α_θ are C^1 functions of θ. First, we prove that also α_θ vanishes asymptotically for large θ's.

LEMMA 2.4. *$\alpha_\theta \to 0$ as $|\theta| \to \infty$.*

Proof. From (14) we get, by taking the inner product with $D_j z_\theta$ (and recalling that $w_\theta \in \mathbb{K}_\theta^\perp$),

$$C\alpha_\theta^j = -\int_{\mathbb{R}^N} a|z_\theta + w_\theta|^{p-1}(z_\theta + w_\theta)D_j z_\theta - b_\infty \int_{\mathbb{R}^N} [|z_\theta + w_\theta|^{p-1}(z_\theta + w_\theta) - z_\theta^p] D_j z_\theta$$

and it is easy to see that the right-hand terms vanish as $|\theta| \to \infty$. ■

To prove that w, α are C^1 functions of θ, we apply the Implicit Function Theorem. For $R > 0$ set $A_R = \mathbb{R}^N \setminus B_R$. Consider the function

$$\Phi = (\Phi_1, \Phi_2) : A_R \times H^1 \times \mathbb{R}^N \to H^1 \times \mathbb{R}^N \tag{15}$$

where

$$\Phi_1(\theta, w, \alpha) = w + \mathcal{K} b_\infty z_\theta^p - \mathcal{K} b |z_\theta + w|^{p-1}(z_\theta + w) - \sum_i \alpha_i D_i z_\theta,$$

$$\Phi_2(\theta, w, \alpha) = ((w|D_1 z_\theta), \dots, (w|D_N z_\theta)).$$

Notice that

$$\Phi(\theta, w_\theta, \alpha_\theta) = (0, 0).$$

We want to solve the equation $\Phi = 0$ near any point $(\theta, w_\theta, \alpha_\theta)$. We will apply the Implicit Function Theorem to get, in a neighborhood of $(\theta, w_\theta, \alpha_\theta)$, functions $\widetilde{w}(\tau), \widetilde{\alpha}(\tau)$ such that $\Phi(\tau, \widetilde{w}(\tau), \widetilde{\alpha}(\tau)) = 0$. Of course we have to compute the differential of Φ with respect to (w, α), and it is not difficult to obtain

$$\begin{aligned} D\Phi(\theta, w_\theta, \alpha_\theta)[v, \beta] &= (D\Phi_1(\theta, w_\theta, \alpha_\theta)[v, \beta], D\Phi_2(\theta, w_\theta, \alpha_\theta)[v, \beta]) \\ &= \Big(v - p\mathcal{K} b|z_\theta + w_\theta|^{p-1} v - \sum_i \beta_i D_i z_\theta; (v|D_1 z_\theta), \dots, (v|D_N z_\theta)\Big). \end{aligned}$$

Here D means the differential with respect to (w, α). In the space $H^1 \times \mathbb{R}^N$, we consider the norm $\|(v, \beta)\|_1 = \|v\| + |\alpha|$. Then we prove the following lemma.

LEMMA 2.5. *There exist $\gamma, R_1 > 0$ such that, for all $|\theta| > R_1$ and all $(v, \beta) \in H^1 \times \mathbb{R}^N$,*

$$\|D\Phi(\theta, w_\theta, \alpha_\theta)[v, \beta]\|_1 \ge \gamma \|(v, \beta)\|_1.$$

Proof. We argue by contradiction. If the conclusion is not true, then there are sequences $\{\theta_k\} \subset \mathbb{R}^N$ and $\{(v_k, \beta_k)\} \subset H^1 \times \mathbb{R}^N$ such that

$$|\theta_k| \to \infty, \quad \|(v_k, \beta_k)\|_1 = 1, \quad \|D\Phi(\theta_k, w_{\theta_k}, \alpha_{\theta_k})[v_k, \beta_k]\|_1 \to 0.$$

Setting $w_k = w_{\theta_k}$ and $z_k = z_{\theta_k}$, we have

$$
\begin{aligned}
(16) \qquad o(1) &= D\Phi_1(\theta_k, w_k, \alpha_{\theta_k})[v_k, \beta_k] \\
&= v_k - p\mathcal{K}b|z_k + w_k|^{p-1}v_k - \sum_i \beta_k^i D_i z_k \\
&= v_k - pb_\infty \mathcal{K} z_k^{p-1} v_k \\
&\quad + pb_\infty \mathcal{K} z_k^{p-1} v_k - p\mathcal{K}b|z_k + w_k|^{p-1}v_k - \sum_i \beta_k^i D_i z_k \\
&= L^1_{\theta_k} v_k - p\mathcal{K}a|z_k + w_k|^{p-1}v_k \\
&\quad - pb_\infty \mathcal{K}[|z_k + w_k|^{p-1} - z_k^{p-1}]v_k - \sum_i \beta_k^i D_i z_k.
\end{aligned}
$$

Since $|\theta_k| \to \infty$, we have $\|w_k\| \to 0$, hence we easily obtain

$$\mathcal{K}a|z_k + w_k|^{p-1}v_k \to 0, \qquad \mathcal{K}[|z_k + w_k|^{p-1} - z_k^{p-1}]v_k \to 0 \qquad \text{as } |\theta_k| \to \infty.$$

Recalling that $L^1_\theta D_j z_k = 0$, from (16) we get, by taking the inner product with $D_j z_k$,

$$\beta_k^j \to 0, \quad \text{hence} \quad \beta_k \to 0 \quad \text{as } k \to \infty.$$

Recall that $D\Phi_2(\theta_k, w_{\theta_k}, \alpha_{\theta_k})v_k \to 0$, that is, $(v_k|D_j z_{\theta_k}) \to 0$, $j = 1, \ldots, N$. Hence, if we write $v_k = v_k^1 + v_k^2$ where $v_k^1 \in \mathbb{K}_{\theta_k}$ and $v_k^2 \in \mathbb{K}^\perp_{\theta_k}$, we obtain $v_k^1 \to 0$. From this we deduce $\|L^1_{\theta_k} v_k\| = \|L^1_{\theta_k} v_k^2\| \geq \gamma_1 \|v_k^2\| = \gamma_1 \|v_k\| + o(1)$. From (16) we get

$$\gamma_1 \|v_k\| \leq o(1),$$

so $v_k \to 0$ and $\beta_k \to 0$, contrary to the hypothesis $\|(v_k, \beta_k)\|_1 = 1$. ∎

Thanks to Lemma 2.5, we can apply the Implicit Function Theorem at any point $(\theta, w_\theta, \alpha_\theta)$ with $|\theta| > R$ (it is not restrictive to assume $R > R_1$), and we get, in the neighborhood of such a point, two C^1 functions $\widetilde{w}(\tau), \widetilde{\alpha}(\tau)$ such that $\Phi(\tau, \widetilde{\alpha}(\tau), \widetilde{w}(\tau)) = 0$. This equation means exactly that $\widetilde{w}(\tau) \in \mathbb{K}^\perp_\tau$ and that $\widetilde{w}(\tau)$ satisfies (9). We know that w_τ is the unique point satisfying these conditions, so we must have $w_\tau = \widetilde{w}(\tau)$ in this entire neighborhood of θ. In particular we get $w_\theta = \widetilde{w}(\theta)$ for all θ and w_θ is a C^1 function of θ. In the same way $\alpha_\theta = \widetilde{\alpha}(\theta)$ is a C^1 function of θ. This gives the C^1 regularity of w_θ (and also of α_θ), and to complete the proof of Theorem 2.1 we just have to prove that the derivatives $D_{\theta_j} w$ vanish as $|\theta| \to \infty$.

LEMMA 2.6. *$\|D_{\theta_i} w\| \to 0$ and $|D_{\theta_i} \alpha| \to 0$ as $|\theta| \to \infty$.*

Proof. We differentiate (14) with respect to θ_j, and recall that $(w(\theta)|D_i z_\theta) = 0$, to obtain the following system of equations satisfied by $D_{\theta_i} w$, $D_{\theta_i} \alpha$:

$$
(17) \qquad \begin{cases}
D_{\theta_j} w + pb_\infty \mathcal{K} z_\theta^{p-1} D_j z_\theta - p\mathcal{K}b|z_\theta + w|^{p-1}(D_j z_\theta + D_{\theta_j} w) \\
\qquad\qquad = \displaystyle\sum_{i=1}^N D_{\theta_j} \alpha^j D_i z_\theta + \alpha^i D_{ij} z_\theta, \\
(w|D_{ij} z_\theta) + (D_{\theta_j} w|D_i z_\theta) = 0.
\end{cases}
$$

Notice that the equality

$$(w|D_{ij}z_\theta) + (D_{\theta_j}w|D_i z_\theta) = 0$$

implies that

$$(D_{\theta_j}w|D_i z_\theta) = o(1) \quad \text{as } |\theta| \to \infty, \tag{18}$$

because $w = w(\theta) = o(1)$ and $D_{ij}z_\theta$ is bounded. We write the first equation of (17) in the following way:

$$\begin{aligned} 0 &= D_{\theta_j}w + p\mathcal{K}b_\infty z_\theta^{p-1}D_j z_\theta - p\mathcal{K}b|z_\theta + w|^{p-1}(D_j z_\theta + D_{\theta_j}w) \\ &\quad - \sum_{i=1}^N [D_{\theta_j}\alpha^j D_i z_\theta + \alpha^i D_{ij}z_\theta] \\ &= D_{\theta_j}w - p\mathcal{K}a|z_\theta + w|^{p-1}D_j z_\theta - pb_\infty\mathcal{K}(|z_\theta + w|^{p-1} - z_\theta^{p-1})D_j z_\theta \\ &\quad - p\mathcal{K}b|z_\theta + w|^{p-1}D_{\theta_j}w - \sum_{i=1}^N [D_{\theta_j}\alpha^j D_i z_\theta + \alpha^i D_{ij}z_\theta] \\ &= D_{\theta_j}w - pb_\infty\mathcal{K}z_\theta^{p-1}D_{\theta_j}w \\ &\quad - pb_\infty\mathcal{K}(|z_\theta + w|^{p-1} - z_\theta^{p-1})(D_j z_\theta + D_{\theta_j}w) \\ &\quad - p\mathcal{K}a|z_\theta + w|^{p-1}(D_j z_\theta + D_{\theta_j}w) - \sum_{i=1}^N [D_{\theta_j}\alpha^i D_i z_\theta + \alpha^i D_{ij}z_\theta]. \end{aligned} \tag{19}$$

Multiplying this equation by $D_j z_\theta$ and recalling that $L_\theta^1 D_j z_\theta = 0$, and that $|z_\theta + w|^{p-1} - z_\theta^{p-1}$ and $a|z_\theta + w|^{p-1}$ go to zero as $|\theta| \to \infty$ in the relevant norms, it is easy to see that

$$|D_{\theta_j}\alpha^i| = o(1)(1 + \|D_{\theta_j}w\|). \tag{20}$$

Write $D_{\theta_j}w = \zeta_{1,\theta} + \zeta_{2,\theta}$ where $\zeta_{1,\theta} \in \mathbb{K}_\theta$, $\zeta_{2,\theta} \in \mathbb{K}_\theta^\perp$. Then (18) shows that

$$\zeta_{1,\theta} = o(1) \quad \text{as } |\theta| \to \infty. \tag{21}$$

Now, substituting (20) and (21) in (19) and recalling that $\|L_\theta^1 \zeta_{2,\theta}\| \geq \gamma_1\|\zeta_{2,\theta}\|$ we get

$$\|\zeta_{2,\theta}\| \leq o(1)(1 + \|\zeta_{2,\theta}\|),$$

hence $\|\zeta_{2,\theta}\| = o(1)$, so $\|D_{\theta_j}w\| = o(1)$, which implies $|D_{\theta_j}\alpha| = o(1)$ and the lemma is proved. ■

Now we can introduce the perturbed manifold that we announced at the beginning of this section. Define

$$\widetilde{Z}_R = \widetilde{Z} = \{z_\theta + w(\theta) : |\theta| > R\}. \tag{22}$$

Thanks to the previous results, $\widetilde{Z}$ is a smooth manifold. We consider the constrained functional $J_{|\widetilde{Z}}$ and we then get the following basic result, which states that the constrained critical points of $J_{|\widetilde{Z}}$ are true critical points.

THEOREM 2.2. *If* $u \in \widetilde{Z}$ *and* $\nabla J_{|\widetilde{Z}}(u) = 0$, *then* $\nabla J(u) = 0$.

Proof. One has just to repeat the arguments of Lemma 2.1 in [3]. ∎

Now our goal is to prove the existence of critical points of $J_{|\widetilde{Z}}$. For this we notice that, by easy computations, the following development holds (for simplicity we set $w = w(\theta)$):

$$
\begin{aligned}
J(z_\theta + w) &= \frac{1}{2}\|z_\theta + w\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} b|z_\theta + w|^{p+1}\,dx \\
&= \frac{1}{2}\|z_\theta\|^2 + (z_\theta|w) + \frac{1}{2}\|w\|^2 - \frac{b_\infty}{p+1}\int_{\mathbb{R}^N} z_\theta^{p+1}\,dx \\
&\quad + \frac{b_\infty}{p+1}\int_{\mathbb{R}^N} z_\theta^{p+1}\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} b|z_\theta + w|^{p+1}\,dx \\
&= c_0 - \frac{1}{p+1}\int_{\mathbb{R}^N} a z_\theta^{p+1}\,dx - \int_{\mathbb{R}^N} a z_\theta^p w\,dx + o(\|w\|),
\end{aligned}
\tag{23}
$$

where we have set

$$
c_0 = \frac{1}{2}\|z_\theta\|^2 - \frac{b_\infty}{p+1}\int_{\mathbb{R}^N} z_\theta^{p+1}\,dx.
$$

Now we have at hand all the abstract machinery that we need: we have reduced problem (1) to the finite-dimensional problem of finding critical points of $J_{|\widetilde{Z}}$, and we also have an asymptotic expansion of $J_{|\widetilde{Z}}$. In the next sections we will prove, using these results and suitable assumptions on b, the existence of infinitely many critical points of $J_{|\widetilde{Z}}$, hence of solutions to (1).

3. Existence results for a with constant sign. To obtain existence of solutions for problem (1), we first estimate the terms in (23).

LEMMA 3.1. *Assume that a does not change sign. Then, as* $|\theta| \to \infty$,

$$
\Big|\int_{\mathbb{R}^N} a z_\theta^p w\,dx\Big| \le o(1)\Big|\int_{\mathbb{R}^N} a z_\theta^{p+1}\,dx\Big|.
\tag{24}
$$

Proof. We have, by the Hölder inequality,

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^N} a z_\theta^p w\,dx\Big| &\le \Big[\int_{\mathbb{R}^N} |a|^{(p+1)/p} z_\theta^{p+1}\,dx\Big]^{p/(p+1)}\|w\| \\
&\le C\Big[\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big]^{p/(p+1)}\|w\|.
\end{aligned}
$$

Hence, by Lemma 2.3 we obtain

$$
\begin{aligned}
(25)\qquad \Big|\int_{\mathbb{R}^N} a z_\theta^p w\,dx\Big| &\le \Big[\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big]^{2p/(p+1)} \\
&= \Big[\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big]\Big[\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big]^{(p-1)/(p+1)} \\
&= o(1)\Big[\int_{\mathbb{R}^N} |a| z_\theta^{p+1}\,dx\Big] = o(1)\Big|\int_{\mathbb{R}^N} a z_\theta^{p+1}\,dx\Big|
\end{aligned}
$$

if a does not change sign. ∎

Lemmas 2.3, 3.1 and formula (23) imply the following theorem.

THEOREM 3.1. *Define* $\Gamma(\theta) = \int_{\mathbb{R}^N} a z_\theta^{p+1}\,dx$. *Then the functional* $J_{|\widetilde{Z}}$ *satisfies the following equation*:

$$J(z_\theta + w(\theta)) = c_0 - \frac{1}{p+1}\,\Gamma(\theta) + o(\Gamma(\theta)).$$

This theorem says that we can study, instead of the functional $J_{|\widetilde{Z}}$, the functional $\Gamma : \mathbb{R}^N \to \mathbb{R}$, which is of course much easier. In the following theorems we will prove that, under suitable hypotheses on a (that is, on b), Γ has infinitely many local maxima or minima, which gives rise to infinitely many solutions of (1).

THEOREM 3.2. *Assume that* $a \ge 0$ *and that the following hypotheses hold.*

(i) *There are* $T, r > 0$ *and sequences* $\{P_k\}_k \subset \mathbb{R}^N$ *and* $\{\alpha_k\}_k \subset \mathbb{R}$ *such that* $\alpha_k > 0$, $|P_k| \to +\infty$ *and*

$$a(x) \ge \alpha_k \quad \forall x \in B(P_k, r), \qquad a(x) \le T\alpha_k \quad \forall |x| \ge |P_k|/2.$$

(ii) *There are sequences* $\{r_k\}_k$, $\{R_k\}_k$, $\{\beta_k\}_k$ *such that* $r_k, R_k > 0$, $\beta_k \ge 0$, $r_k \to +\infty$, $R_k - r_k \to +\infty$, $R_k < |P_k|/2$ *and*

$$a(x) \le \beta_k \quad \forall x \in \Sigma_k = \{x \in \mathbb{R}^N : r_k \le |x| \le R_k\}.$$

(iii) *There exist* $\mu > 1$, $\delta \in]0, p+1[$ *and* $C > 0$ *such that for large* k,

$$\alpha_k > \mu c_3 \beta_k + C\exp\Big(-\delta\,\frac{R_k - r_k}{2}\Big),$$

where $c_3 = c_2/c_1$, $c_2 = \int_{\mathbb{R}^N} z^{p+1}(x)\,dx$, $c_1 = \int_{|x|\le r} z^{p+1}(x)\,dx$.

Then there are infinitely many solutions of (1).

Proof. The proof is in two steps: we first prove that Γ has infinitely many local maxima; then we prove that this oscillatory behavior is preserved when passing from Γ to $J_{|\tilde{Z}}$, which is a perturbation of Γ.

STEP 1. By (i) we have

$$\begin{aligned}\Gamma(P_k) &= \int_{\mathbb{R}^N} z^{p+1}(x-P_k)a\,dx \geq \int_{B(P_k,r)} z^{p+1}(x-P_k)a\,dx \\ &\geq \alpha_k \int_{B(P_k,r)} z^{p+1}(x-P_k)\,dx = \alpha_k \int_{B(0,r)} z^{p+1}(x)\,dx = c_1\alpha_k.\end{aligned}$$

On the other hand, pick any $\xi \in \mathbb{R}^N$ such that $|\xi| = (R_k + r_k)/2$. We get

$$\begin{aligned}\Gamma(\xi) &= \int_{\Sigma_k} z^{p+1}(x-\xi)a\,dx + \int_{|x|<r_k} z^{p+1}(x-\xi)a\,dx + \int_{|x|>R_k} z^{p+1}(x-\xi)a\,dx \\ &\leq \beta_k \int_{\Sigma_k} z^{p+1}(x-\xi)\,dx + C\int_{|x|<r_k} z^{p+1}(x-\xi)\,dx + C\int_{|x|>R_k} z^{p+1}(x-\xi)\,dx.\end{aligned}$$

We estimate $\int_{\Sigma_k} z^{p+1}(x-\xi)\,dx < \int_{\mathbb{R}^N} z^{p+1}(x-\xi)\,dx = c_2$. On the other hand, $\int_{|x|\leq r_k} z^{p+1}(x-\xi)\,dx = \int_{|y+\xi|\leq r_k} z^{p+1}(y)\,dy$. But it is easy to see that $|y+\xi| \leq r_k$ and $|\xi| = (R_k + r_k)/2$ imply $|y| \geq (R_k - r_k)/2$ so that

$$\int_{|y+\xi|\leq r_k} z^{p+1}(y)\,dy \leq \int_{|y|\geq (R_k-r_k)/2} z^{p+1}(y)\,dy = O\left(\exp\left(-\delta_1 \frac{R_k - r_k}{2}\right)\right),$$

where $\delta < \delta_1 < p+1$. In the same way

$$\begin{aligned}\int_{|x|\geq R_k} z^{p+1}(x-\xi)\,dx &= \int_{|x+\xi|\geq R_k} z^{p+1}(x)\,dx \leq \int_{|y|\geq (R_k-r_k)/2} z^{p+1}(y)\,dy \\ &= O\left(\exp\left(-\delta_1 \frac{R_k - r_k}{2}\right)\right).\end{aligned}$$

Hence

$$\Gamma(\xi) \leq c_2\beta_k + C\exp\left(-\delta_1 \frac{R_k - r_k}{2}\right), \quad \text{while} \quad \Gamma(P_k) \geq c_1\alpha_k.$$

These estimates and (iii) prove that

$$\Gamma(P_k) > \Gamma(\xi) \tag{26}$$

when $|\xi| = (R_k + r_k)/2$.

STEP 2. We now want to prove that (26) is stable under perturbations, that is,

$$\Gamma(P_k) + o(\Gamma(P_k)) > \Gamma(\xi) + o(\Gamma(\xi)). \tag{27}$$

For this we have to estimate $\Gamma(P_k)$ from above. We have

$$\begin{aligned}0 < \Gamma(P_k) &= \int_{\mathbb{R}^N} az^{p+1}(x-P_k)\,dx \\ &= \int_{|x|<|P_k|/2} az^{p+1}(x-P_k)\,dx + \int_{|x|\ge|P_k|/2} az^{p+1}(x-P_k)\,dx \\ &\le C \int_{|x+P_k|<|P_k|/2} z^{p+1}(x)\,dx + T\alpha_k c_2 \\ &\le C\exp(-\delta_1|P_k|/2) + T\alpha_k c_2.\end{aligned}$$

Hence we can say that there is a vanishing sequence $\{\varepsilon_k\}_k$ such that

$$\Gamma(P_k) + o(\Gamma(P_k)) \ge c_1\alpha_k(1 - c_3T\varepsilon_k) - \varepsilon_k C\exp(-\delta_1|P_k|/2).$$

On the other hand we trivially get

$$\Gamma(\xi) + o(\Gamma(\xi)) \le \left[c_2\beta_k + C\exp\left(-\delta_1\,\frac{R_k - r_k}{2}\right)\right](1+\varepsilon_k),$$

so to obtain (27) it is enough to prove

$$\begin{aligned}&c_1\alpha_k(1 - c_3T\varepsilon_k) \\ &\quad\ge \left[c_2\beta_k + C\exp\left(-\delta_1\,\frac{R_k - r_k}{2}\right)\right](1+\varepsilon_k) + C\,\varepsilon_k\exp\left(-\delta_1\,\frac{|P_k|}{2}\right).\end{aligned}$$

But we have

$$\begin{aligned}&\left[c_2\beta_k + C\exp\left(-\delta_1\,\frac{R_k - r_k}{2}\right)\right](1+\varepsilon_k) + C\,\varepsilon_k\exp(-\delta_1|P_k|/2) \\ &\qquad\le c_2(1+\varepsilon_k)\beta_k + C\exp\left(-\delta_1\,\frac{R_k - r_k}{2}\right)\end{aligned}$$

(recall that $|P_k|/2 > R_k$ and that C is any positive constant); hence it is enough to prove

$$\alpha_k \ge c_3\,\frac{1+\varepsilon_k}{1 - c_3T\varepsilon_k}\,\beta_k + \frac{C}{1 - c_3T\varepsilon_k}\exp\left(-\delta_1\,\frac{R_k - r_k}{2}\right). \tag{28}$$

But now recall that we assume (iii), that $\varepsilon_k \to 0$ and that $\delta_1 > \delta$; hence it is easy to prove that (28) holds, and so also (27) is proved.

Now we can conclude. Define

$$\phi(\theta) = c_0 - J(z_\theta + w(\theta)) = \frac{1}{p+1}\,\Gamma(\theta) + o(\Gamma(\theta)).$$

From (27) we see that ϕ has infinitely many local maximum points. Indeed, we have

$$\phi(P_k) > \phi(\xi) > 0 \qquad \left(\text{when } |\xi| = \frac{R_k + r_k}{2}\right),$$

$$|P_k| > R_k > \frac{R_k + r_k}{2} \quad \text{and} \quad \phi(\theta) \to 0 \quad \text{as } |\theta| \to \infty.$$

Hence $J_{|\widetilde{Z}}$ has infinitely many local maximum points, which gives rise to infinitely many solutions of (1). Notice that the solutions $\{u_k\}$ we find have the following form:

$$u_k = z_{\theta_k} + w(\theta_k), \tag{29}$$

where the θ_k's are the critical points of $J_{|\widetilde{Z}}$. By construction $\{\theta_k\}_k$ is not bounded. Passing to a subsequence we can assume that $|\theta_k - \theta_{k-1}| > 1$ and $|\theta_k| \to \infty$, hence $|w(\theta_k)| \to 0$. This implies that all the u_k's given by (29) are distinct, for large k's. ■

We have a similar result for the case $a \le 0$. We do not give the proof, which is the same as that of the previous theorem.

THEOREM 3.3. *Assume that $a \le 0$ and that the following hypotheses hold.*

(i) *There are $T, r > 0$ and sequences $\{P_k\}_k \subset \mathbb{R}^N$ and $\{\alpha_k\}_k \subset \mathbb{R}$ such that $\alpha_k < 0$, $|P_k| \to +\infty$ and*

$$a(x) \le \alpha_k \quad \forall x \in B(P_k, r), \qquad a(x) \ge T\alpha_k \quad \forall |x| \ge |P_k|/2.$$

(ii) *There are sequences $\{r_k\}_k$, $\{R_k\}_k$, $\{\beta_k\}_k$ such that $r_k, R_k > 0$, $\beta_k \le 0$, $r_k \to +\infty$, $R_k - r_k \to +\infty$, $R_k < |P_k|/2$ and*

$$a(x) \ge \beta_k \quad \forall x \in \Sigma_k = \{x \in \mathbb{R}^N : r_k \le |x| \le R_k\}.$$

(iii) *There are $\mu > 1$, $\delta \in \,]0, p+1[$ and $C > 0$ such that for large k,*

$$\alpha_k < \mu c_0 \beta_k + C \exp\left(-\delta \frac{R_k - r_k}{2}\right).$$

Then there are infinitely many solutions of (1).

EXAMPLE. Let us give a more concrete example of a function b satisfying the assumptions of Theorem 3.3. Set $b_\infty = 1$ and consider $b_0 \in C(\mathbb{R}^N)$ such that $b_0(x) = 1 - 1/|x|^\alpha$ if $|x| \ge 1$. We do not require anything about b_0 in B_1, it can also assume negative values there. Let $\varphi \in C_0(B_2)$ be such that $0 \le \varphi \le 1$ and $\varphi(x) = 1$ for all $x \in B_1$. Define $P_k = 4e^k e_1$, $\alpha_k = -1/k^\alpha$, $\varphi_k(x) = \varphi(x - P_k)$. Then we consider the function

$$b(x) = b_0(x) + \sum_{k=2}^{\infty} \alpha_k \varphi_k(x).$$

It is easy to see that the hypotheses of Theorem 3.3 are satisfied if we set, for $k \ge 3$, $r_k = e^k + 3$, $R_k = e^{k+1} - 3$, $\beta_k = -2/|r_k|^\alpha = -2/|e^k + 3|^\alpha$, $r = 2$,

$T = 3$. We then obtain infinitely many solutions for (1). Notice that in this case $(b-b_\infty)^-$ does not decay exponentially, so the results of [6, 7] cannot be applied, and that we allow b to assume negative values (in B_1). By a similar construction it would be simple to prove some density results analogous to those of [1].

4. Existence results for a with changing sign. The previous results can be extended to the case in which a changes sign. However, we need to make some more precise assumptions on the asymptotic behavior of a. We begin by recalling a known result (Proposition 1.2 in [6]).

LEMMA 4.1. *Let $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\psi \in C(\mathbb{R}^N)$ be radially symmetric and satisfy, for some $\alpha \ge 0$, $\beta \ge 0$, $\gamma \in \mathbb{R}$,*

$$\begin{gathered} \varphi(x)\exp(\alpha|x|)|x|^\beta \to \gamma \quad \text{as } |x| \to \infty, \\ \int_{\mathbb{R}^N} |\psi(x)|\exp(\alpha|x|)(1+|x|^\beta)\,dx < \infty. \end{gathered} \tag{30}$$

Then

$$\int_{\mathbb{R}^N} \varphi(x+y)\psi(x)\,dx\exp(\alpha|y|)|y|^\beta \to \gamma \int_{\mathbb{R}^N} \psi(x)\exp(-\alpha|x_1|)\,dx \qquad \text{as } |y| \to \infty.$$

We now need to prove a result analogous to Lemma 3.1. For this we use the previous lemma, and, as a is not radial, we introduce some radial functions linked to it. Set

$$a_1(x) = \inf_{|y|=|x|} a(y), \quad a_2(x) = \sup_{|y|=|x|} a(y),$$

and denote by a_i^+, a_i^- the positive and negative parts of a_i. We then assume that there are $0 \le \alpha < p+1$, $\beta \ge 0$ and $\gamma_i^+, \gamma_i^- \in [0, +\infty[$ $(i = 1, 2)$ such that

$$|a_i^\pm(x)|\exp(\alpha|x|)|x|^\beta \to \gamma_i^\pm. \tag{31}$$

We obtain the following lemma.

LEMMA 4.2. *Assume that* (31) *holds and also*

$$\gamma_1^+ > \gamma_1^- \quad \text{or} \quad \gamma_2^+ < \gamma_2^-. \tag{32}$$

Then the same conclusion of Lemma 3.1 *holds*, *that is*,

$$\Big|\int_{\mathbb{R}^N} az_\theta^p w\,dx\Big| \le o(1)\Big|\int_{\mathbb{R}^N} az_\theta^{p+1}\,dx\Big|. \tag{33}$$

Proof. From (25) we have

$$\Big|\int_{\mathbb{R}^N} az_\theta^p w\,dx\Big| = o(1)\int_{\mathbb{R}^N} |a|z_\theta^{p+1}\,dx.$$

We apply Lemma 4.1 with $\varphi = a_i^{\pm}$, $\psi = z^{p+1}$ to obtain

$$\int_{\mathbb{R}^N} z^{p+1}(x+\theta) a_i^{\pm}\, dx \exp(\alpha|\theta|)|\theta|^{\beta} \to \gamma_i^{\pm} \int_{\mathbb{R}^N} a_i^{\pm} \exp(-\alpha|x_1|)\, dx \quad \text{as } |\theta| \to \infty,$$

which implies

$$\int_{\mathbb{R}^N} z^{p+1}(x+\theta)|a_i(x)|\, dx \exp(\alpha|\theta|)|\theta|^{\beta} \to (\gamma_i^{+} + \gamma_i^{-}) \int_{\mathbb{R}^N} |a_i(x)| \exp(-\alpha x_1)\, dx \quad \text{as } |\theta| \to \infty. \tag{34}$$

and

$$\int_{\mathbb{R}^N} z^{p+1}(x+\theta) a_i(x)\, dx \exp(\alpha|\theta|)|\theta|^{\beta} \to (\gamma_i^{+} - \gamma_i^{-}) \int_{\mathbb{R}^N} a_i(x) \exp(-\alpha x_1)\, dx \quad \text{as } |\theta| \to \infty. \tag{35}$$

As $a_1 \le a \le a_2$, (32) and (35) imply that there are $0 < c_1 \le c_2$ such that

$$c_1 \exp(-\alpha|\theta|)|\theta|^{-\beta} \le \Big| \int_{\mathbb{R}^N} z^{p+1}(x+\theta) a(x)\, dx \Big| \le c_2 \exp(-\alpha|\theta|)|\theta|^{-\beta}. \tag{36}$$

On the other hand, $|a(x)| \le \max\{|a_1(x)|, |a_2(x)|\}$, hence (34) implies that

$$\int_{\mathbb{R}^N} z^{p+1}(x+\theta)|a(x)|\, dx = O(\exp(-\alpha|\theta|)|\theta|^{-\beta}).$$

From this and from (36) we easily obtain (33). ∎

We then get the following theorem:

THEOREM 4.1. *Under the hypotheses of Lemma* 4.2, *suppose moreover that*:

(i) *There are sequences* $\{P_k\}_k \subset \mathbb{R}^N$ *and* $\{\varrho_k\}, \{\alpha_k\}_k \subset \mathbb{R}$ *such that* $\alpha_k > 0$, $\varrho_k, |P_k| \to \infty$ *and*

$$a(x) \ge \alpha_k \quad \forall x \in B(P_k, \varrho_k).$$

(ii) *There are sequences* $\{r_k\}_k$, $\{R_k\}_k$, $\{\beta_k\}_k$ *such that* $r_k, R_k > 0$, $\beta_k \le 0$, $r_k \to +\infty$, $R_k - r_k \to +\infty$ *and*

$$a(x) \le \beta_k \quad \forall x \in \Sigma_k = \{x \in \mathbb{R}^N : r_k \le |x| \le R_k\}.$$

(iii) *There are* $\mu > 1$, $\delta \in \,]0, p+1[$ *and* $C > 0$ *such that*, *for large* k,

$$\alpha_k > \mu|\beta_k| + C \exp\left(-\delta\, \frac{R_k - r_k}{2}\right).$$

Then there are infinitely many solutions of (1).

Proof. Notice that we can always assume, passing to subsequences if necessary, that $\varrho_k > R_k$ for all k. Using arguments similar to those for the lemmas of the previous section, it is not difficult to see that

$$\Gamma(P_k) + o(\Gamma(P_k)) \geq \alpha_k \sigma_k + \exp(-\delta_1 \varrho_k),$$

where $\delta_1 > \delta$ and $\sigma_k \to \int_{\mathbb{R}^N} z^{p+1}\,dx$. We also obtain, for any ξ with $|\xi| = (r_k + R_k)/2$,

$$\Gamma(\xi) + o(\Gamma(\xi)) \leq \beta_k \sigma_k + C \exp\biggl(-\delta_1 \frac{r_k + R_k}{2}\biggr).$$

The result can be easily derived from these estimates. ■

RMARK 4.1. Notice that if a is radial, then $a_1 = a_2$ and $\gamma_1^+ = \gamma_2^+$, $\gamma_1^- = \gamma_2^-$. Hence in this case the hypotheses (32) just mean that $\gamma_1^+ \neq \gamma_1^-$. Roughly speaking, this forbids a to have "symmetric" oscillations. Also notice that (31) in any case allows a to have a polynomial decay at infinity.

References

[1] F. Alessio, P. Caldiroli and P. Montecchiari, *Genericity of the existence of infinitely many solutions for a class of semilinear elliptic equations in* $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa 27 (1998), 47–68.

[2] A. Ambrosetti and M. Badiale, *Homoclinics: Poincaré–Melnikov type results via a variational approach*, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), 233–252.

[3] —, —, *Variational perturbative methods and bifurcation of bound states from the essential spectrum*, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 1131–1161.

[4] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. 140 (1997), 285–300.

[5] M. Badiale and G. Citti, *Concentration compactness principle and quasilinear elliptic equations in* $\mathbb{R}^n$, Comm. Partial Differential Equations 16 (1991), 1795–1818.

[6] A. Bahri and Y. Y. Li, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^N$, Rev. Mat. Iberoamericana 6 (1990), 1–15.

[7] A. Bahri and P. L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997), 365–413.

[8] V. Benci and G. Cerami, *Positive solutions of semilinear elliptic equations in exterior domains*, Arch. Rational Mech. Anal. 99 (1987), 283–300.

[9] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I. Existence of ground state*, ibid. 82 (1983), 315–345.

[10] —, —, *Nonlinear scalar field equations, II. Existence of infinitely many solutions*, ibid. 82 (1983), 347–375.

[11] D. M. Cao, *Positive solutions and bifurcation from the essential spectrum of a semilinear elliptic equation in* $\mathbb{R}^N$, Nonlinear Anal. 15 (1990), 1045–1052.

[12] —, *Existence of positive solutions of semilinear elliptic equations in* $\mathbb{R}^N$, Differential Integral Equations 6 (1993), 655–661.

[13] J. Chabrowski, *Variational Methods for Potential Operator Equations*, de Gruyter, Berlin, 1997.

[14] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. 105 (1989), 243–266.

[15] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 109–145.
[16] —, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 2*, ibid. 1 (1984), 223–283.
[17] —, *The concentration-compactness principle in the calculus of variations: the limit case, part 1*, Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.
[18] —, *The concentration-compactness principle in the calculus of variations: the limit case, part 2*, ibid. 1 (1985), no. 2, 46–121.
[19] Y. J. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. 131 (1990), 223–253.

Dipartimento di Matematica
Università di Torino
via Carlo Alberto 10
10123 Torino, Italy
E-mail: marino.badiale@unito.it

Reçu par la Rédaction le 1.9.2001 (1282)