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Linear differential polynomials sharing the same 1-points with weight two

by Indrajit Lahiri (Kalyani)

Abstract. We prove a uniqueness theorem for meromorphic functions involving differential polynomials which improves some previous results and provides a better answer to a question of C. C. Yang.

1. Introduction and definitions. Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g *share the value a CM* (counting multiplicities) and if we do not consider the multiplicities, f and g are said to *share the value a IM* (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as those are available in [2].

In [9] C. C. Yang asked: *What can be said if two nonconstant entire functions f, g share the value 0 CM and their first derivatives share the value 1 CM?*

A number of authors have worked on this question of Yang (e.g. [3, 6, 7, 10, 11]). To answer the question of Yang, K. Shibazaki [7] proved the following result.

Theorem A. *Let f and g be two entire functions of finite order. If f' and g' share the value 1 CM with $\delta(0; f) > 0$ and 0 being lacunary for g then either $f \equiv g$ or $f'g' \equiv 1$.*

Improving Theorem A, H. X. Yi [12] obtained the following theorem.

Theorem B. *Let f, g be two entire functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

For meromorphic functions H. X. Yi and C. C. Yang [13] proved the following result.

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THEOREM C. *Let f and g be two meromorphic functions such that $\Theta(\infty; f) = \Theta(\infty; g) = 1$. If $f^{(n)}$ and $g^{(n)}$ share the value 1 CM with $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

In [3] the following question was asked: *What can be said if two linear differential polynomials generated by two meromorphic functions f and g share the value 1 CM?*

We denote by $\Psi(D)$ a linear differential operator with constant coefficients of the form

$$\Psi(D) = \sum_{i=1}^{p} \alpha_i D^i,$$

where $D = d/dz$.

Also we denote by $N_k(r, a; f)$ the counting function of a-points of f where an a-point of multiplicity μ is counted μ times if $\mu \le k$ and k times if $\mu > k$, where k is a positive integer. We put

$$\delta_k(a; f) = 1 - \limsup_{r\to\infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

Clearly $\delta(a; f) \le \delta_k(a; f) \le \delta_{k-1}(a; f) \le \ldots \le \delta_1(a; f) = \Theta(a; f)$.

In [3] the following two theorems were proved.

THEOREM D. *Let f and g be two meromorphic functions such that*

(i) *$\Psi(D)f$, $\Psi(D)g$ are nonconstant and share 1 CM, and*

(ii)
$$\frac{\sum_{a\ne\infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a\ne\infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{4(1 - \Theta(\infty; f))}{\sum_{a\ne\infty} \delta_p(a; f)} + \frac{4(1 - \Theta(\infty; g))}{\sum_{a\ne\infty} \delta_p(a; g)},$$

where $\sum_{a\ne\infty}\delta_p(a; f) > 0$ and $\sum_{a\ne\infty}\delta_p(a; g) > 0$. Then either $[\Psi(D)f][\Psi(D)g] \equiv 1$ or $f - g \equiv s$ where $s = s(z)$ is a solution of the differential equation $\Psi(D)w = 0$.

THEOREM E. *If f and g are of finite order then Theorem* D *still holds if condition* (ii) *is replaced by the following weaker one*:

$$\frac{\sum_{a\ne\infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a\ne\infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{2(1 - \Theta(\infty; f))}{\sum_{a\ne\infty} \delta_p(a; f)} + \frac{2(1 - \Theta(\infty; g))}{\sum_{a\ne\infty} \delta_p(a; g)},$$

where $\sum_{a\ne\infty} \delta_p(a; f) > 0$ and $\sum_{a\ne\infty} \delta_p(a; g) > 0$.

H. X. Yi [10] also answered the question of Yang and proved the following result.

Theorem F. *Let f and g be two nonconstant entire functions. Assume that f, g share 0 CM and $f^{(n)}$, $g^{(n)}$ share 1 CM, where n is a nonnegative integer. If $\delta(0;f) > 1/2$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

As an application of Theorem D, in [3] the following answer to the question of Yang was given.

Theorem G. *Let f and g be two nonconstant meromorphic functions with $\Theta(\infty;f) = \Theta(\infty;g) = 1$. Suppose that $f^{(n)}$, $g^{(n)}$ $(n \geq 1)$ share 1 CM and f, g share a value b $(\neq \infty)$ IM. If $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

The following example shows that in Theorems D and E sharing the value 1 cannot be relaxed from CM to IM.

Example 1. Let $f = -ie^z$, $g = 2^{-p}e^{2z} - 2ie^z$ and $\Psi(D) = D^p$. Then $\Psi(D)f, \Psi(D)g$ share the value 1 IM and $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) = 3/2$ but neither $f \equiv g + Q$ nor $[\Psi(D)f][\Psi(D)g] \equiv 1$ where Q is a polynomial of degree at most $p-1$.

Now one may ask the following question: *Is it possible in any way to relax the nature of sharing the value* 1 *in Theorems* D *and* E*?*

The purpose of the paper is to study this problem. We shall not only relax the nature of sharing the value 1 but also weaken the condition on deficiencies. To this end we consider a gradation of sharing of values which measures how close a shared value is to being shared IM or being shared CM and is called weighted sharing of values as introduced in [4, 5].

Definition 1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with *weight* k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity m $(\leq k)$ if and only if z_0 is a zero of $g-a$ with multiplicity m $(\leq k)$, and z_0 is a zero of $f-a$ with multiplicity m $(> k)$ if and only if z_0 is a zero of $g-a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write "f, g share (a,k)" to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 2. We denote by $N(r,a;f\,|\!=\!1)$ the counting function of simple a-points of f.

Definition 3. If s is a positive integer, we denote by $\overline{N}(r,a;f\,|\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.

Definition 4. Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the counting function of those a-points of f whose multiplicities are not equal to multiplicities of the corresponding a-points of g, where each a-point is counted only once.

Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$.

Definition 5 (cf. [1]). For a meromorphic function f we put

$$T_0(r,f)=\int_1^r\frac{T(t,f)}{t}\,dt,\qquad N_0(r,a;f)=\int_1^r\frac{N(t,a;f)}{t}\,dt,$$

$$N_k^0(r,a;f)=\int_1^r\frac{N_k(t,a;f)}{t}\,dt,\qquad m_0(r,f)=\int_1^r\frac{m(t,f)}{t}\,dt,$$

$$S_0(r,f)=\int_1^r\frac{S(t,f)}{t}\,dt.$$

Definition 6. If f is a meromorphic function, we put, for $a\in\mathbb{C}\cup\{\infty\}$,

$$\delta_0(a;f)=1-\limsup_{r\to\infty}\frac{N_0(r,a;f)}{T_0(r,f)},$$

$$\Theta_0(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}_0(r,a;f)}{T_0(r,f)},$$

$$\delta_k^0(a;f)=1-\limsup_{r\to\infty}\frac{N_k^0(r,a;f)}{T_0(r,f)}.$$

2. Lemmas. In this section we present some lemmas which will be needed in what follows. Let f, g be two nonconstant meromorphic functions and we put

$$h=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

Lemma 1. *If f, g share $(1,1)$ and $h\not\equiv 0$ then*

(i) $N(r,1;f\,|=1)\leq N(r,h)+S(r,f)+S(r,g)$,

(ii) $N(r,1;g\,|=1)\leq N(r,h)+S(r,f)+S(r,g)$.

Proof. Since f, g share $(1,1)$, it follows that a simple 1-point of f is a simple 1-point of g and conversely. Let z_0 be a simple 1-point of f and g. Then by a simple calculation we see that in some neighbourhood of z_0,

$$h=(z-z_0)\phi(z),$$

where ϕ is analytic at z_0.

Hence by the first fundamental theorem and the Milloux theorem [2, p. 47] we get

$$N(r,1;f\,|\,{=}1)\le N(r,0;h)\le N(r,h)+S(r,f)+S(r,g),$$

which is (i).

Now (ii) follows from (i) because $N(r,1;f\,|\,{=}1)\equiv N(r,1;g\,|\,{=}1)$. This proves the lemma. ∎

LEMMA 2. *Let f, g share $(1,0)$ and $h\not\equiv 0$. Then for any number b $(\neq 0,1,\infty)$,*

$$\begin{aligned}N(r,h)\le{}&\overline{N}(r,\infty;f\,|\,{\ge}2)+\overline{N}(r,0;f\,|\,{\ge}2)+\overline{N}(r,b;f\,|\,{\ge}2)\\&+\overline{N}(r,\infty;g\,|\,{\ge}2)+\overline{N}(r,0;g\,|\,{\ge}2)+\overline{N}_*(r,1;f,g)\\&+\overline{N}_\oplus(r,0;f')+\overline{N}_\otimes(r,0;g'),\end{aligned}$$

where $\overline{N}_\oplus(r,0;f')$ is the reduced counting function of those zeros of f' which are not zeros of $f(f-1)(f-b)$, and $\overline{N}_\otimes(r,0;g')$ is the reduced counting function of those zeros of g' which are not zeros of $g(g-1)$.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f, g; (ii) multiple poles of f, g; (iii) zeros of $f-1$, $g-1$; (iv) multiple zeros of $f-b$; (v) zeros of f' which are not zeros of $f(f-1)(f-b)$; (vi) zeros of g' which are not zeros of $g(g-1)$.

Let z_0 be a zero of $f-1$ with multiplicity m (≥ 1) and of $g-1$ with multiplicity n (≥ 1). Then in some neighbourhood of z_0 we get

$$h=\frac{(n-m)\psi}{z-z_0}+\phi,$$

where ϕ,ψ are analytic at z_0 and $\psi(z_0)\neq 0$.

This shows that if $m=n$ then z_0 is not a pole of h and if $m\neq n$ then z_0 is a simple pole of h. Since all the poles of h are simple, the lemma is proved. ∎

LEMMA 3. *If f, g share $(1,2)$ then*

$$\begin{aligned}&N_\otimes(r,0;g')+\overline{N}(r,1;g\,|\,{\ge}2)+\overline{N}_*(r,1;f,g)\\&\qquad\le\overline{N}(r,\infty;g)+\overline{N}(r,0;g)+S(r,g),\end{aligned}$$

where $N_\otimes(r,0;g')$ is the counting function of those zeros of g' which are not zeros of $g(g-1)$.

Proof. Since f, g share $(1,2)$, it follows that $\overline{N}_*(r,1;f,g)\le\overline{N}(r,1;g\,|\,{\ge}3)$. So remembering the definition of $N_\otimes(r,0;g')$ we get

$$
\begin{aligned}
(1)\qquad & N_{\otimes}(r,0;g') + \overline{N}(r,1;g\,|\,{\geq}2) + \overline{N}_{*}(r,1;f,g) \\
&\qquad + N(r,0;g) - \overline{N}(r,0;g) \\
&\leq N_{\otimes}(r,0;g') + \overline{N}(r,1;g\,|\,{\geq}2) + \overline{N}(r,1;g\,|\,{\geq}3) \\
&\qquad + N(r,0;g) - \overline{N}(r,0;g) \\
&\leq N(r,0;g').
\end{aligned}
$$

By the first fundamental theorem and the Milloux theorem [2, p. 55] we get

$$
\begin{aligned}
(2)\qquad N(r,0;g') &\leq N(r,0;g'/g) + N(r,0;g) - \overline{N}(r,0;g) \\
&\leq N(r,g'/g) + N(r,0;g) - \overline{N}(r,0;g) + S(r,g) \\
&= \overline{N}(r,\infty;g) + \overline{N}(r,0;g) + N(r,0;g) - \overline{N}(r,0;g) + S(r,g) \\
&= \overline{N}(r,\infty;g) + N(r,0;g) + S(r,g).
\end{aligned}
$$

Now the lemma follows from (1) and (2). ∎

LEMMA 4 (see [1]). $\lim_{r\to\infty} S_0(r,f)/T_0(r,f) = 0$ *through all values of* r.

LEMMA 5 (see [3]). *For* $a \in \mathbb{C}\cup\{\infty\}$, $\delta(a;f) \leq \delta_0(a;f)$, $\Theta(a;f) \leq \Theta_0(a;f)$ *and* $\delta_k(a;f) \leq \delta_k^0(a;f)$.

LEMMA 6 (see [3]).

$$
\text{(i)}\ \liminf_{r\to\infty} \frac{T_0(r,\Psi(D)f)}{T_0(r,f)} \geq \sum_{a\neq\infty} \delta_p^0(a;f),
$$

$$
\text{(ii)}\ \delta_0(0;\Psi(D)f) \geq \frac{\sum_{a\neq\infty}\delta_0(a;f)}{1+p(1-\Theta_0(\infty;f))}.
$$

LEMMA 7 (see [3]). *If* $\sum_{a\neq\infty}\delta_p^0(a;f) > 0$ *then*

$$
\Theta_0(\infty;\Psi(D)f) \geq 1 - \frac{1-\Theta_0(\infty;f)}{\sum_{a\neq\infty}\delta_p^0(a;f)}.
$$

LEMMA 8 (see [8]). *If* f *is transcendental then* $\lim_{r\to\infty} T_0(r,f)/(\log r)^2 = \infty$ *through all values of* r.

3. The main result. In this section we discuss the main result of the paper.

THEOREM 1. *Let* f, g *be two meromorphic functions such that*

(i) $\Psi(D)f$, $\Psi(D)g$ *are transcendental and share* $(1,2)$ *and*

$$\text{(ii)}\qquad \frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))}+\frac{\sum_{a\neq\infty}\delta(a;g)}{1+p(1-\Theta(\infty;g))}$$
$$+\min\{\delta_2(b;\Psi(D)f),\delta_2(b;\Psi(D)g)\}$$
$$>1+\frac{2(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta_p(a;f)}+\frac{2(1-\Theta(\infty;g))}{\sum_{a\neq\infty}\delta_p(a;g)}$$

for some $b\neq 0,1,\infty,1/2,2,-\omega,-\omega^2$, *with* $\sum_{a\neq\infty}\delta_p(a;f)>0$, $\sum_{a\neq\infty}\delta_p(a;g)>0$ *and* ω *being the imaginary cube root of unity.*

Then either $[\Psi(D)f][\Psi(D)g]\equiv 1$ *or* $f-g\equiv s$, *where* $s=s(z)$ *is a solution of the differential equation* $\Psi(D)w=0$.

The following example shows that Theorem 1 is sharp.

Example 2. Let $f=\frac{1}{2}e^z(e^z-1)$, $g=\frac{1}{2}e^{-z}\bigl(\frac{1}{2}-\frac{1}{5}e^{-z}\bigr)$ and $\Psi(D)=D^2-3D$. Then $\Psi(D)f=e^z(1-e^z)$, $\Psi(D)g=e^{-z}(1-e^{-z})$, $\sum_{a\neq\infty}\delta(a;f)=\sum_{a\neq\infty}\delta(a;g)=1/2$, $\Theta(\infty;f)=\Theta(\infty;g)=1$, $\delta_2(b;\Psi(D)f)=\delta_2(b;\Psi(D)g)=0$ for $b\neq 0,\infty$ and $\Psi(D)f,\Psi(D)g$ share $(1,2)$. It is easily seen that neither $[\Psi(D)f][\Psi(D)g]\equiv 1$ nor $f-g\equiv c_1-c_2e^{3z}$ for any constants c_1 and c_2.

Proof of Theorem 1. Let $F=\Psi(D)f$ and $G=\Psi(D)g$. Then in view of Lemmas 5–7 condition (ii) implies

$$(3)\qquad \delta_0(0;F)+\delta_0(0;G)+2\Theta_0(\infty;F)+2\Theta_0(\infty;G)$$
$$+\min\{\delta_2^0(b;F),\delta_2^0(b;G)\}>5.$$

We put

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

Suppose $H\not\equiv 0$. Then by Lemmas 1–3 we get

$$(4)\qquad N(r,1;F\,|\,{=}1)\le \overline{N}(r,\infty;F\,|\,{\ge}2)+\overline{N}(r,0;F\,|\,{\ge}2)+\overline{N}(r,b;F\,|\,{\ge}2)$$
$$+\overline{N}(r,\infty;G\,|\,{\ge}2)+\overline{N}(r,0;G\,|\,{\ge}2)+\overline{N}_{\oplus}(r,0;F')$$
$$+\overline{N}(r,\infty;G)+\overline{N}(r,0;G)-\overline{N}(r,1;G\,|\,{\ge}2)$$
$$+S(r,F)+S(r,G).$$

By the second fundamental theorem we get

$$(5)\qquad 2T(r,F)\le\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+\overline{N}(r,b;F)$$
$$+\overline{N}(r,0;F)-N_{\oplus}(r,0;F')+S(r,F),$$

where $N_{\oplus}(r,0;F')$ is the counting function of those zeros of F' which are not zeros of $F(F-1)(F-b)$.

Since F, G share $(1,2)$, we see that

$$(6)\qquad \overline{N}(r,1;F)=\overline{N}(r,1;F\,|\,{=}1)+\overline{N}(r,1;F\,|\,{\ge}2)$$
$$=\overline{N}(r,1;F\,|\,{=}1)+\overline{N}(r,1;G\,|\,\ge 2).$$

Since $N_2(r,\infty;F) \le 2\overline{N}(r,\infty;F)$ and $N_2(r,\infty;G) \le 2\overline{N}(r,\infty;G)$, we get from (4)–(6) on integration

$$(7)\qquad 2T_0(r,F) \le N_2^0(r,0;F) + N_2^0(r,b;F) + N_2^0(r,0;G) + 2\overline{N}_0(r,\infty;F) + 2\overline{N}_0(r,\infty;G) + S_0(r,F) + S_0(r,G).$$

Similarly we obtain

$$(8)\qquad 2T_0(r,G) \le N_2^0(r,0;F) + N_2^0(r,b;G) + N_2^0(r,0;G) + 2\overline{N}_0(r,\infty;F) + 2\overline{N}_0(r,\infty;G) + S_0(r,F) + S_0(r,G).$$

From (7) and (8) we get

$$(9)\qquad 2T_0(r) \le N_2^0(r,0;F) + N_2^0(r,0;G) + N_2^0(r,b) + 2\overline{N}_0(r,\infty;F) + 2\overline{N}_0(r,\infty;G) + S_0(r,F) + S_0(r,G),$$

where $T_0(r) = \max\{T_0(r,F), T_0(r,G)\}$ and $N_2^0(r,b) = \max\{N_2^0(r,b;F), N_2^0(r,b;G)\}$.

Since (9) contradicts (3), it follows that $H \equiv 0$. Then

$$(10)\qquad F = \frac{AG+B}{CG+D},$$

where A, B, C, D are complex numbers such that $AD - BC \ne 0$.

In view of (10) we get

$$(11)\qquad T_0(r,F) = T_0(r,G) + O(\log r).$$

Now we consider the following cases.

Case 1: $AC \ne 0$. Then

$$(12)\qquad F - \frac{A}{C} = \frac{B - \frac{AD}{C}}{CG+D}.$$

Subcase 1.1: $A/C \ne b$. Then by the second fundamental theorem we get on integration

$$2T_0(r,F) \le \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,A/C;F) + \overline{N}_0(r,b;F) + S_0(r,F)$$
$$= \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,b;F) + \overline{N}_0(r,\infty;G) + S_0(r,F),$$

which implies (9) in view of (11) and Lemma 8 and finally contradicts (3).

Subcase 1.2: $A/C = b$. Also we suppose that $BD \ne 0$. Then $B/D \ne b$ because $AD - BC \ne 0$. So by the second fundamental theorem we get on integration

$$2T_0(r,F) \le \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,b;F) + \overline{N}_0(r,B/D;F) + S_0(r,F)$$
$$= \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,b;F) + \overline{N}_0(r,0;G) + S_0(r,F),$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

Let $B = 0$. Then $D \neq 0$ because F is nonconstant. Now from (12) we get

$$F - b = \frac{-b}{\alpha G + 1}, \tag{13}$$

where $\alpha = C/D$.

Let 1 be a Picard exceptional value (e.v.P.) of F and so of G. Then by the second fundamental theorem we get on integration

$$2T_0(r,F) \leq \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,b;F) + S_0(r,F),$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

Let 1 be not an e.v.P. of F and G. Then from (13) we get $\alpha = \frac{1}{b-1}$ so that

$$F = \frac{bG}{(b-1)+G}.$$

Since $b \neq 1/2$, by the second fundamental theorem we get on integration

$$\begin{aligned} 2T_0(r,G) & \\ &\leq \overline{N}_0(r,\infty;G) + \overline{N}_0(r,0;G) + \overline{N}_0(r,b;G) + \overline{N}_0(r,1-b;G) + S_0(r,G) \\ &= \overline{N}_0(r,\infty;G) + \overline{N}_0(r,0;G) + \overline{N}_0(r,b;G) + \overline{N}_0(r,\infty;F) + S_0(r,G), \end{aligned}$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

Let $B \neq 0$, $D = 0$. Then from (12) we obtain

$$F = b + \frac{\beta}{G}, \tag{14}$$

where $\beta = B/C$.

If 1 is an e.v.P. of F and so of G, by the second fundamental theorem we get on integration

$$2T_0(r,F) \leq \overline{N}_0(r,\infty;F) + \overline{N}_0(r,0;F) + \overline{N}_0(r,b;F) + S_0(r,F),$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

Suppose 1 is not an e.v.P. of F and G. Then from (14) we get $\beta = 1 - b$ so that

$$F = b + \frac{1-b}{G}.$$

Since $b \neq -\omega, -\omega^2$, by the second fundamental theorem we get on integration

$$\begin{aligned} 2T_0(r,G) & \\ &\leq \overline{N}_0(r,\infty;G) + \overline{N}_0(r,0;G) + \overline{N}_0(r,b;G) + \overline{N}_0(r,1-1/b;G) + S_0(r,G) \\ &= \overline{N}_0(r,\infty;G) + \overline{N}_0(r,0;G) + \overline{N}_0(r,b;G) + \overline{N}_0(r,0;F) + S_0(r,G), \end{aligned}$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

Case 2: $AC = 0$. Since F is nonconstant, it follows that A and C are not simultaneously zero.

Subcase 2.1: $A = 0$ and $C \neq 0$. Then $B \neq 0$ and from (10) we get

$$\frac{1}{F} = \alpha G + \beta, \tag{15}$$

where $\alpha = C/B$ and $\beta = D/B$.

If 1 is an e.v.P. of F and G, by the second fundamental theorem we get on integration

$$2T_0(r, F) \leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + S_0(r, F),$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

Suppose 1 is not an e.v.P. of F and G. Then from (15) we get $\alpha + \beta = 1$ so that

$$\frac{1}{F} = \alpha G + 1 - \alpha.$$

If $\alpha \neq 1, 1 - 1/b$, by the second fundamental theorem we get on integration

$$\begin{aligned} &2T_0(r, F) \\ &\leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + \overline{N}_0(r, 1/(1 - \alpha); F) + S_0(r, F) \\ &= \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + \overline{N}_0(r, 0; G) + S_0(r, F), \end{aligned}$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

If $\alpha = 1$ then $FG \equiv 1$, i.e. $[\Psi(D)f][\Psi(D)g] \equiv 1$.

If $\alpha = 1 - 1/b$ then

$$F = \frac{b}{1 + (b - 1)G}.$$

Since $b \neq -\omega, -\omega^2$, by the second fundamental theorem we get on integration

$$\begin{aligned} &2T_0(r, G) \\ &\leq \overline{N}_0(r, \infty; G) + \overline{N}_0(r, 0; G) + \overline{N}_0(r, b; G) + \overline{N}_0(r, 1/(1 - b); G) + S_0(r, G) \\ &= \overline{N}_0(r, \infty; G) + \overline{N}_0(r, 0; G) + \overline{N}_0(r, b; G) + \overline{N}_0(r, \infty; F) + S_0(r, G), \end{aligned}$$

which by (11) and Lemma 8 implies (9) and so contradicts (3).

Subcase 2.2: $A \neq 0$ and $C = 0$. Then $D \neq 0$ and from (10) we get

$$F = \alpha G + \beta, \tag{16}$$

where $\alpha = A/D$, $\beta = B/D$.

If 1 is an e.v.P. of F and G, by the second fundamental theorem we get on integration

$$2T_0(r, F) \leq \overline{N}_0(r, \infty; F) + \overline{N}_0(r, 0; F) + \overline{N}_0(r, b; F) + S_0(r, F),$$

which implies (9) by (11) and Lemma 8 and so contradicts (3).

Suppose 1 is not an e.v.P. of F and G. Then from (16) we get $\alpha+\beta=1$ and so

$$F=\alpha G+1-\alpha.$$

If $\alpha\neq 1,1-b$, by the second fundamental theorem we get on integration

$$\begin{aligned}2T_0(r,F)&\\ &\le \overline{N}_0(r,\infty;F)+\overline{N}_0(r,0;F)+\overline{N}_0(r,b;F)+\overline{N}_0(r,1-\alpha;F)+S_0(r,F)\\ &=\overline{N}_0(r,\infty;F)+\overline{N}_0(r,0;F)+\overline{N}_0(r,b;F)+\overline{N}_0(r,0;G)+S_0(r,F),\end{aligned}$$

which implies (9) in view of (11) and Lemma 8 and so contradicts (3).

If $\alpha=1$ then $F\equiv G$ and so $f-g\equiv s$, where $s=s(z)$ is a solution of the differential equation $\Psi(D)w=0$.

If $\alpha=1-b$ then

$$F=(1-b)G+b.$$

Since $b\neq 2$, by the second fundamental theorem we get on integration

$$\begin{aligned}2T_0(r,G)&\\ &\le \overline{N}_0(r,\infty;G)+\overline{N}_0(r,0;G)+\overline{N}_0(r,b;G)+\overline{N}_0(r,b/(b-1);G)+S_0(r,G)\\ &=\overline{N}_0(r,\infty;G)+\overline{N}_0(r,0;G)+\overline{N}_0(r,b;G)+\overline{N}_0(r,0;F)+S_0(r,G),\end{aligned}$$

which by (11) and Lemma 8 implies (9) and so contradicts (3). This proves the theorem. ∎

4. Applications. In this section we discuss two applications of the main theorem, the first of which improves a result of Yi and Yang [13] and the second gives a better answer to the question of Yang [9] mentioned in the introduction.

THEOREM 2. *Let f, g be two nonconstant meromorphic functions with $\Theta(\infty;f)=\Theta(\infty;g)=1$. If for $n\ge 1$ the derivatives $f^{(n)}$, $g^{(n)}$ share* $(1,2)$ *and*

$$\text{(i)}\ \sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)+\min\{\delta_2(b;f^{(n)}),\delta_2(b;g^{(n)})\}>1$$

for some $b\neq 0,1,\infty,1/2,2,-\omega,-\omega^2$, and

(ii) $\Theta(\alpha;f)+\Theta(\alpha;g)>1$

for some $\alpha\neq\infty$, then either (I) $f^{(n)}g^{(n)}\equiv 1$ *or* (II) $f\equiv g$.

Proof. From the given condition it follows that f, g are transcendental and so $f^{(n)}$, $g^{(n)}$ are transcendental. Choosing $\Psi(D)=D^n$ in Theorem 1 we get either $f^{(n)}g^{(n)}\equiv 1$ or $f-g\equiv Q$, where Q is a polynomial of degree at most $n-1$. If possible let $Q\not\equiv 0$. Then by Nevanlinna's theorem on three small functions [2, p. 47] we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,\alpha;f)+\overline{N}(r,\alpha+Q;f)+\overline{N}(r,\infty;f)+S(r,f)\\ &=\overline{N}(r,\alpha;f)+\overline{N}(r,\alpha;g)+\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Since $f-g\equiv Q$, it follows that $T(r,f)=T(r,g)+O(\log r)$. So $\Theta(\alpha;f)+\Theta(\alpha;g)\le 1$, which is a contradiction. Therefore $Q\equiv 0$ and so $f\equiv g$. This proves the theorem. ∎

The following examples show that the condition $\Theta(\alpha;f)+\Theta(\alpha;g)>1$ is necessary for the validity of case (II).

Example 3. Let $f=1+e^z$ and $g=e^z$. Then

$$\sum_{a\ne\infty}\delta(a;f)+\sum_{a\ne\infty}\delta(a;g)+\min\{\delta_2(b;f^{(n)}),\delta_2(b;g^{(n)})\}=2$$

for any $b\ne 0,\infty$, $\Theta(\infty;f)=\Theta(\infty;g)=1$, $\Theta(0;f)+\Theta(0;g)=1$, $\Theta(1;f)+\Theta(1;g)=1$, $\Theta(\alpha;f)+\Theta(\alpha;g)<1$ for $\alpha\ne 0,1,\infty$ and $f^{(n)}$, $g^{(n)}$ share $(1,2)$ but $f-g\equiv 1$.

Example 4. Let $f=1+e^z$ and $g=(-1)^ne^{-z}$. Then $\sum_{a\ne\infty}\delta(a;f)+\sum_{a\ne\infty}\delta(a;g)+\min\{\delta_2(b;f^{(n)}),\delta_2(b;g^{(n)})\}=2$ for any $b\ne 0,\infty$, $\Theta(\infty;f)=\Theta(\infty;g)=1$, $\Theta(0;f)+\Theta(0;g)=1$, $\Theta(1;f)+\Theta(1;g)=1$, $\Theta(\alpha;f)+\Theta(\alpha;g)<1$ for $\alpha\ne 0,1,\infty$ and $f^{(n)}$, $g^{(n)}$ share $(1,2)$ but $f^{(n)}g^{(n)}\equiv 1$.

Remark 1. Theorem 2 improves Theorem C, a result of Yi and Yang [13] and also a recent result of Lahiri [3].

In the following theorem we provide a better answer to a question of Yang [9] than those given in Theorems F and G.

Theorem 3. *Let f and g be two meromorphic functions such that $f^{(n)}$, $g^{(n)}$ $(n\ge 1)$ share $(1,2)$, f, g share $(\alpha,0)$ for some $\alpha\ne\infty$ and*

$$\begin{aligned} &\frac{\sum_{a\ne\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))}+\frac{\sum_{a\ne\infty}\delta(a;g)}{1+p(1-\Theta(\infty;g))}+\min\{\delta_2(b;f^{(n)}),\delta_2(b;g^{(n)})\}\\ &\qquad\qquad >1+\frac{2(1-\Theta(\infty;f))}{\sum_{a\ne\infty}\delta_p(a;f)}+\frac{2(1-\Theta(\infty;g))}{\sum_{a\ne\infty}\delta_p(a;g)} \end{aligned}$$

for some $b\ne 0,1,\infty,1/2,2,-\omega,-\omega^2$, with $\sum_{a\ne\infty}\delta_p(a;f)>0$, $\sum_{a\ne\infty}\delta_p(a;g)>0$ and ω being the imaginary cube root of unity. Then either $f^{(n)}g^{(n)}\equiv 1$ or $f\equiv g$.

Proof. From the assumption it follows that f and g are transcendental and so $f^{(n)}$ and $g^{(n)}$ are transcendental. Choosing $\Psi(D)=D^n$ we see from Theorem 1 that either $f-g\equiv Q$ or $f^{(n)}g^{(n)}\equiv 1$, where Q is a polynomial of degree at most $n-1$. If possible, let $Q\not\equiv 0$. Since f,g share $(\alpha,0)$, it follows that $\overline{N}(r,\alpha;f) = \overline{N}(r,\alpha;g) \le \overline{N}(r,0;Q) = O(\log r)$. Now by Nevanlinna's theorem on three small functions [2, p. 47] we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,\alpha;f) + \overline{N}(r,\alpha+Q;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,\alpha;f) + \overline{N}(r,\alpha;g) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,\infty;f) + O(\log r) + S(r,f), \end{aligned}$$

which implies that $\Theta(\infty;f) = 0$. Similarly we see that $\Theta(\infty;g) = 0$. Since this contradicts the assumption, it follows that $Q \equiv 0$ and so $f \equiv g$. This proves the theorem. ∎

The following example shows that Theorem 3 is sharp.

Example 5. Let $f = -2^{-n}e^{2z} + (-1)^{n+1}2^{-n}e^{z}$ and $g = (-1)^{n+1}2^{-n}e^{-2z} - 2^{-n}e^{-z}$. Then $f^{(n)}$, $g^{(n)}$ share $(1,2)$, f,g share $(0,0)$, $\Theta(\infty;f) = \Theta(\infty;g) = 1$ and $\sum_{a\ne\infty}\delta(a;f) + \sum_{a\ne\infty}\delta(a;g) + \min\{\delta_2(b;f^{(n)}),\delta_2(b;b^{(n)})\} = 1$ for any $b \ne 0,\infty$ but neither $f \equiv g$ nor $f^{(n)}g^{(n)} \equiv 1$.

Concluding Remark. Since Example 1 shows that in Theorem 1 sharing $(1,2)$ cannot be relaxed to sharing $(1,0)$, we conclude the paper with the following question: *Is it possible in Theorem* 1 *to relax sharing* $(1,2)$ *to sharing* $(1,1)$*?*

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Department of Mathematics
University of Kalyani
West Bengal 741235, India
E-mail: indrajit@cal2.vsnl.net.in

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